Graph Theory and Social Networks - part 5

EE599: Social Network Systems

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Overview

- Last part of deterministic graph models for social networks
  - Finish partitioning
  - Segregation, polarization (Easley & Kleinberg, Ch. 4, 5)
    - Context for social networks
    - Homophily
    - Positive/negative links

Network Partitioning Approaches

- Multiple approaches [Jackson 13.2]
  - Edge removal - e.g., Girvan-Neman
  - Hierarchical Clustering
    - *Group like nodes according to some similarity measure*
  - Iterated correlation (CONCOR)
    - *Iterate a correlation measure on rows of adjacency matrix*
  - Maximum Likelihood approaches (EM algorithm)
    - *Find partition that maximizes probability of observed graph given intra-community and inter-community edge probabilities*

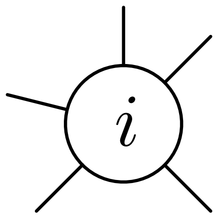
Network Partitioning

- When to stop?
  - Need a measure of how good a partition is…
  - One measure is *modularity* — compare how much larger intra-community edge frequencies with those that would be expected under random connections
    - Example: Girvan-Newman
      - *stop removing edges when modularity decreases*
      - *go through entire process and select partition with highest modularity*

Modularity

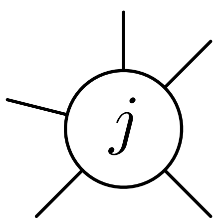
- Measures how much larger the fraction of intra-community edges is relative to the fraction expected under random placement
  - Positive modularity => captures community structure
  - Zero modularity => same as expected under random grouping
  - Negative modularity => worse than random grouping

Modularity - Random Model



k_i “edge stubs”

Randomly select from the 2L edge stubs to connect with the L edges



k_j “edge stubs”

This model attempts to preserve edge degree while having random connections

Prob(selecting an edge stub connected to node i) = $\dfrac{k_i}{2L}$

Prob(selecting an edge stub connected to node j) = $\dfrac{k_j}{2L}$

Prob(randomly connecting i & j) = $\dfrac{k_i k_j}{4L^2}$

(This may have some issues since it allows for nodes to connect to themselves)

Modularity

fraction of all edges connecting nodes in same community

fraction of edges connecting nodes in community m to other nodes in community m

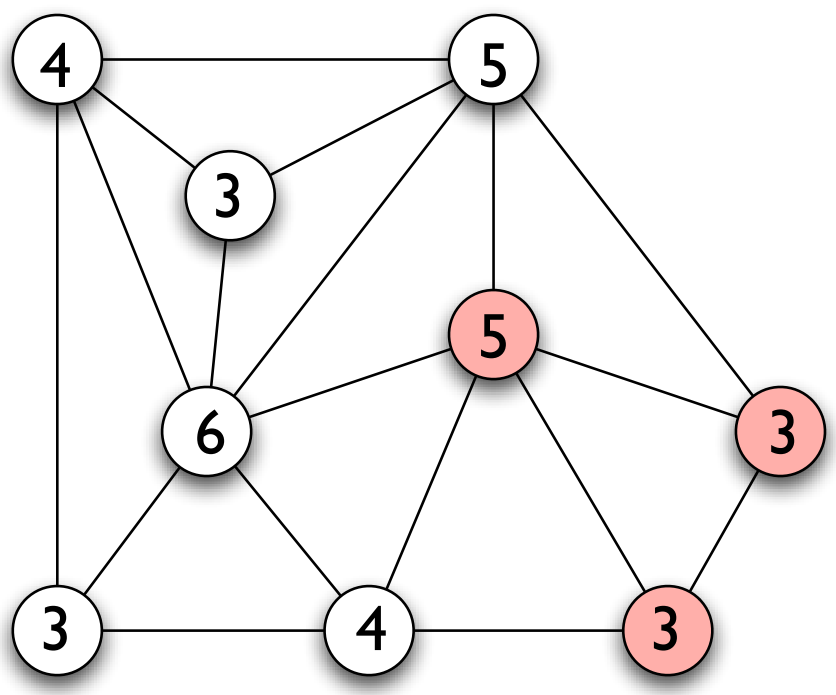
$$Q = \sum_{m=1: i,j \in C_m}^{M} \frac{a_{ij}}{2L} - \sum_{m=1: i,j \in C_m}^{M} \frac{k_i k_j}{2L} = \sum_{m=1}^{M} e_{mm} - r_m^2 \qquad r_m = \frac{1}{2L} \sum_{i \in C_m} k_i$$

sum over nodes in the same communities

fraction of all edges connecting nodes in same community under random model

sum over all communities

Network Partitioning - Modularity



$$e_{ww} = \frac{10}{18} \qquad e_{pp} = \frac{3}{18}$$

$$r_w = \frac{25}{36} \qquad r_p = \frac{11}{36}$$

$$Q = \frac{13}{18} - \left(\frac{25}{36}\right)^2 - \left(\frac{11}{36}\right)^2 = 0.147 > 0$$

I suspect that there may be some other reasonable (better?) ways to define the quality of a partition

Network Partitioning & Betweenness

- Community Detection/Partitioning
  - http://arxiv.org/abs/0908.1062 - comparison of partitioning algorithms http://arxiv.org/abs/0908.1062
  - http://compbio.cs.uic.edu/~chayant/work/phd_thesis.pdf - thesis on community detection
- Brandes Algorithm [77] in Easley & Kleinberg
  - http://www.inf.uni-konstanz.de/algo/publications/b-fabc-01.pdf
- Code available to compute many of these properties and algorithms
  - MatLabBGL (www.mathworks.com/matlabcentral/fileexchange/10922-matlabbgl)
  - MIT MatLab routines (http://strategic.mit.edu/downloads.php?page=matlab_networks)
- Modularity & Girvan-Newman
  - Original: http://journals.aps.org/pre/abstract/10.1103/PhysRevE.69.026113
  - More from Newman: http://www.pnas.org/content/103/23/8577.full.pdf+html

Overview

- Last part of deterministic graph models for social networks
  - Finish partitioning
  - Segregation, polarization (Easley & Kleinberg, Ch. 4, 5)
    - Context for social networks
    - Homophily
    - Positive/negative links

Homophily

- *Homophily* is the property of "like" nodes to connect in a social network
  - Like ~ race, interest, age, etc.
  - Two predominant mechanisms
    - *Selection*: people tend to befriend people similar to themselves
    - *Social Influence* (peer pressure): once in with a group, people tend to conform

Homophily

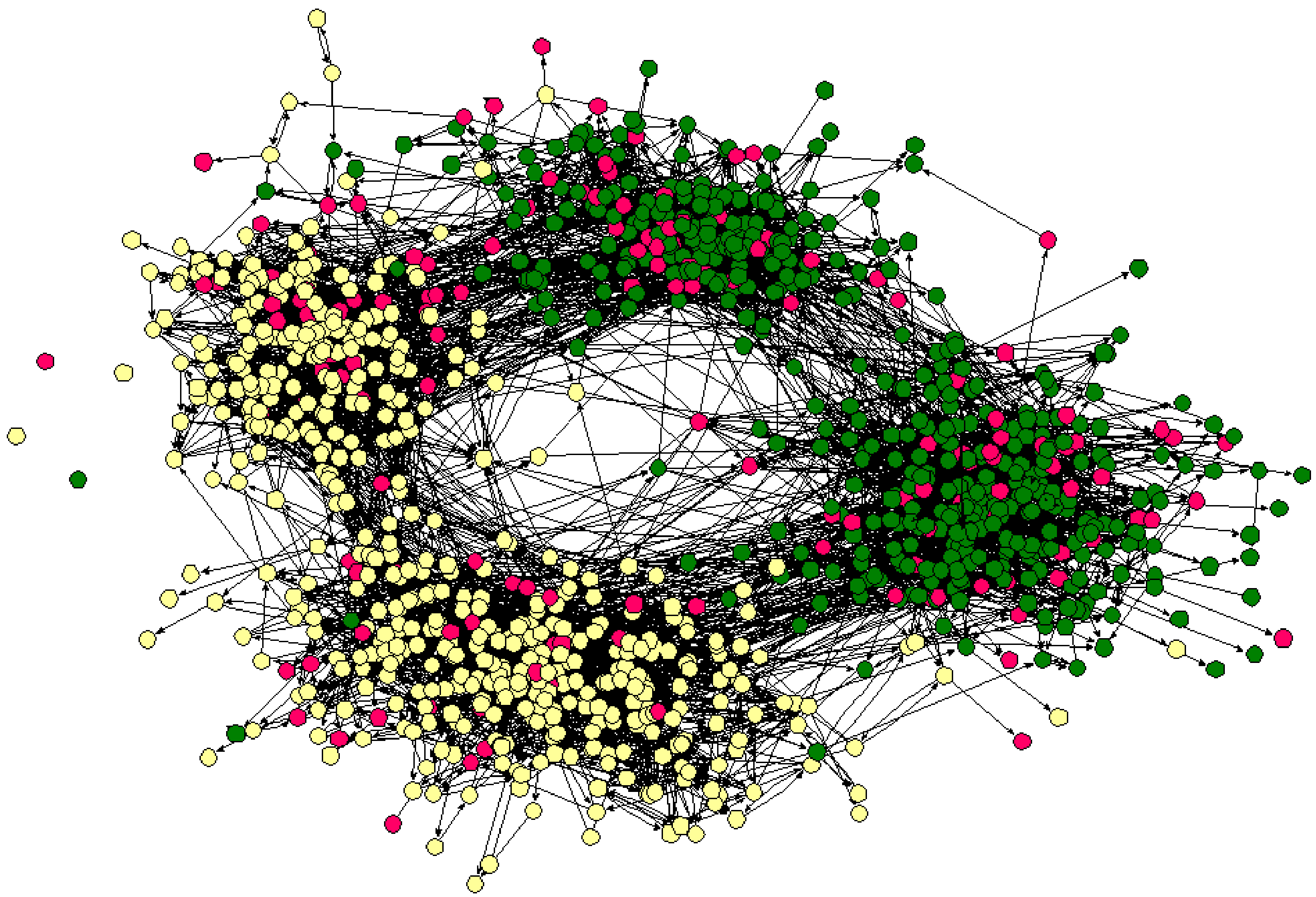


Figure 4.1: Homophily can produce a division of a social network into densely-connected, homogeneous parts that are weakly connected to each other. In this social network from a town's middle school and high school, two such divisions in the network are apparent: one based on race (with students of different races drawn as differently colored circles), and the other based on friendships in the middle and high schools respectively [304].

Easley & Kleinberg

- Observed in real social network data

Homophily Test

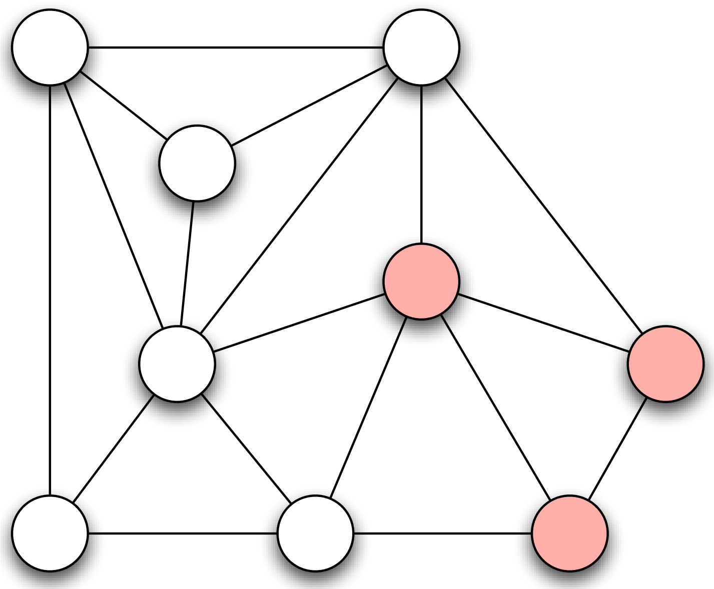


Figure 4.2: Using a numerical measure, one can determine whether small networks such as this one (with nodes divided into two types) exhibit homophily.

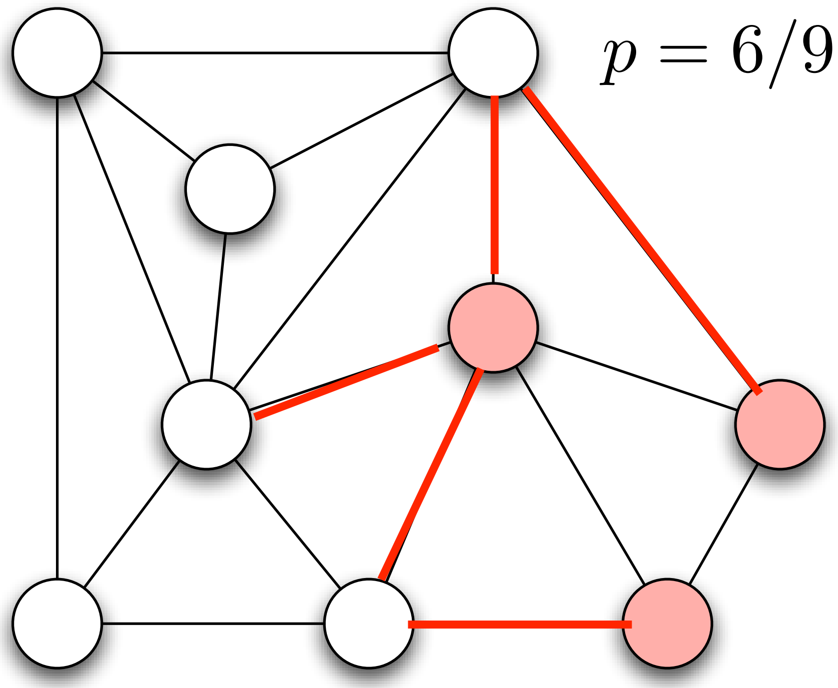
Easley & Kleinberg

- Observed frequency of inter-type links as compared to what would occur with random edge placement

- Expected inter-type link frequency under random connections: $2pq = 4/9 = 8/18$
- Observed inter-type links frequency: $5/18$

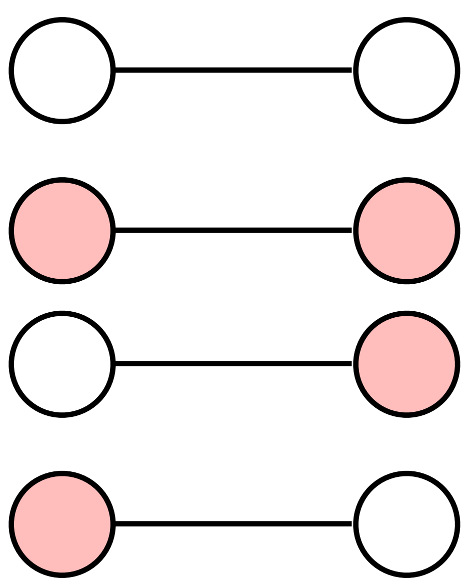
"some evidence of homophily"

