Information Diffusion in Social Networks

EE599: Social Network Systems

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Overview

- Network robustness/resilience and percolation theory
  - Cascades
- Information diffusion and epidemics
  - Network search   Easley & Kleinberg Ch 14 & others
- Learning and consensus formation

  Jackson Ch. 8

Opinion Diffusion & Consensus

- DeGroot Model for influence
  - Each person (node) starts with a belief on a subject
    - Represented by a probability (opinion)
      - $p_i(k)$ = p(agree with a specific idea @ time k)
  - Update opinion by a weighted sum of the opinion of others (e.g., neighbors)

DeGroot, Morris H. 1974. "Reaching a Consensus." *Journal of the American Statistical Association*, Vol. 69, No. 345, pp. 118-121.

DeGroot Model - Example

$$\mathbf{p}(k) = \mathbf{T}\mathbf{p}(k-1)$$

$$\begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} p_1(k-1) \\ p_2(k-1) \\ p_3(k-1) \end{bmatrix}$$

$t_{ij} =$ weight node i places on node j's opinion

DeGroot Model - Example

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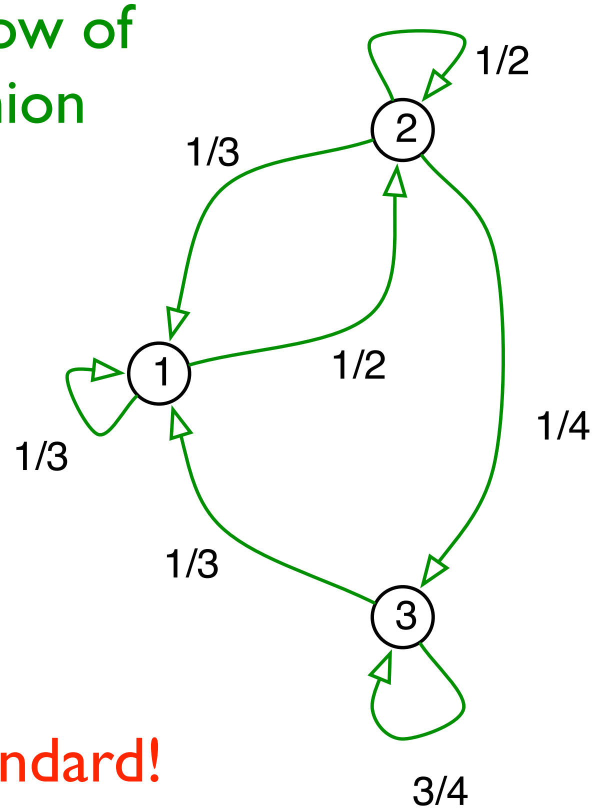
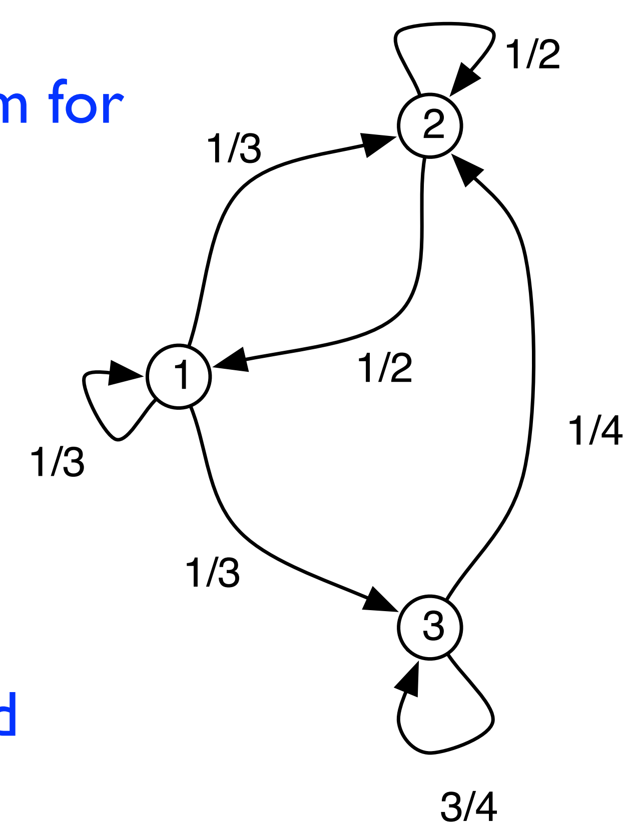
$t_{ij} =$ weight node i places on node j's opinion

the flow of opinion



not standard!

Standard diagram for DeGroot



Same as related DTMC

Opinion Diffusion & Consensus

- Will people come to a consensus?
- Do some people hold more influence in shaping group opinion
  - Opinion leaders, social influence, PageRank
- How quickly does "learning" occur?
- How does the network topology affect this process?

Understanding of the DeGroot model for opinion dynamics can be obtained from discrete time Markov chain (DTMC) results

Probability Review Items

- Some important random variables
  - Bernoulli, Binomial, Poisson, Gaussian
- Bayes Law & Theorem of Total Probability
- Moments and (Moment) Generating Functions
- Linear MMSE estimation
- Statistics
  - Law of Large Numbers
  - Central Limit Theorem
  - Confidence Intervals
  - Linear Regression
- **Markov Chains**

Reference:

A. Leon-Garcia, Probability, Statistics, and Random Processes for Electrical Engineer- ing, 3rd Edition, Addison Wesley, 2012.

DTMC Summary/Review

$$p(X_k = j | X_{k-1} = i, X_{k-2} = i_{k-2}, \ldots X_0 = i_0) = p(X_k = j | X_{k-1} = i) = p_{ij}$$

Process evolution depends on only one step in the past (state)

$$\pi_j(k) = p(X_k = j) = \sum_i \pi_i(k-1) p_{ij}$$

$$\boldsymbol{\pi}(k) = \boldsymbol{\pi}(k-1)\mathbf{P} = \boldsymbol{\pi}(0)\mathbf{P}^k$$

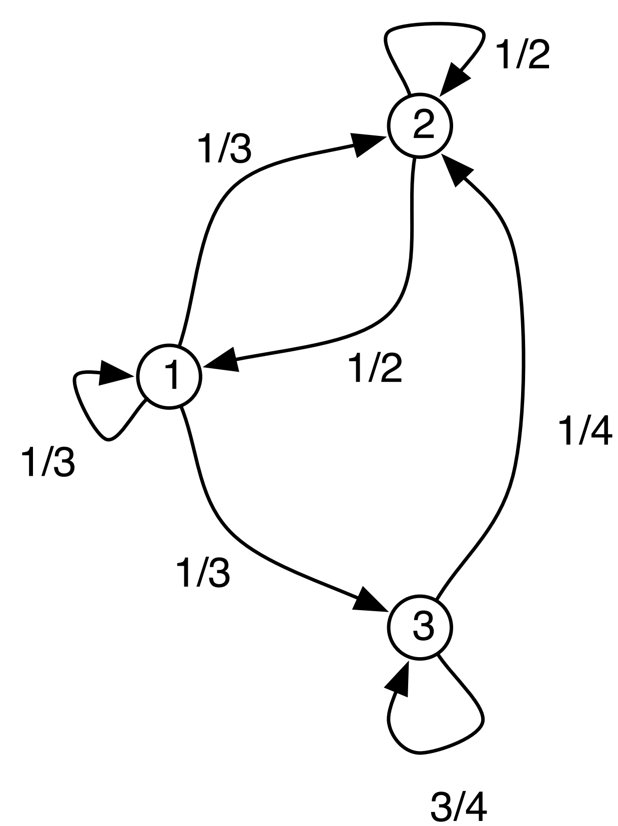
Probability mass function for state occupancy and state transition probability matrix

DTMC Model - Example

$$\begin{bmatrix} \pi_1(k) & \pi_2(k) & \pi_3(k) \end{bmatrix} = \begin{bmatrix} \pi_1(k-1) & \pi_2(k-1) & \pi_3(k-1) \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix}$$

$p_{ij} =$ probability of transitioning to i, given in j now

DTMC State Transition Diagram



DTMC Summary/Review

$$\boldsymbol{\pi}(k) = \boldsymbol{\pi}(k-1)\mathbf{P}$$

When will this have a fixed point?