Homophily Test



$p = 6/9 = 2/3$

- White nodes occur with freq: p=2/3
- Pink nodes occur with freq: q=1-p=1/3



p^2

q^2

pq

pq

- Expected frequencies for random connections:
  - white-pink/pink-white $2pq$

Homophily Measure ~ 5/18 - 2(2/3)(1/3) = -0.17 < 0

Social Network Context

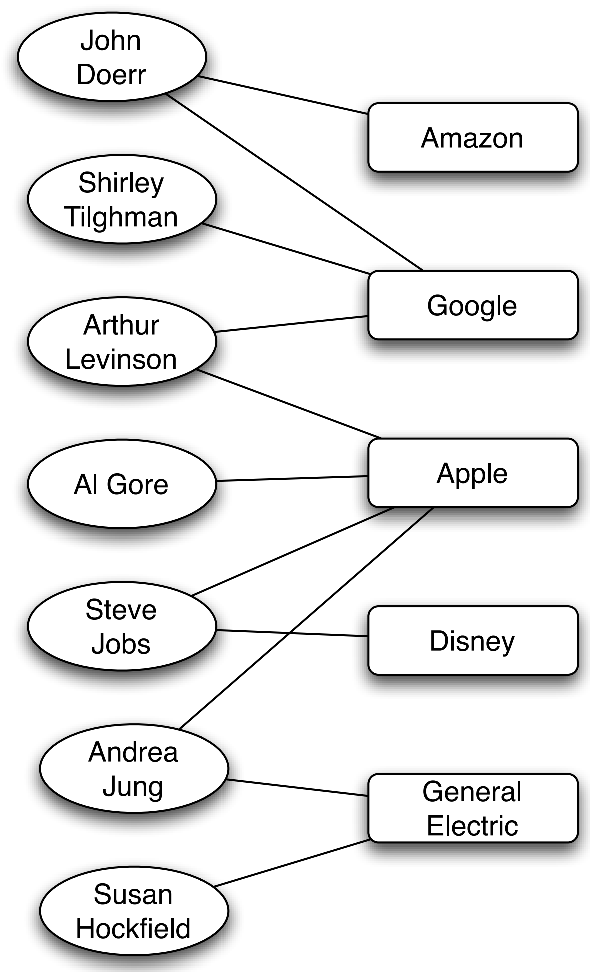


Figure 4.4: One type of affiliation network that has been widely studied is the memberships of people on corporate boards of directors [301]. A very small portion of this network (as of mid-2009) is shown here. The structural pattern of memberships can reveal subtleties in the interactions among both the board members and the companies.

Easley & Kleinberg

- *Affilition Network:* what foci are people members of?

Social Network Context

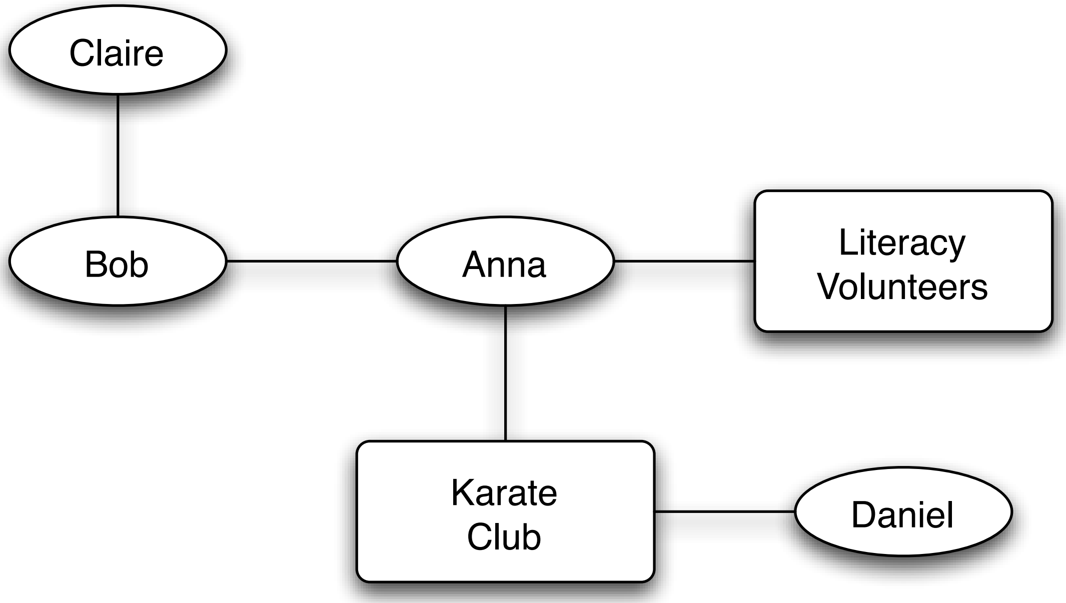


Figure 4.5: A social-affiliation network shows both the friendships between people and their affiliation with different social foci.

Easley & Kleinberg

- Social network and can be combined with affiliations to provide context

Social Network Context

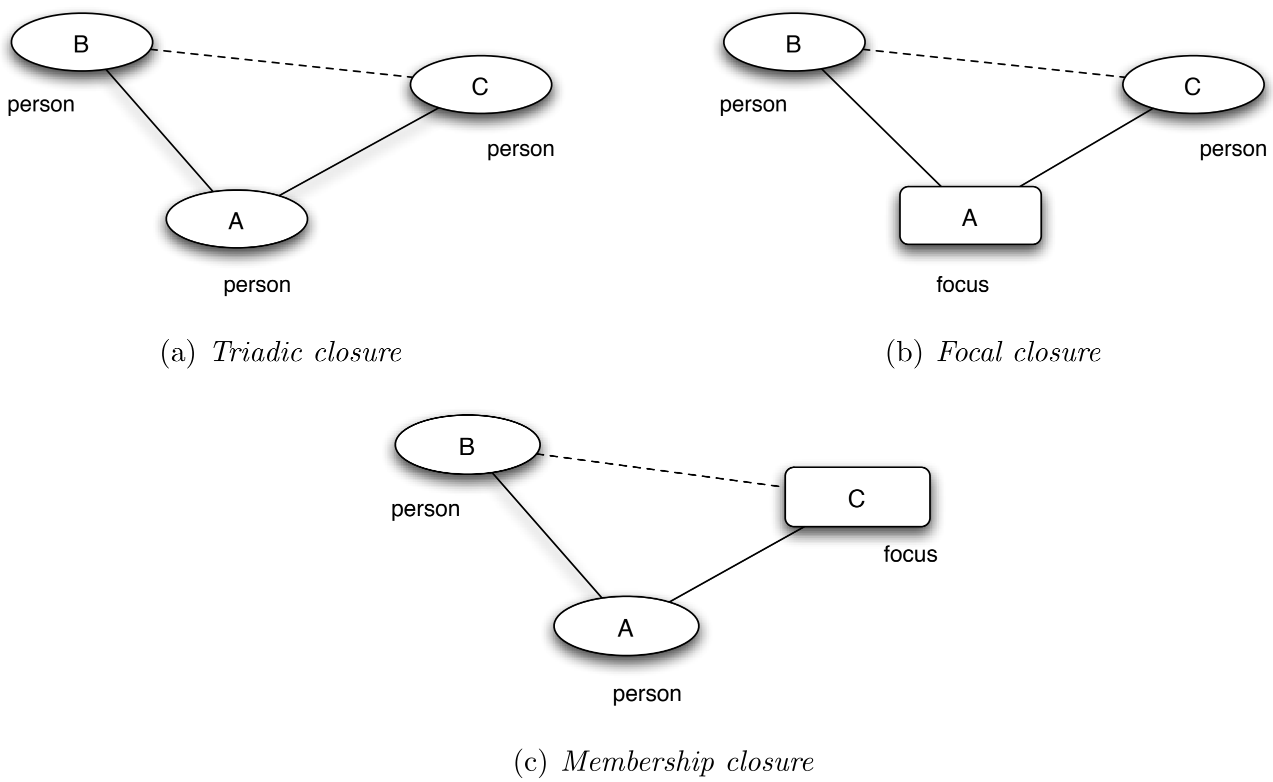


Figure 4.6: Each of triadic closure, focal closure, and membership closure corresponds to the closing of a triangle in a social-affiliation network.

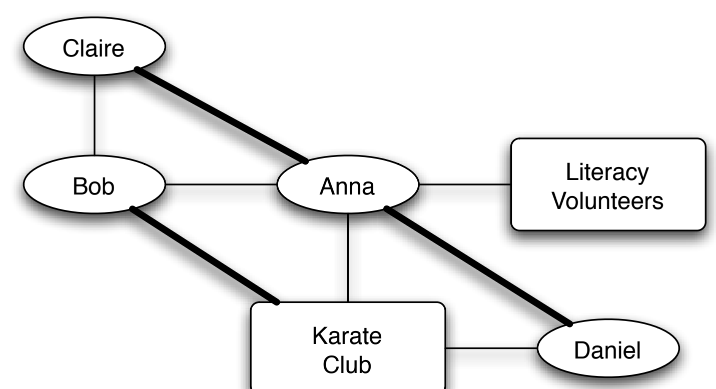
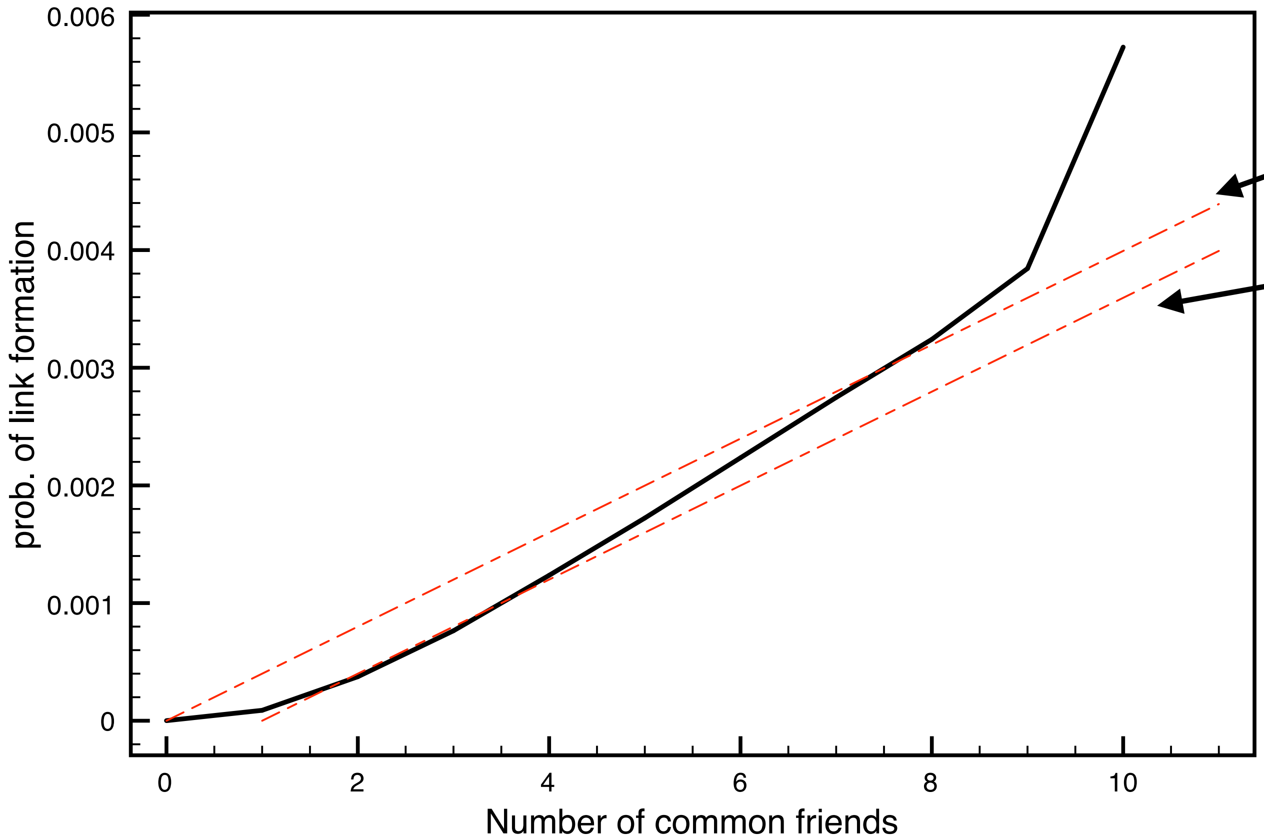


Figure 4.7: In a social-affiliation network containing both people and foci, edges can form under the effect of several different kinds of closure processes: two people with a friend in common, two people with a focus in common, or a person joining a focus that a friend is already involved in.

Easley & Kleinberg

Examples from Data



$1-(1-p)^k$

$1-(1-p)^{k-1}$

- Triadic closure as a function of number of common friends

Figure 4.9: Quantifying the effects of triadic closure in an e-mail dataset [259]. The curve determined from the data is shown in the solid black line; the dotted curves show a comparison to probabilities computed according to two simple baseline models in which common friends provide independent probabilities of link formation.