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$$

When will it have a fixed point that is unique?

Doesn't matter what the initial distribution is….

DTMC Summary/Review

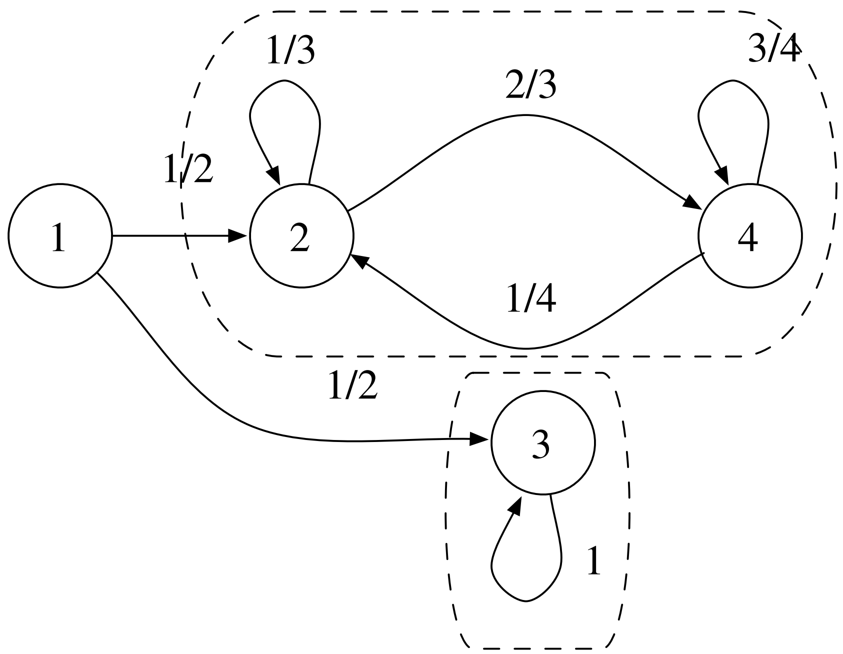


Figure 19: Two irreducible subchains exist in this DTMC.

for any $\boldsymbol{\pi}$ with $\pi_1 = \pi_3 = 0$ we have a unique solution for $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ and it is

$$\boldsymbol{\pi}(0) = \begin{pmatrix} 0 & p & 0 & (1-p) \end{pmatrix} \implies \lim_{n\to\infty} \boldsymbol{\pi}(n) = \begin{pmatrix} 0 & \frac{3}{11} & 0 & \frac{8}{11} \end{pmatrix} \tag{66}$$

If the process starts in state 3, it just stays there so we have

$$\boldsymbol{\pi}(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \implies \lim_{n\to\infty} \boldsymbol{\pi}(n) = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \tag{67}$$

If the process starts in state 1, then let A be the event it enters the 2-state subchain (*i.e.*, $P(A) = 1/2$). It follows that

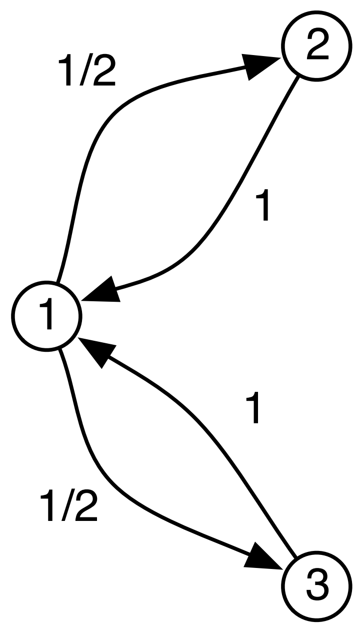
$$\boldsymbol{\pi}(n) = P(A)\boldsymbol{\pi}_A(n) + P(A^c)\boldsymbol{\pi}_{A^c}(n) \tag{68}$$

where $\boldsymbol{\pi}_A(n)$ is the state probability vector given that the state at time $n = 2$ is state 2 and $\boldsymbol{\pi}_{A^c}(n)$ has all of its mass at state 3 (not a function of n). It follows that for $\boldsymbol{\pi}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \boldsymbol{\pi}(n) &= \lim_{n\to\infty} \left(P(A)\boldsymbol{\pi}_A(n) + P(A^c)\boldsymbol{\pi}_{A^c}(n)\right) & (69)\\
&= P(A) \lim_{n\to\infty} \boldsymbol{\pi}_A(n) + P(A^c) \lim_{n\to\infty} \boldsymbol{\pi}_{A^c}(n) & (70)\\
&= P(A)\boldsymbol{\pi}_A + P(A^c)\boldsymbol{\pi}_{A^c} & (71)
\end{aligned}$$