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Examples from Data

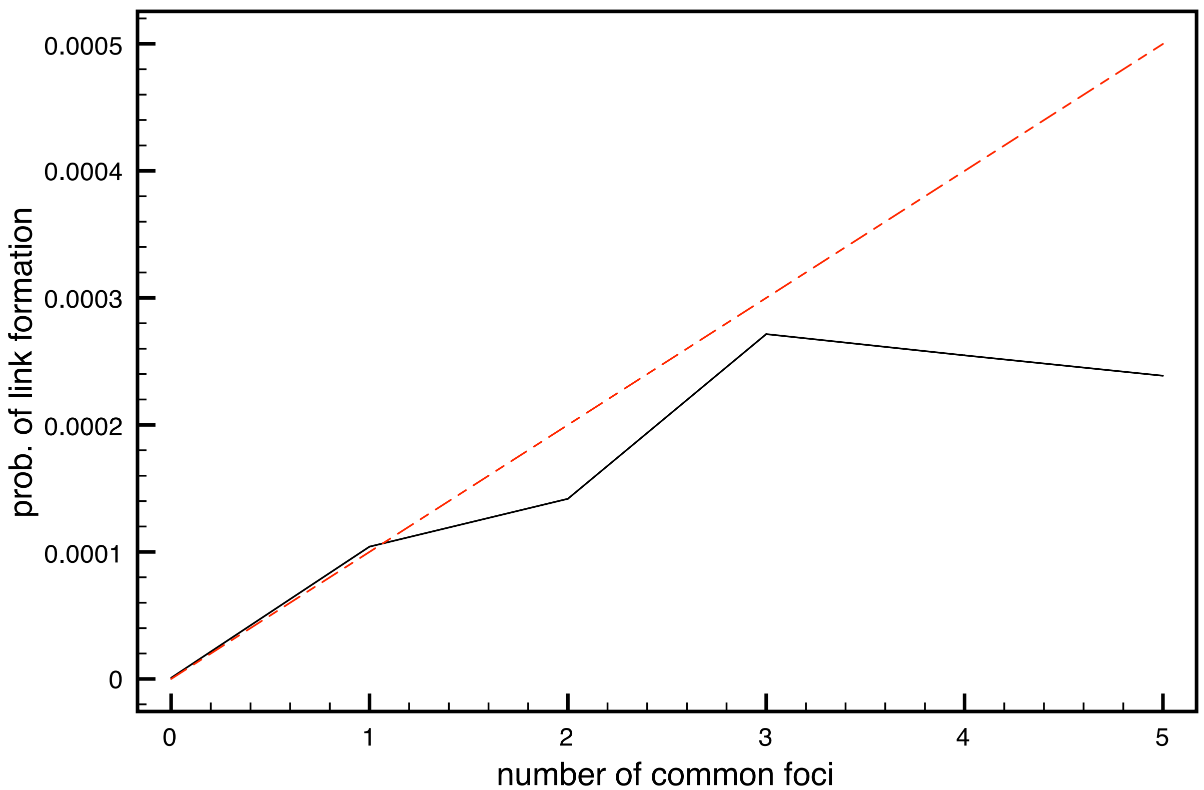

Figure 4.10: Quantifying the effects of focal closure in an e-mail dataset [259]. Again, the curve determined from the data is shown in the solid black line, while the dotted curve provides a comparison to a simple baseline.

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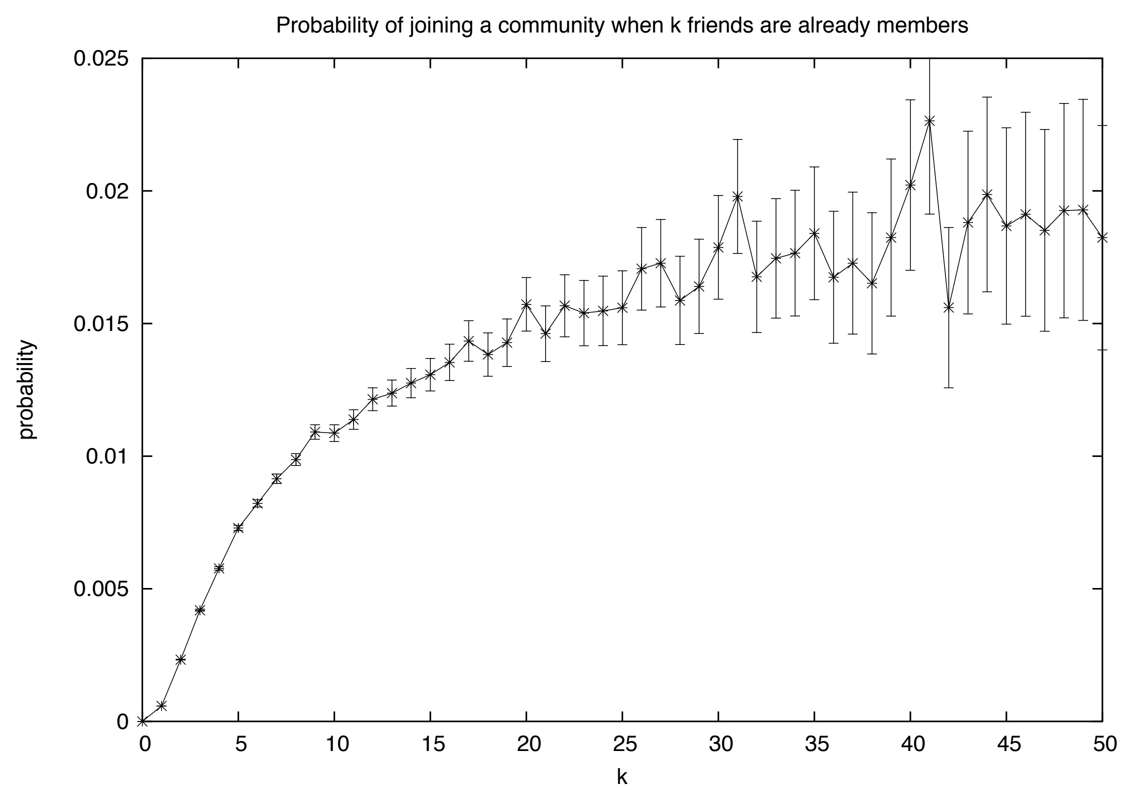
- Foci closure as a function of number of common foci
- Lesser effect of many shared foci as compared to many shared friends

Examples from Data



- Membership closure as a function of number of common memberships in online communities
- Lesser effect of many shared foci as compared to many shared friends

Figure 4.11: Quantifying the effects of membership closure in a large online dataset: The plot shows the probability of joining a LiveJournal community as a function of the number of friends who are already members [32].

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Examples from Data

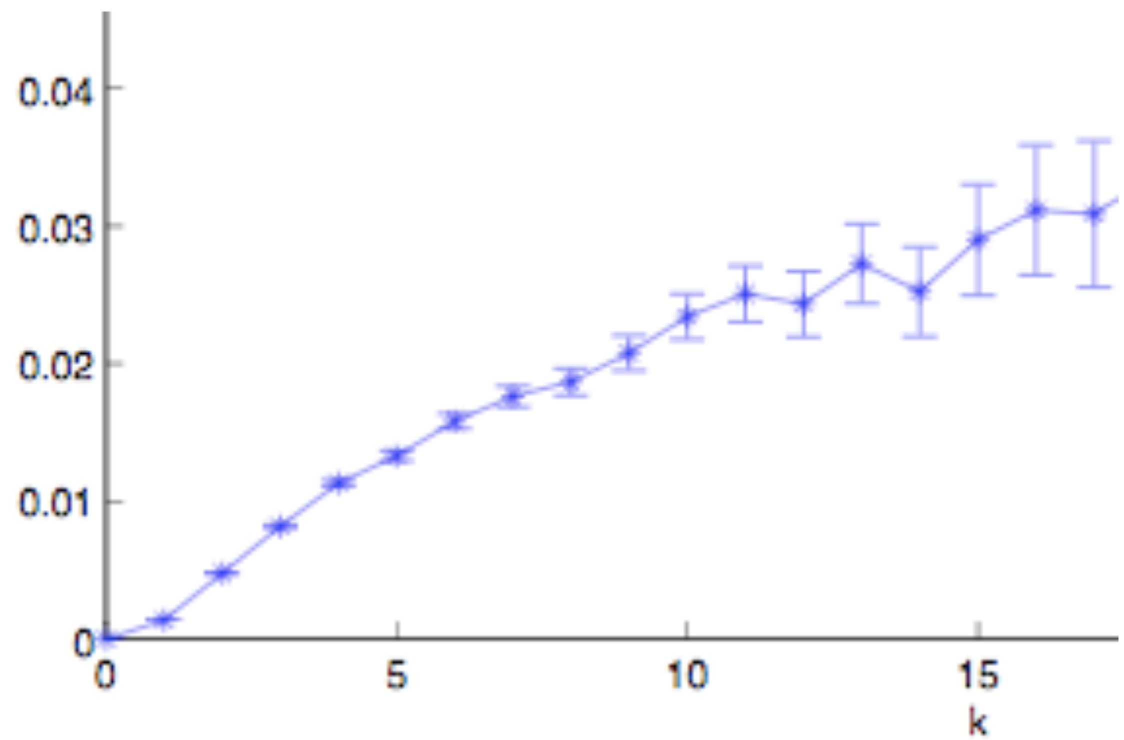

Figure 4.12: Quantifying the effects of membership closure in a large online dataset: The plot shows the probability of editing a Wikipedia articles as a function of the number of friends who have already done so [122].

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- Membership closure as a function of number of common pages edited on Wikipedia
- Lesser effect of many shared foci as compared to many shared friends

Examples from Data

- Many shared friends is the strongest force for closure
- In all cases, there is a significant jump in closure rates from having 1 entity in common to having 2 entities in common

Relative Cause of Homophily

$$\frac{\text{number of articles edited by } \textit{both } A \text{ and } B}{\text{number of articles edited by } \textit{at least one of } A \text{ or } B}$$

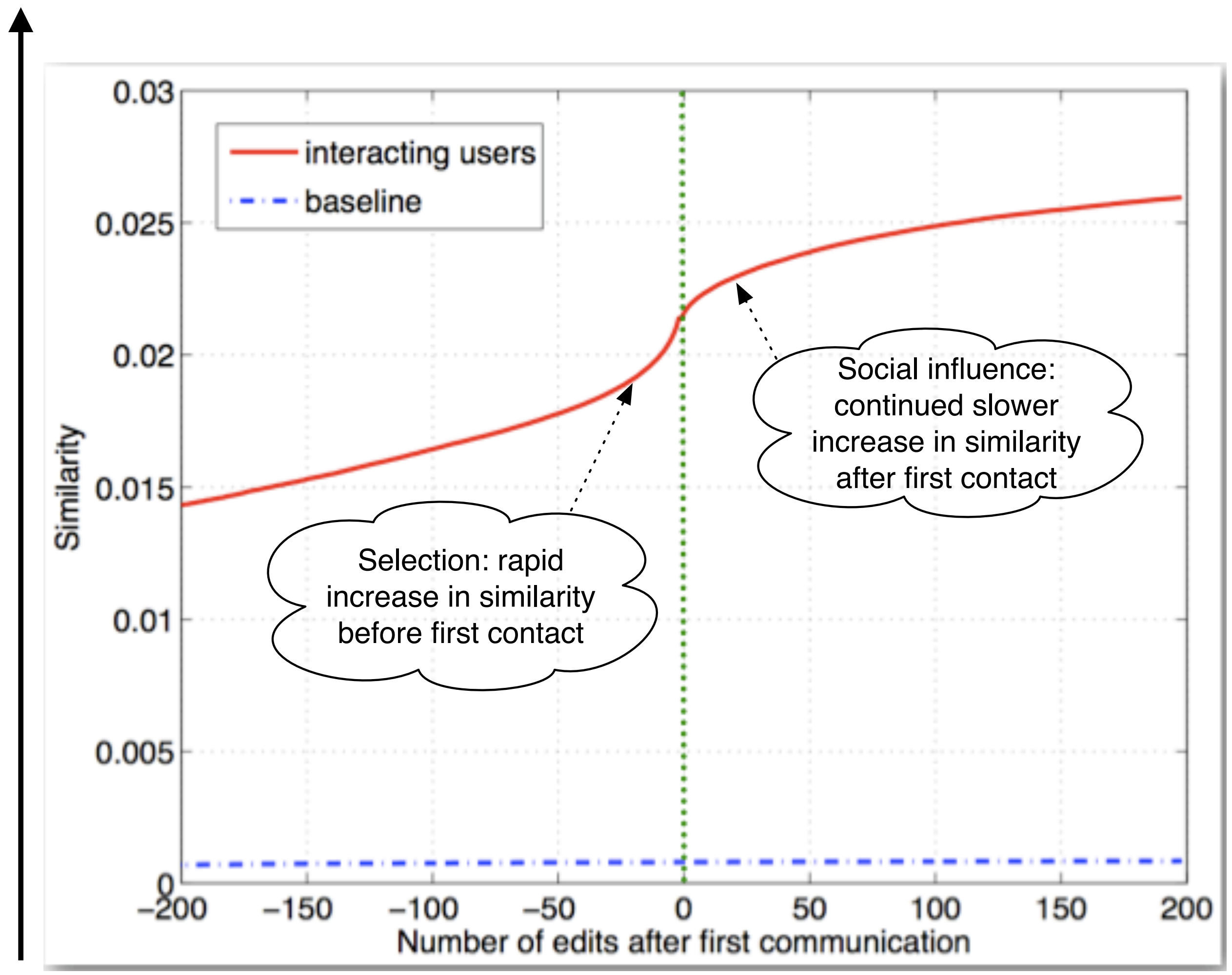

Figure 4.13: The average similarity of two editors on Wikipedia, relative to the time (0) at which they first communicated [122]. Time, on the x-axis, is measured in discrete units, where each unit corresponds to a single Wikipedia action taken by either of the two editors. The curve increases both before and after the first contact at time 0, indicating that both selection and social influence play a role; the increase in similarity is steepest just before time 0.

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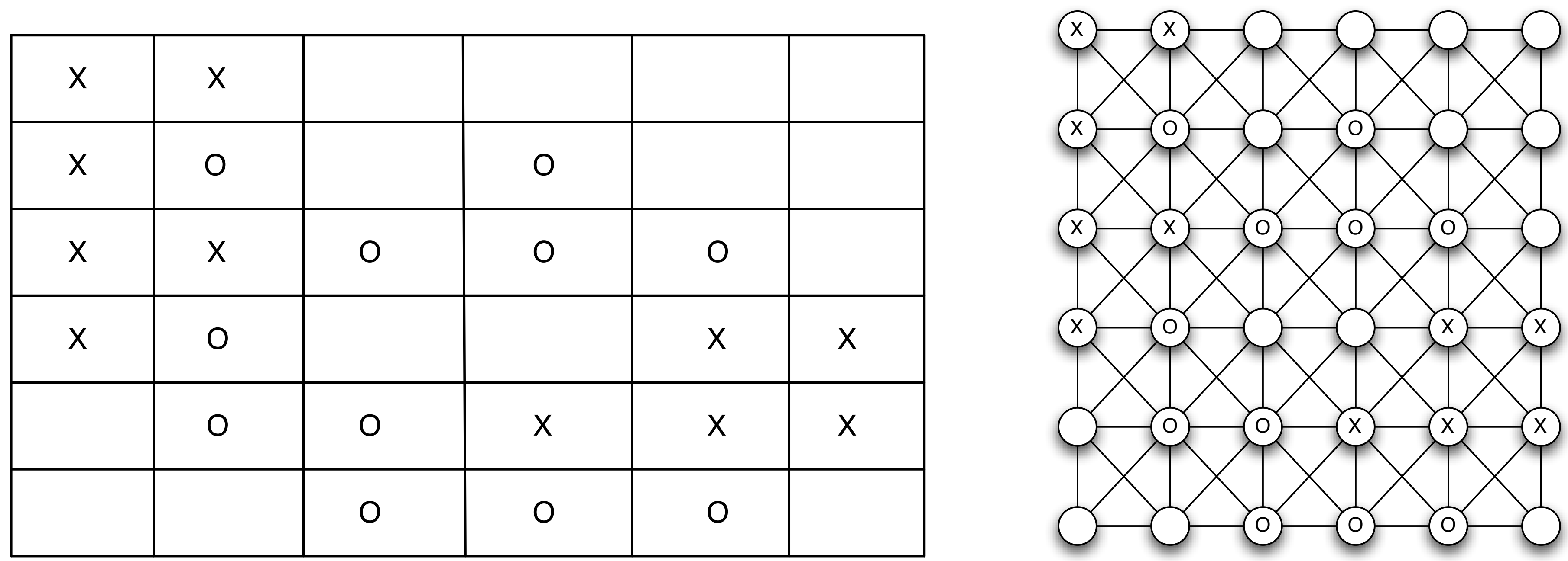
- Selection is a stronger cause for homophily in this experiment

Shelling's Model for Segregation



(a) *Agents occupying cells on a grid.* (b) *Neighbor relations as a graph.*

Figure 4.15: In Schelling's segregation model, agents of two different types (X and O) occupy cells on a grid. The neighbor relationships among the cells can be represented very simply as a graph. Agents care about whether they have at least some neighbors of the same type.

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- People require that *t* of their 8 neighbors are like them or else they move

Shelling's Model for Segregation

X1*	X2*				
X3	O1*		O2		
X4	X5	O3	O4	O5*	
X6*	O6			X7	X8
	O7	O8	X9*	X10	X11
		O9	O10	O11*	

(a) *An initial configuration.*

X3	X6	O1	O2		
X4	X5	O3	O4		
	O6	X2	X1	X7	X8
O11	O7	O8	X9	X10	X11
	O5	O9	O10*		

(b) *After one round of movement.*

Figure 4.16: After arranging agents in cells of the grid, we first determine which agents are *unsatisfied*, with fewer than t other agents of the same type as neighbors. In one round, each of these agents moves to a cell where they will be satisfied; this may cause other agents to become unsatisfied, in which case a new round of movement begins.
Easley & Kleinberg

- If a node is not satisfied (<t like neighbors), then she moves to an empty spot

Shelling's Model for Segregation

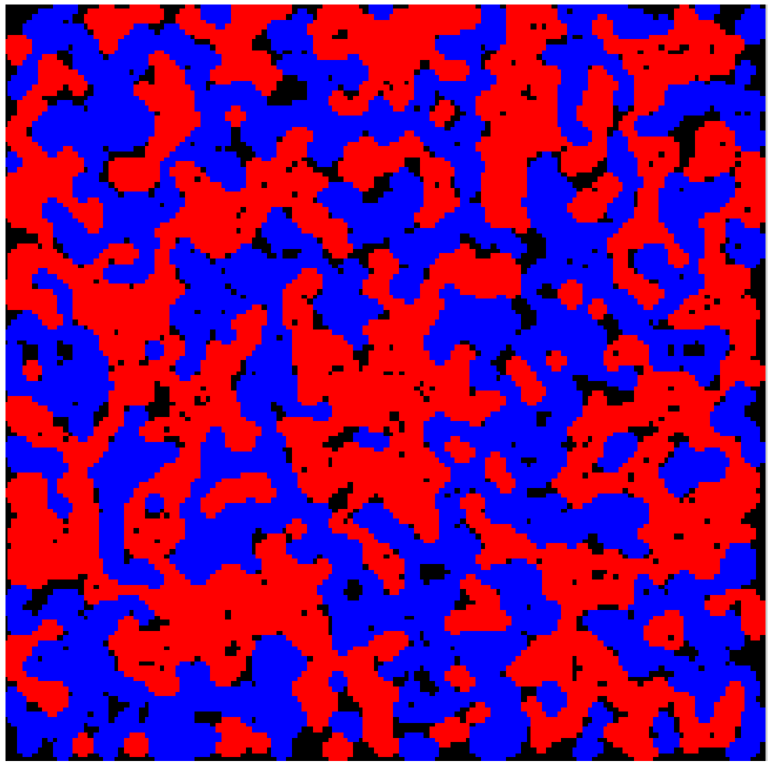
X	X	O	O	X	X
X	X	O	O	X	X
O	O	X	X	O	O
O	O	X	X	O	O
X	X	O	O	X	X
X	X	O	O	X	X

Figure 4.18: With a threshold of 3, it is possible to arrange agents in an integrated pattern: all agents are satisfied, and everyone who is not on the boundary on the grid has an equal number of neighbors of each type.

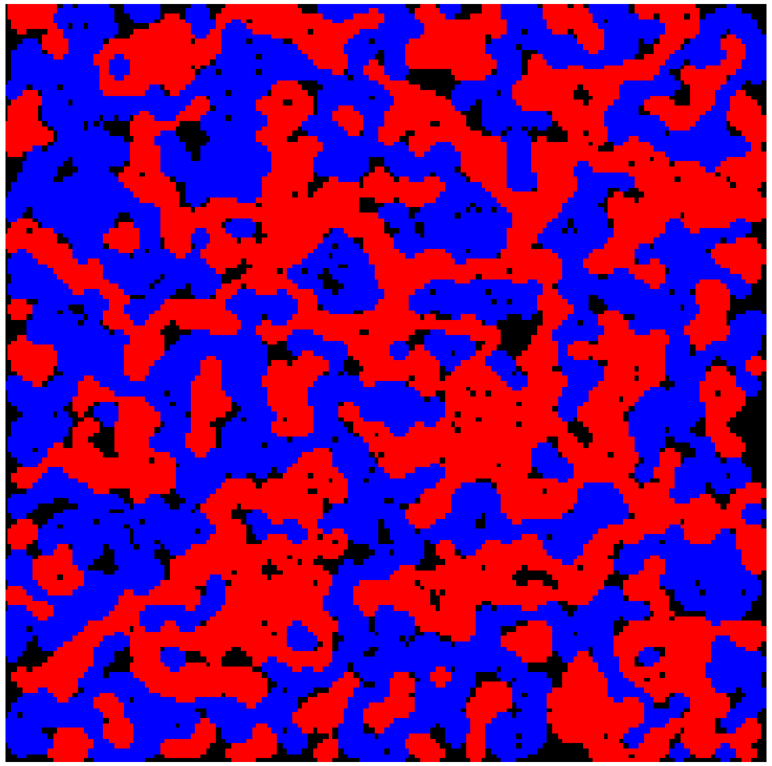
Easley & Kleinberg

- t=3 condition can be satisfied, but these are not likely to be found by randomly moving when dissatisfied.

Shelling's Model for Segregation



(a) *A simulation with threshold 3.*



(b) *Another simulation with threshold 3.*

Figure 4.17: Two runs of a simulation of the Schelling model with a threshold t of 3, on a 150-by-150 grid with 10,000 agents of each type. Each cell of the grid is colored red if it is occupied by an agent of the first type, blue if it is occupied by an agent of the second type, and black if it is empty (not occupied by any agent).

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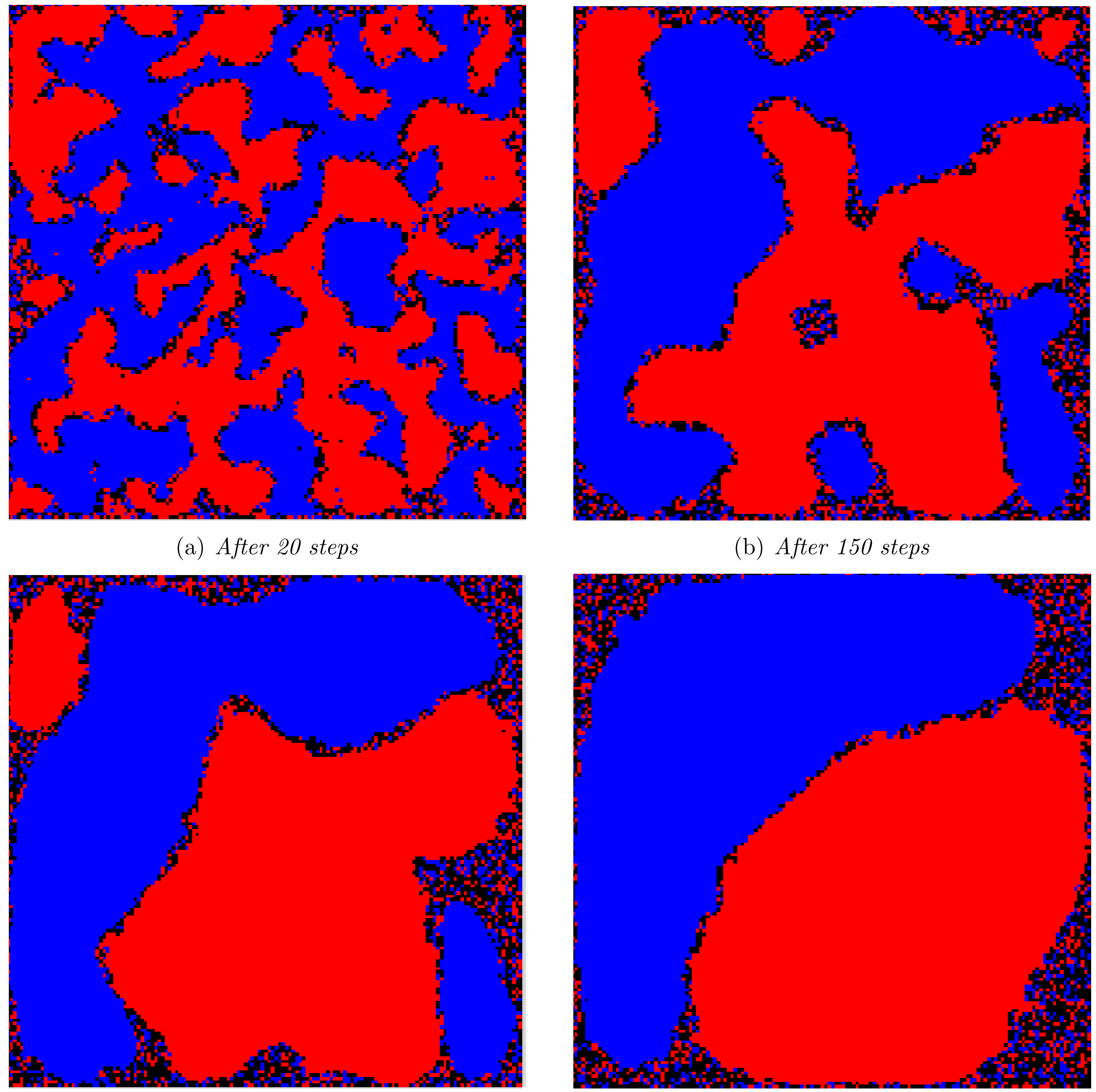
- t=3 is "tolerant" at the level of the individual, but still results in segregation with very high probability!

Shelling's Model for Segregation



(a) *After 20 steps* (b) *After 150 steps*

(c) *After 350 steps* (d) *After 800 steps*

Figure 4.19: Four intermediate points in a simulation of the Schelling model with a threshold t of 4, on a 150-by-150 grid with 10,000 agents of each type. As the rounds of movement progress, large homogeneous regions on the grid grow at the expense of smaller, narrower regions.

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- t=4 is even more pronounced

Shelling's Model for Segregation

- Many on-line simulators for this and related models
  - e.g., http://nifty.stanford.edu/2014/mccown-schelling-model-segregation/

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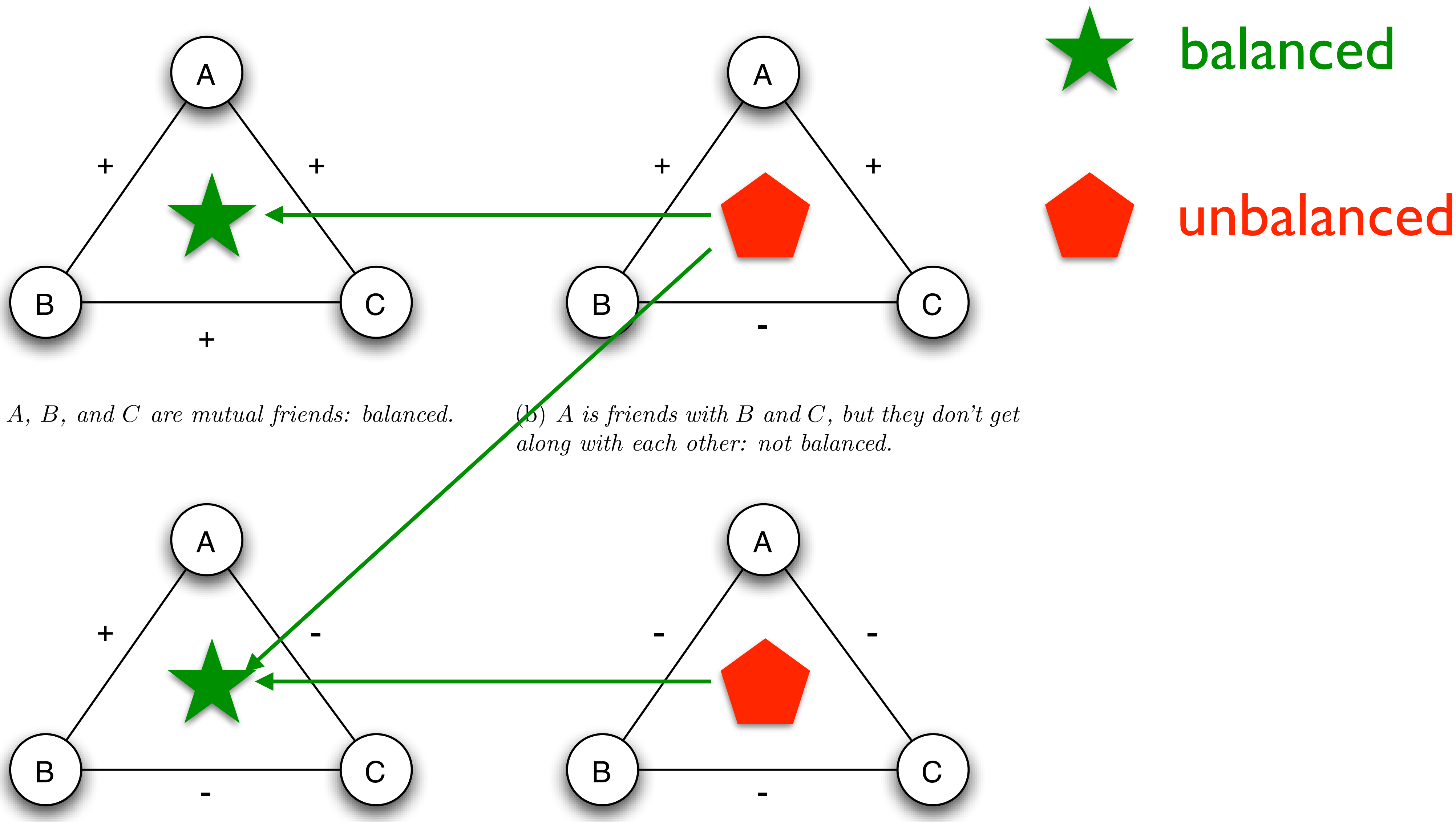
Positive & Negative Relationships

- Not all connections are positive friendships
  - What can we learn from modeling interactions as either positive (friends) or negative (enemies)?