Necessary: the DTMC must be irreducible (strongly connected)

DTMC Summary/Review



$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}^{2k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \qquad \mathbf{P}^{2k+1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

period = gcd(directed loop lengths)

Necessary: the DTMC must be aperiodic

DTMC Review

- An ergodic DTMC is one that is *aperiodic* and *irreducible*
- An ergodic DTMC has a unique stationary distribution
  - All rows of P^k (or T^k) converge to this stationary distribution
  - π_i ~ fraction of time spent in state i ~ inverse of mean return time
- Conclusion for DeGroot model
  - Consensus emerges for aperiodic, strongly connected models
    - π_i is the "social influence" of node i

$$p(\infty) = \boldsymbol{\pi}\mathbf{p}(0)$$

DeGroot Model - Example

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$$\mathbf{T}^k = \mathbf{P}^k = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix}^k \longrightarrow \begin{bmatrix} \boldsymbol{\pi} \\ \boldsymbol{\pi} \\ \boldsymbol{\pi} \end{bmatrix}$$

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \qquad \boldsymbol{\pi} = \begin{bmatrix} 2/5 & 2/5 & 1/5 \end{bmatrix}$$

Consensus: all nodes have same opinion in the limit

$$p(\infty) = \boldsymbol{\pi}\mathbf{p}(0) = \frac{1}{5}\left(2p_1(0) + 2p_2(0) + p_3(0)\right)$$

$$\mathbf{p}(\infty) = p(\infty) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

DeGroot Model - Example

$$\begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} p_1(k-1) \\ p_2(k-1) \\ p_3(k-1) \end{bmatrix}$$

Consensus: all nodes have same opinion in the limit

$$\mathbf{p}(\infty) = p(\infty) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$p(\infty) = \boldsymbol{\pi}\mathbf{p}(0) = \frac{1}{5}\left(2p_1(0) + 2p_2(0) + p_3(0)\right)$$

$$\mathbf{p}(0) = \begin{bmatrix} 0.9 \\ 0.9 \\ 0.1 \end{bmatrix} \longrightarrow \mathbf{p}(\infty) = \frac{37}{50} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

consensus in favor

$$\mathbf{p}(0) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \longrightarrow \mathbf{p}(\infty) = \frac{3}{10} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

consensus in opposition

Relation to PageRank

Basic PageRank is π — random web-surfer model
(Katz Prestige on directed graph)

Example worked in class

Solving for π

repeated multiplication
(eigen-vector amplification; numerical)

cut-set equations
(analytical)

Relation to PageRank

Dynamics — Eigen-decomposition of P/T matrix

$$\mathbf{P}^k = (\mathbf{E}\Lambda\mathbf{E}^{-1})^k = \mathbf{E}\Lambda^k\mathbf{E}^{-1} \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$$

$$\mathbf{P}(\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_n\mathbf{e}_n) = (\lambda_1\alpha_1\mathbf{e}_1 + \lambda_2\alpha_2\mathbf{e}_2 + \lambda_3\alpha_n\mathbf{e}_n)$$

$$\mathbf{P}^k(\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_n\mathbf{e}_n) = (\lambda_1^k\alpha_1\mathbf{e}_1 + \lambda_2^k\alpha_2\mathbf{e}_2 + \lambda_3^k\alpha_n\mathbf{e}_n)$$

$$\frac{1}{\lambda_1^k}\mathbf{P}^k\mathbf{x} \to \alpha_1\mathbf{e}_1$$

repeated multiplication by **P** will pull out the e-vector corresponding to the largest e-value

for row-stochastic P, the max lambda is one…

Extensions of DeGroot

- Non-linear and/or time-varying updates of opinions
  - Leads to Belief Propagation and related message-passing algorithms