Positive & Negative Relationships



(a) *A, B, and C are mutual friends: balanced.*

(b) *A is friends with B and C, but they don't get along with each other: not balanced.*

(c) *A and B are friends with C as a mutual enemy: balanced.*

(d) *A, B, and C are mutual enemies: not balanced.*

Figure 5.1: Structural balance: Each labeled triangle must have 1 or 3 positive edges.

Easley & Kleinberg

Structural Balance

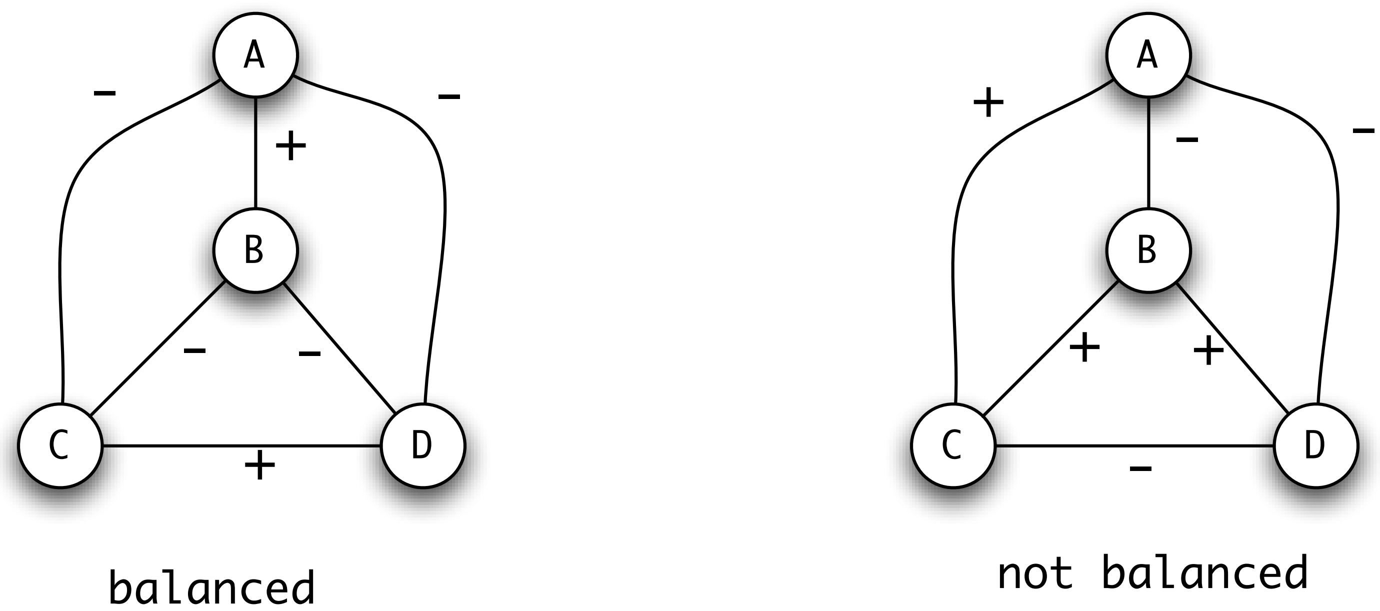


Figure 5.2: The labeled four-node complete graph on the left is balanced; the one on the right is not.

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- Balanced network = all triangles have 3 or 1 +'s

Balanced Complete Networks

Claim: If all triangles in a labeled complete graph are balanced, then either

(a) all pairs of nodes are friends, or else

(b) the nodes can be divided into two groups, X and Y, such that

- *(i) every pair of nodes in X like each other,*
- *(ii) every pair of nodes in Y like each other, and*
- *(iii) everyone in X is the enemy of everyone in Y.*

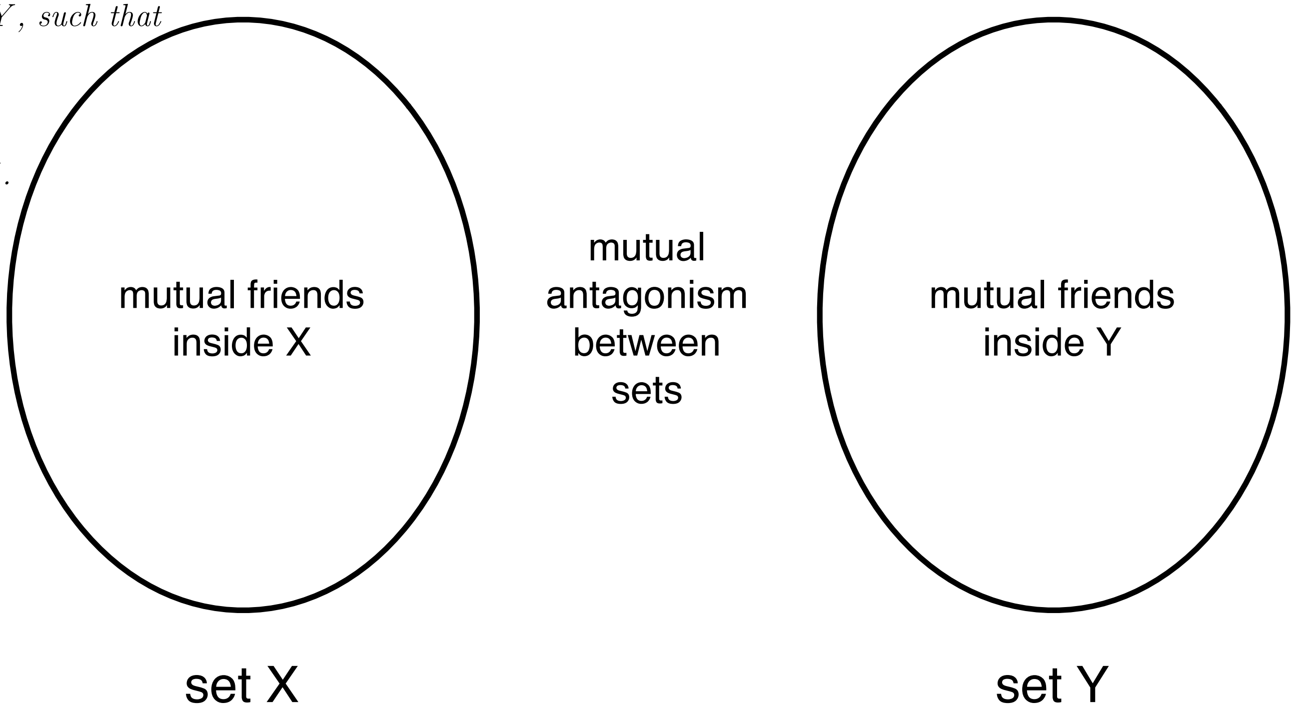

Figure 5.3: If a complete graph can be divided into two sets of mutual friends, with complete mutual antagonism between the two sets, then it is balanced. Furthermore, this is the only way for a complete graph to be balanced.

Easley & Kleinberg

- Balance Theorem: any <u>**complete**</u> balanced network has the above structure — pure two-party polarization. Other option is that everybody is friends (Y is empty set)

Proof of Balance Theorem

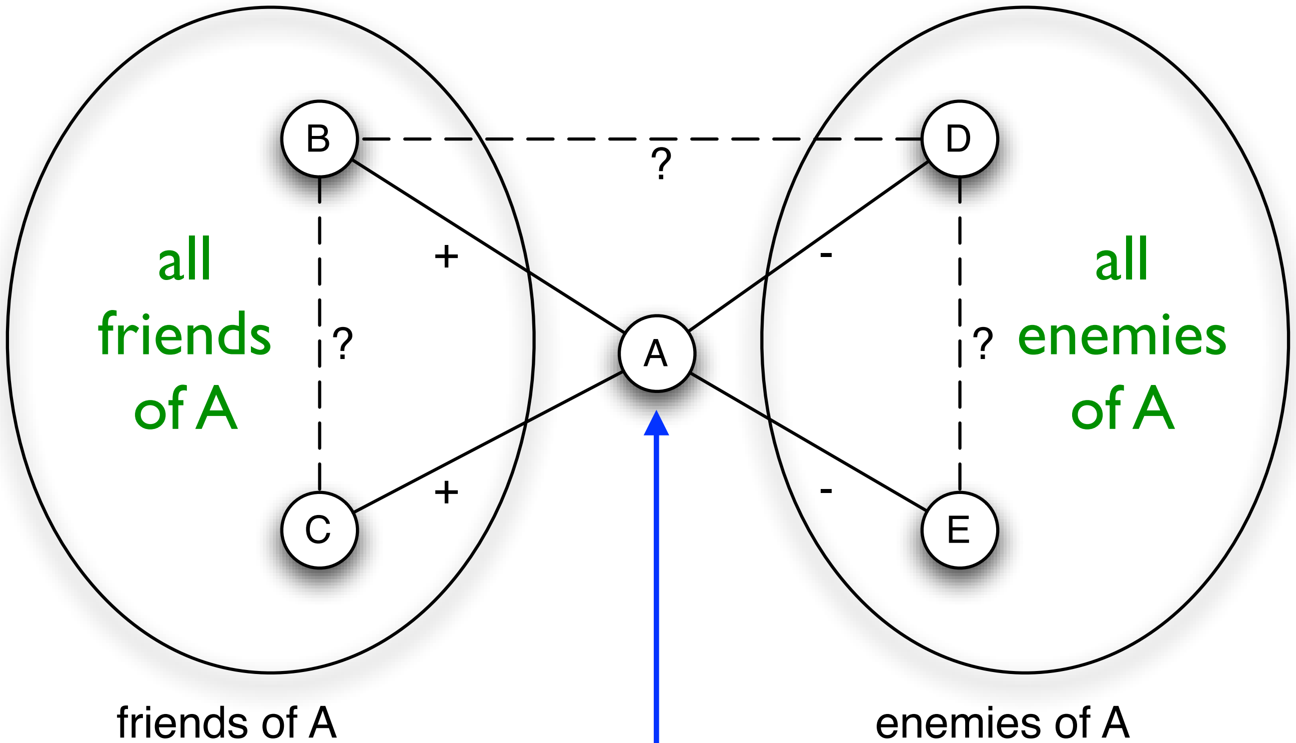

Figure 5.4: A schematic illustration of our analysis of balanced networks. (There may be other nodes not illustrated here.)

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node A selected arbitrarily

Proof of Balance Theorem

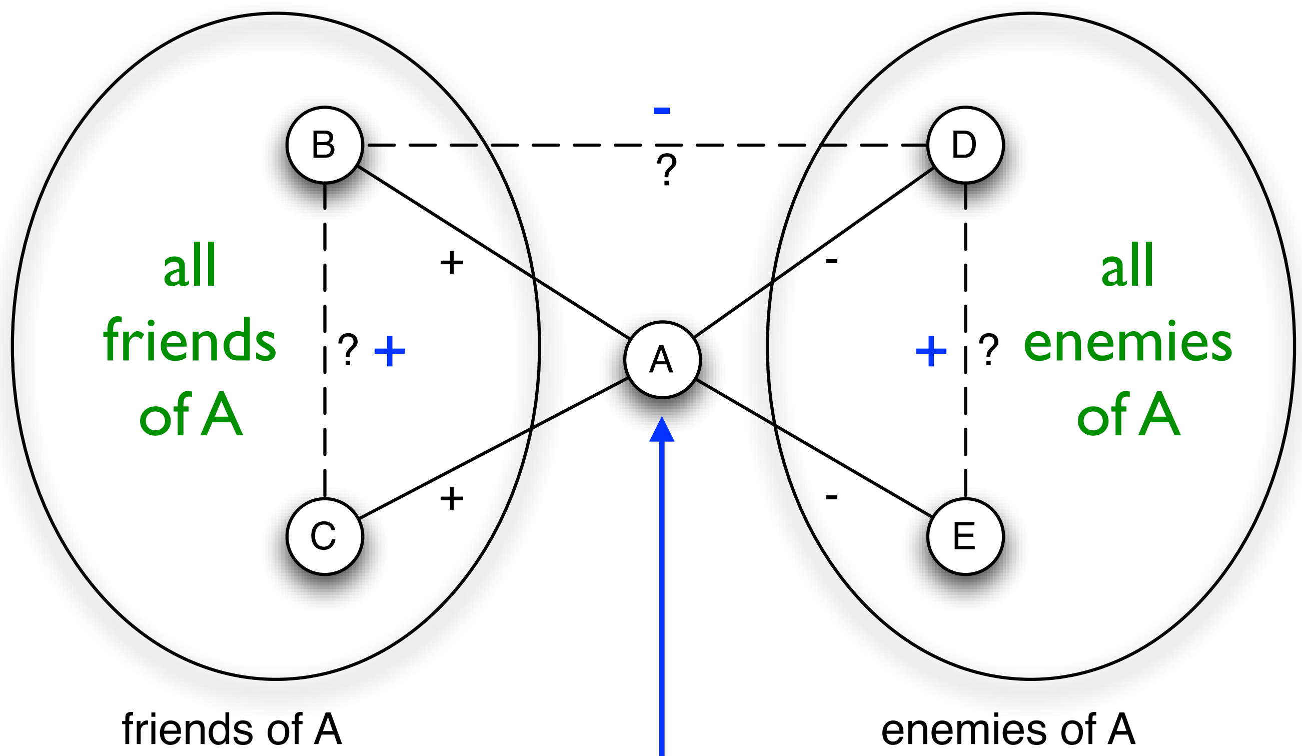

Figure 5.4: A schematic illustration of our analysis of balanced networks. (There may be other nodes not illustrated here.)

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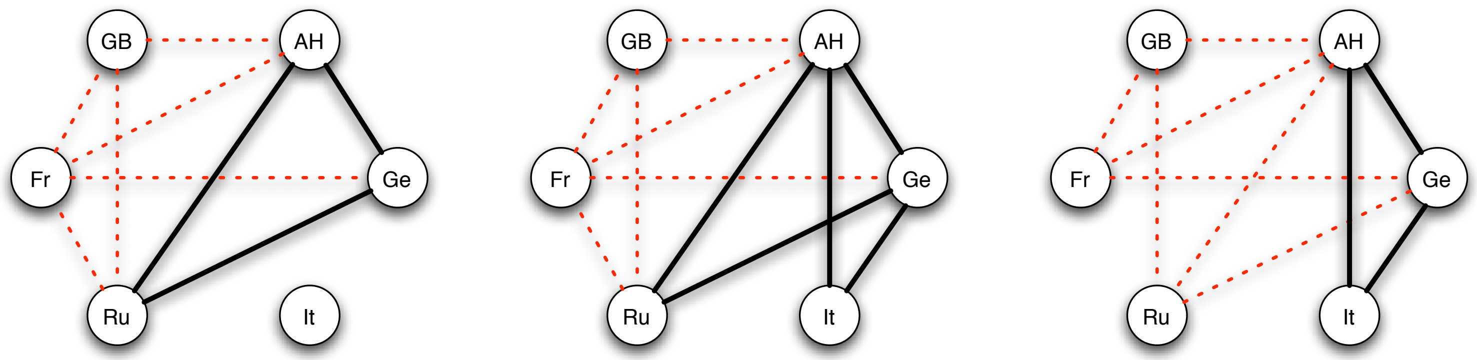
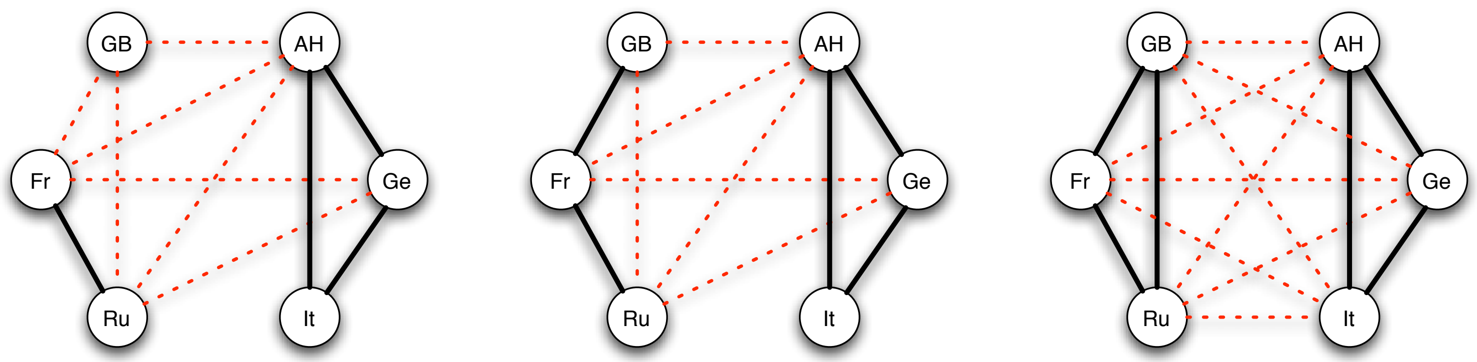
node A selected arbitrarily

- Other option is all

Example: Alliance Evolution Leading to WWI



(a) *Three Emperors' League 1872–81* (b) *Triple Alliance 1882* (c) *German-Russian Lapse 1890*



(d) *French-Russian Alliance 1891–94* (e) *Entente Cordiale 1904* (f) *British Russian Alliance 1907*

Figure 5.5: The evolution of alliances in Europe, 1872-1907 (the nations GB, Fr, Ru, It, Ge, and AH are Great Britain, France, Russia, Italy, Germany, and Austria-Hungary respectively). Solid dark edges indicate friendship while dotted red edges indicate enmity. Note how the network slides into a balanced labeling — and into World War I. This figure and example are from Antal, Krapivsky, and Redner [20].

Easley & Kleinberg

Generalizations of Structural Balance

- Approximately balanced (complete) networks
- Structural balance for incomplete graphs
  - *Definition and algorithmic test*
- Weak balance (complete networks)
  - *Only prohibit 2 + per triangle*
  - *Yields multiparty, mutually antagonistic structure*

Approximately Balanced Complete Networks

general statement

Claim: Let ε be any number such that $0 \leq \varepsilon < \frac{1}{8}$, and define $\delta = \sqrt[3]{\varepsilon}$. If at least $1 - \varepsilon$ of all triangles in a labeled complete graph are balanced, then either

(a) there is a set consisting of at least $1 - \delta$ of the nodes in which at least $1 - \delta$ of all pairs are friends, or else

(b) the nodes can be divided into two groups, X and Y, such that

(i) at least $1 - \delta$ of the pairs in X like each other,

(ii) at least $1 - \delta$ of the pairs in Y like each other, and

(iii) at least $1 - \delta$ of the pairs with one end in X and the other end in Y are enemies.

Easley & Kleinberg

example

Claim: If at least 99.9% of all triangles in a labeled complete graph are balanced, then either

(a) there is a set consisting of at least 90% of the nodes in which at least 90% of all pairs are friends, or else

(b) the nodes can be divided into two groups, X and Y, such that

(i) at least 90% of the pairs in X like each other,

(ii) at least 90% of the pairs in Y like each other, and

(iii) at least 90% of the pairs with one end in X and the other end in Y are enemies.

- Almost all triangles are balanced implies almost completely segregated

Approximately Balanced Complete Networks

- Outline of proof
  - Find a “good” node A
    - Find average number of unbalanced triangles in which nodes are involved (combinatorics)
    - Pigeon-hole principle implies that there is at least one node involved in at most this average number
  - Split network into friends/enemies of this good node A
    - Bound number of unbalanced conditions inside friend group, enemy group, and between these groups

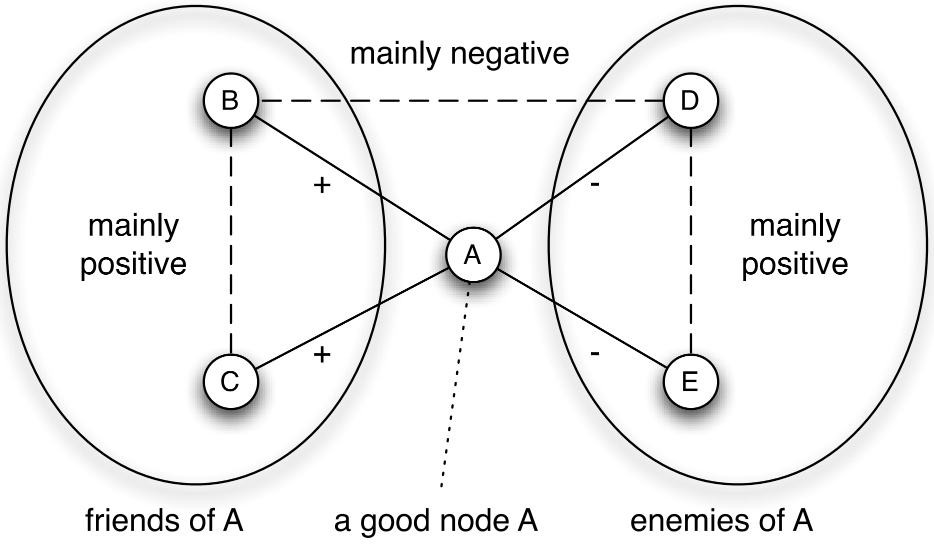


Figure 5.17: The characterization of approximately balanced complete graphs follows from an analysis similar to the proof of the original Balance Theorem. However, we have to be more careful in dividing the graph by first finding a “good” node that isn’t involved in too many unbalanced triangles.

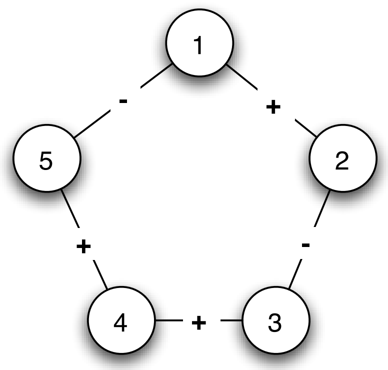
Easley & Kleinberg

Generalizations of Structural Balance

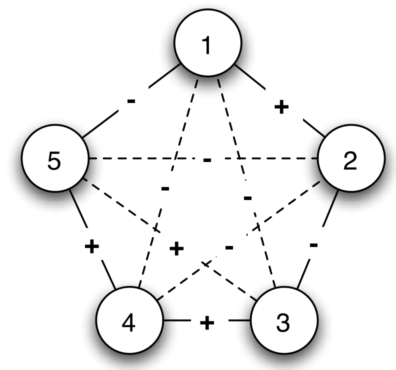
- Approximately balanced (complete) networks
- Structural balance for incomplete graphs
  - *Definition and algorithmic test*
- Weak balance (complete networks)
  - *Only prohibit 2 + per triangle*
  - *Yields multiparty, mutually antagonistic structure*

Structural Balance in Incomplete Networks

Is this network balanced?

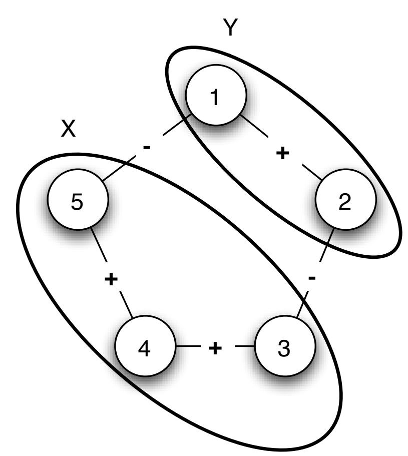


(a) *A graph with signed edges.*



(b) *Filling in the missing edges to achieve balance.*

Yes: can fill in missing edges consistent with a complete, balanced network



(c) *Dividing the graph into two sets.*

Yes: can partition nodes in two mutually agnostic groups of friends

Figure 5.9: There are two equivalent ways to define structural balance for general (non-complete) graphs. One definition asks whether it is possible to fill in the remaining edges so as to produce a signed complete graph that is balanced. The other definition asks whether it is possible to divide the nodes into two sets X and Y so that all edges inside X and inside Y are positive, and all edges between X and Y are negative.

Easley & Kleinberg

- These two "definitions" are equivalent — this is structural balance in incomplete networks

Structural Balance in Incomplete Networks

Is this network balanced?

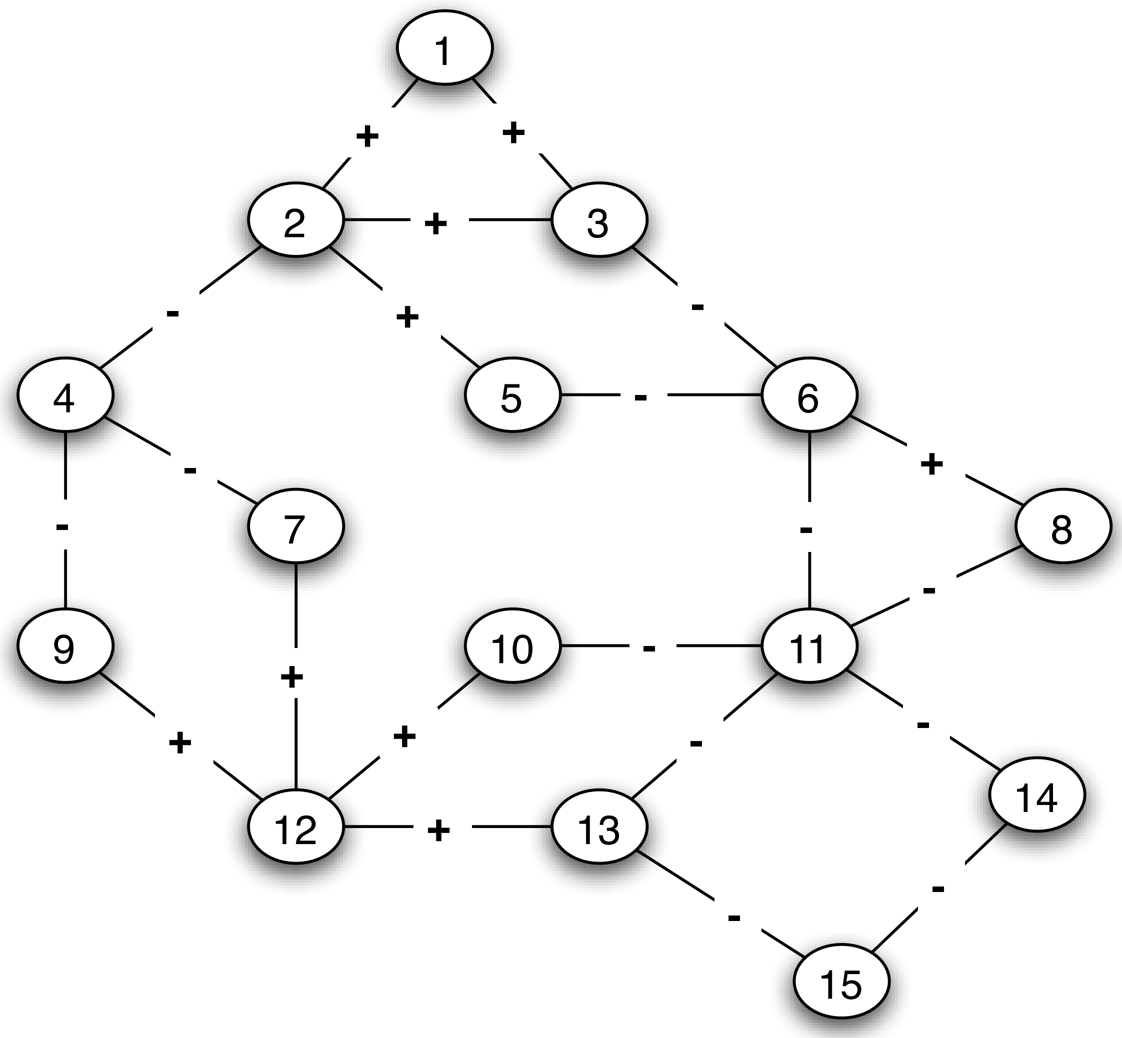


Figure 5.8: In graphs that are not complete, we can still define notions of structural balance when the edges that are present have positive or negative signs indicating friend or enemy relations.

Easley & Kleinberg

- Difficult to ascertain as networks becomes nontrivial

Structural Balance in Incomplete Networks

Balanced if this sign is changed to +

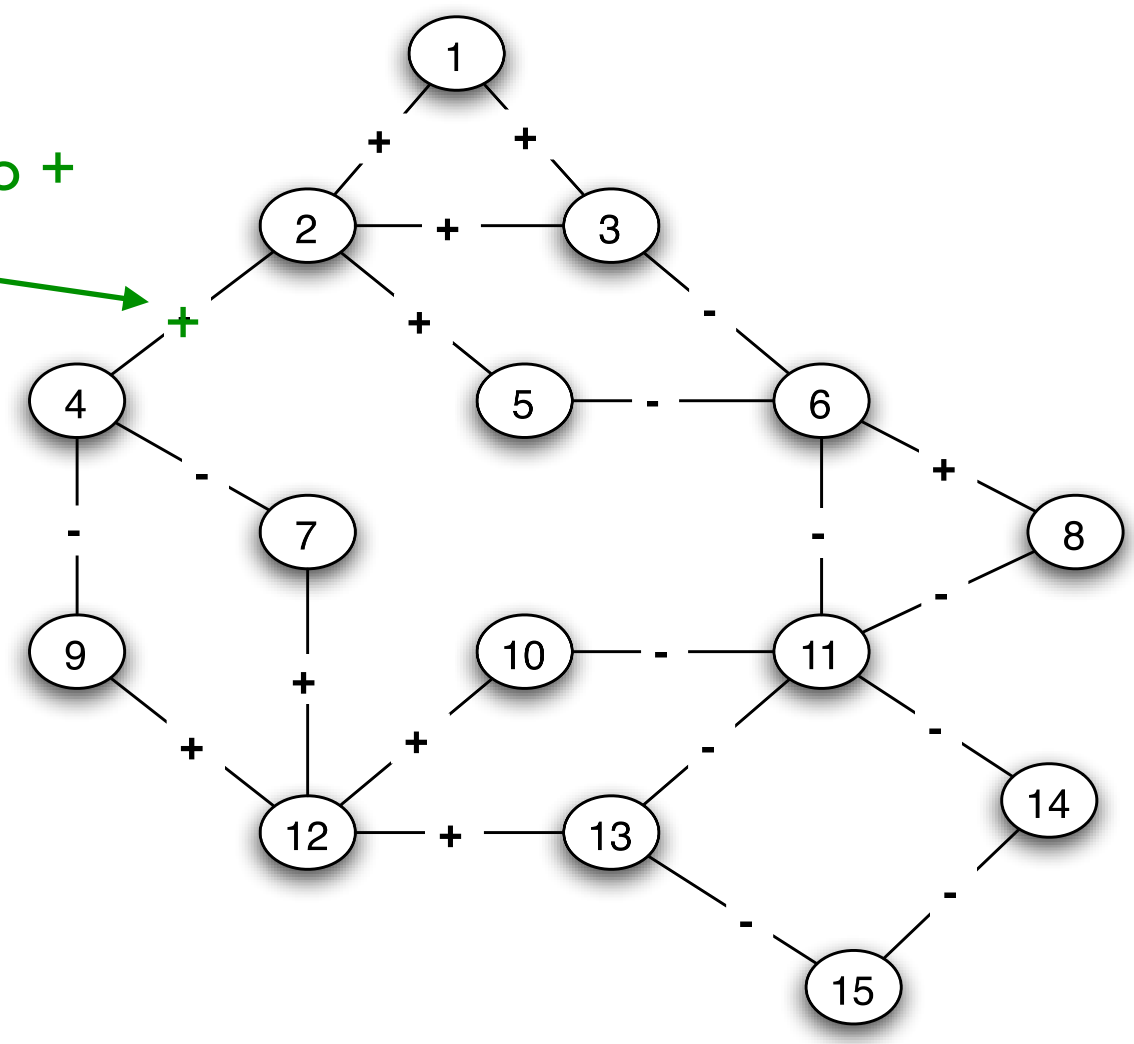
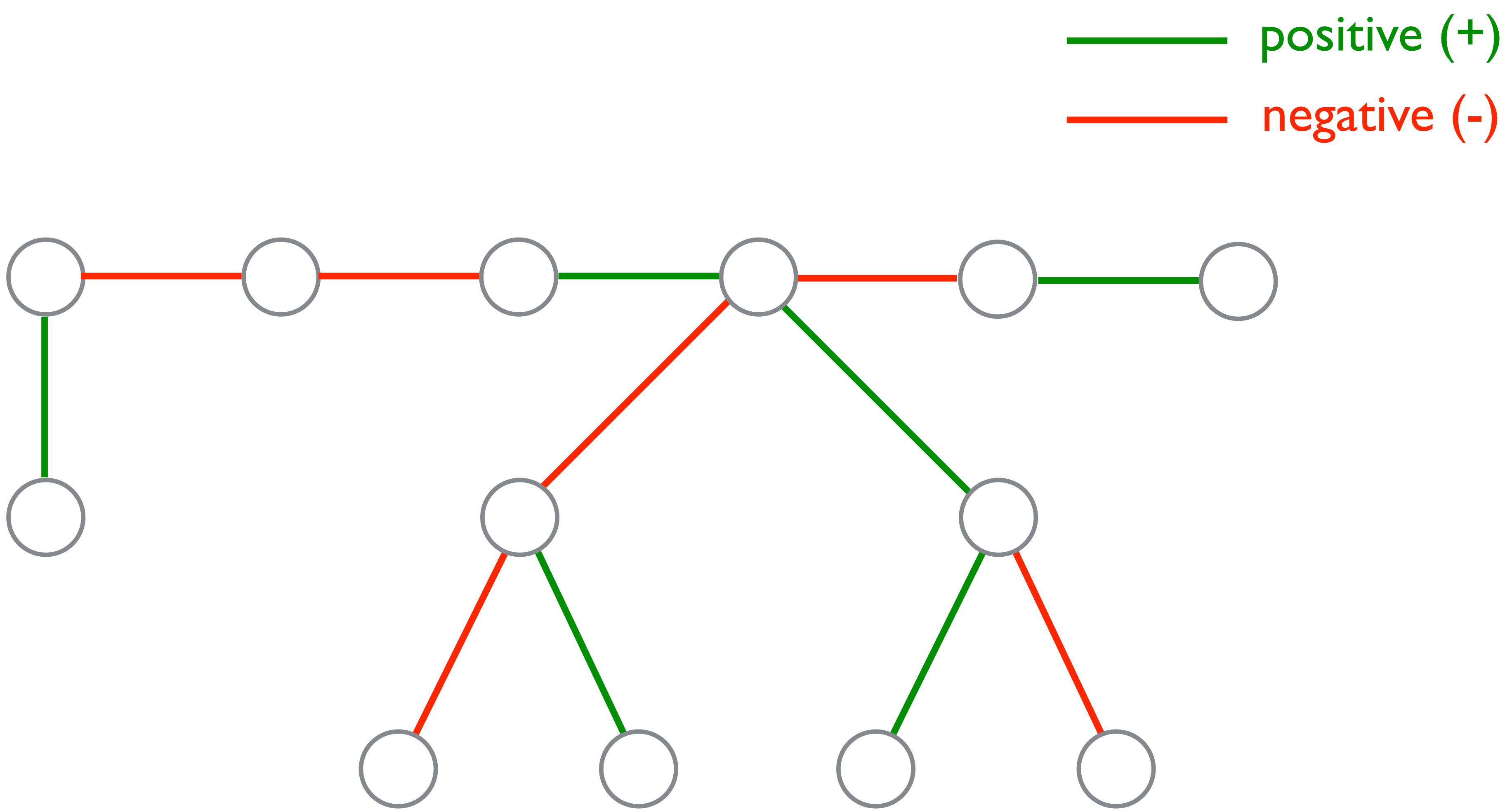


Figure 5.8: In graphs that are not complete, we can still define notions of structural balance when the edges that are present have positive or negative signs indicating friend or enemy relations.

Easley & Kleinberg

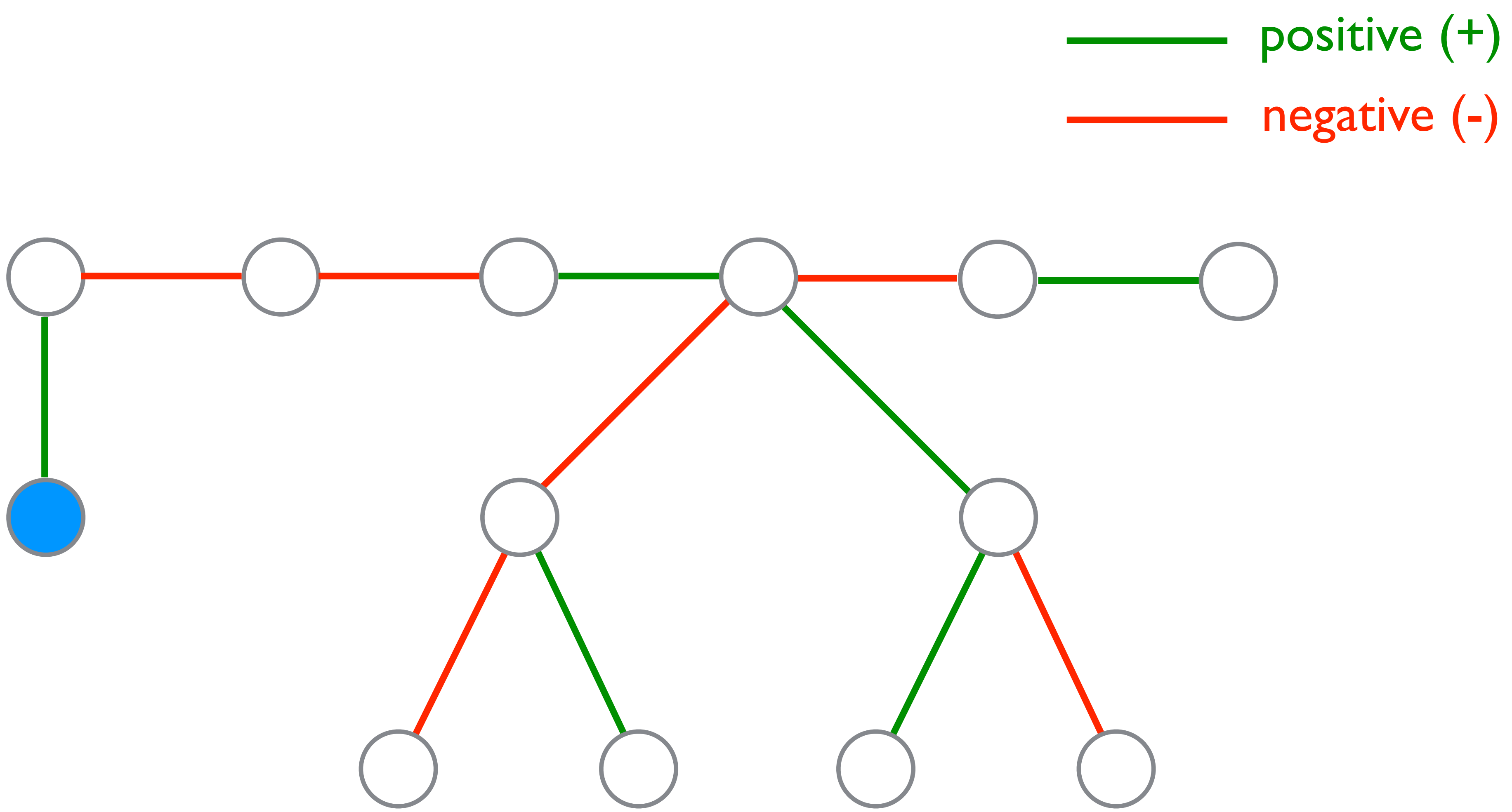
- Difficult to ascertain as networks becomes nontrivial

Structural Balance in Incomplete Networks



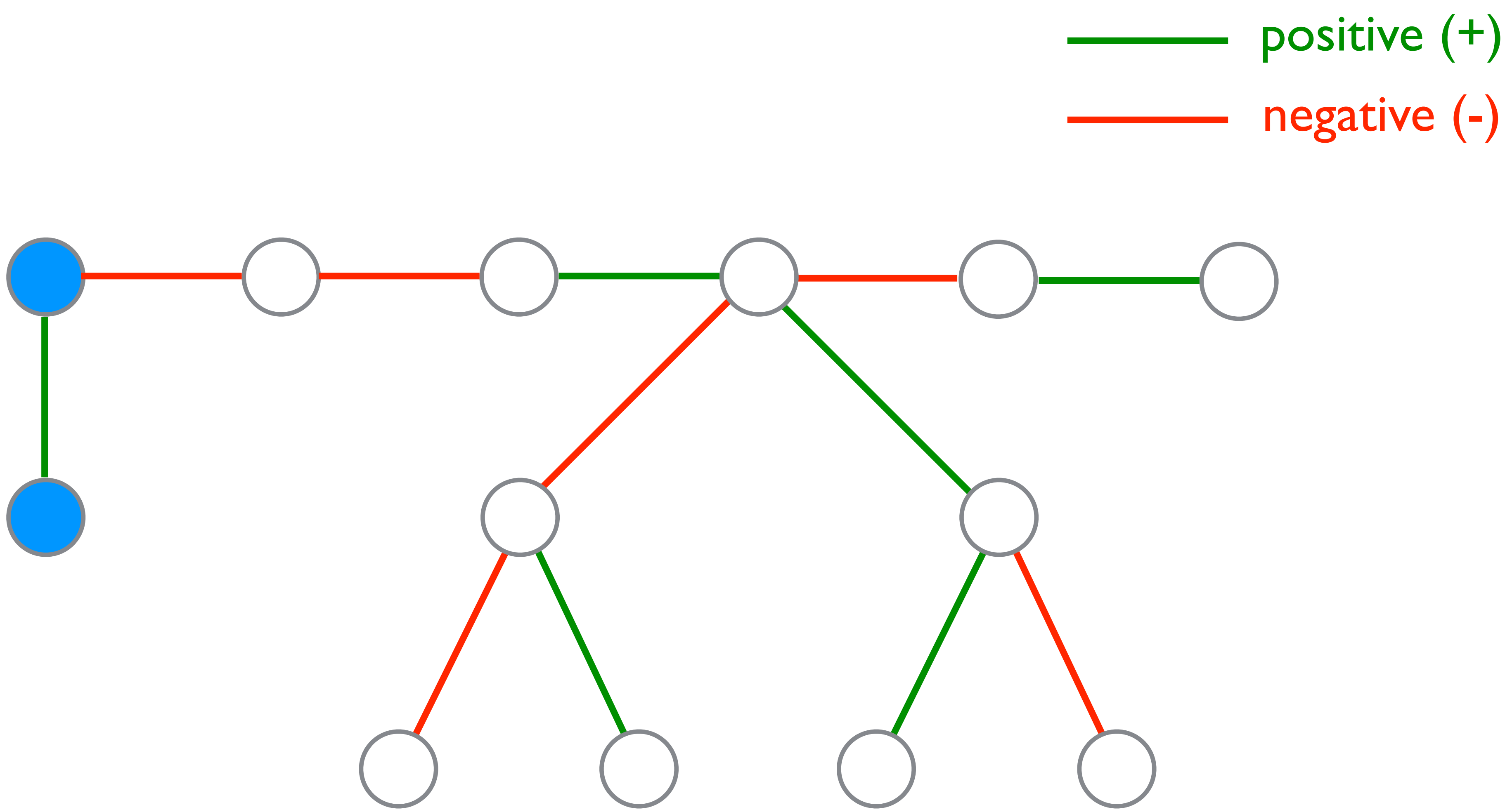
- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks



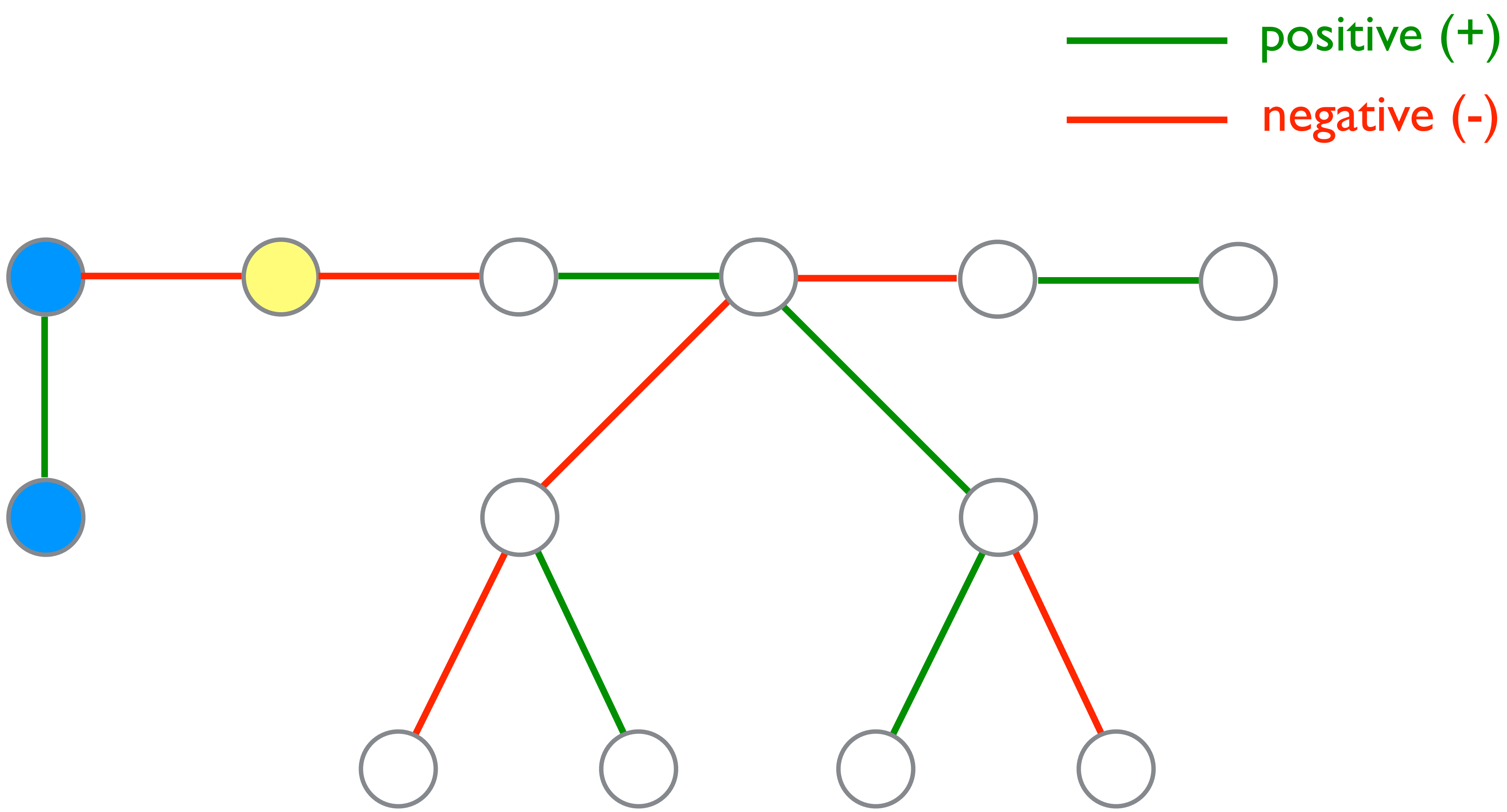
- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks



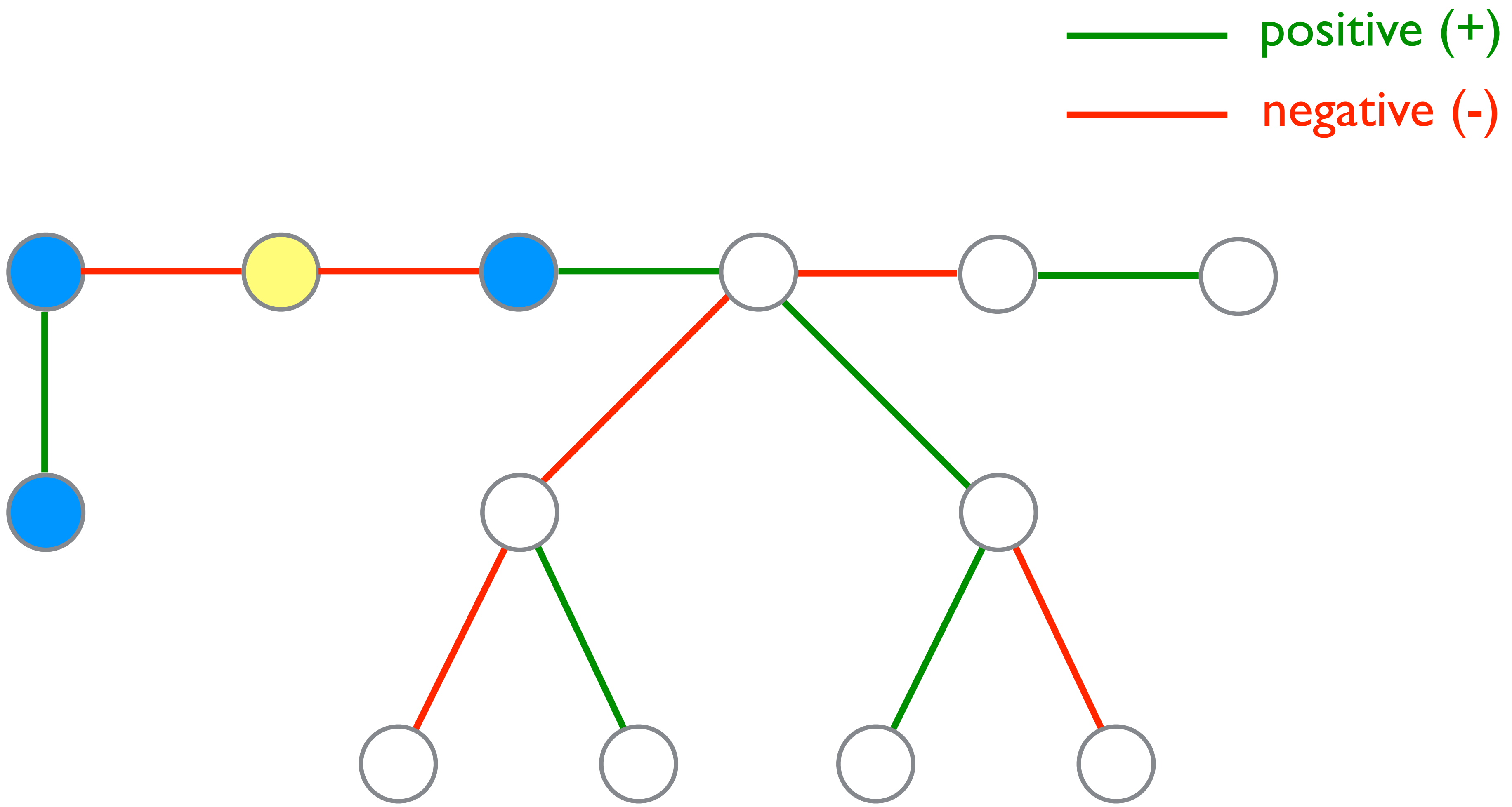
- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks



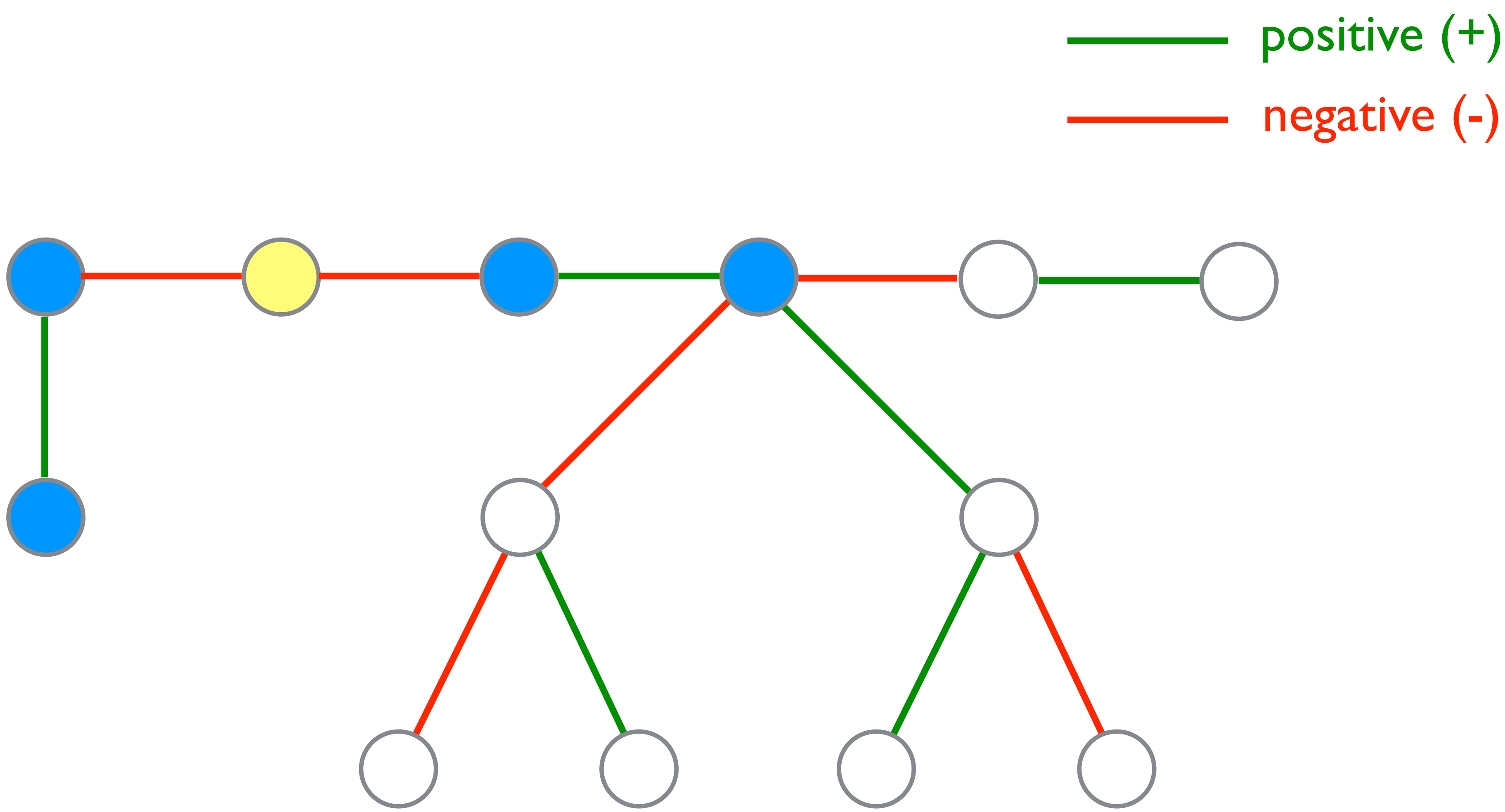
- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks



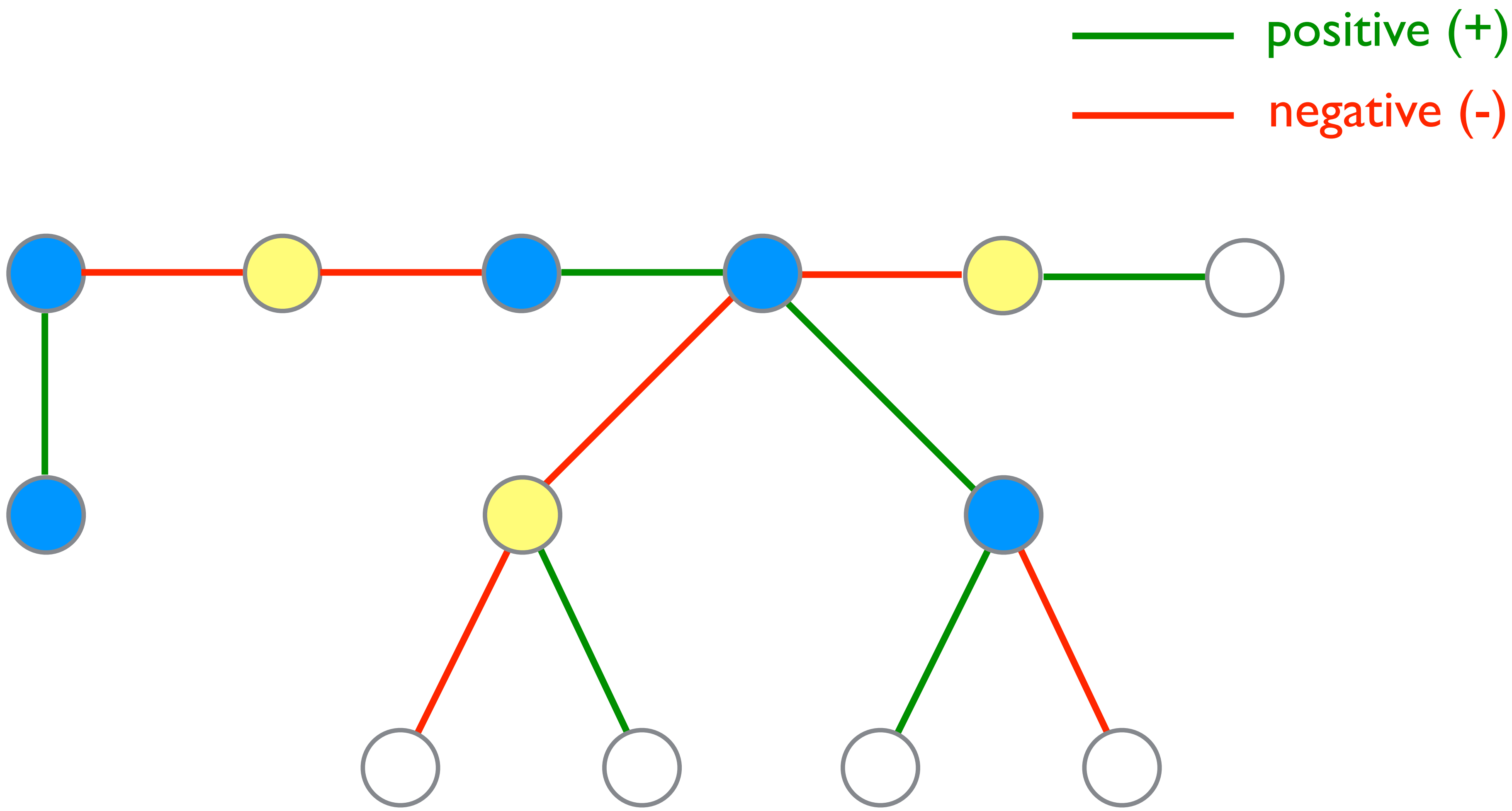
- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks



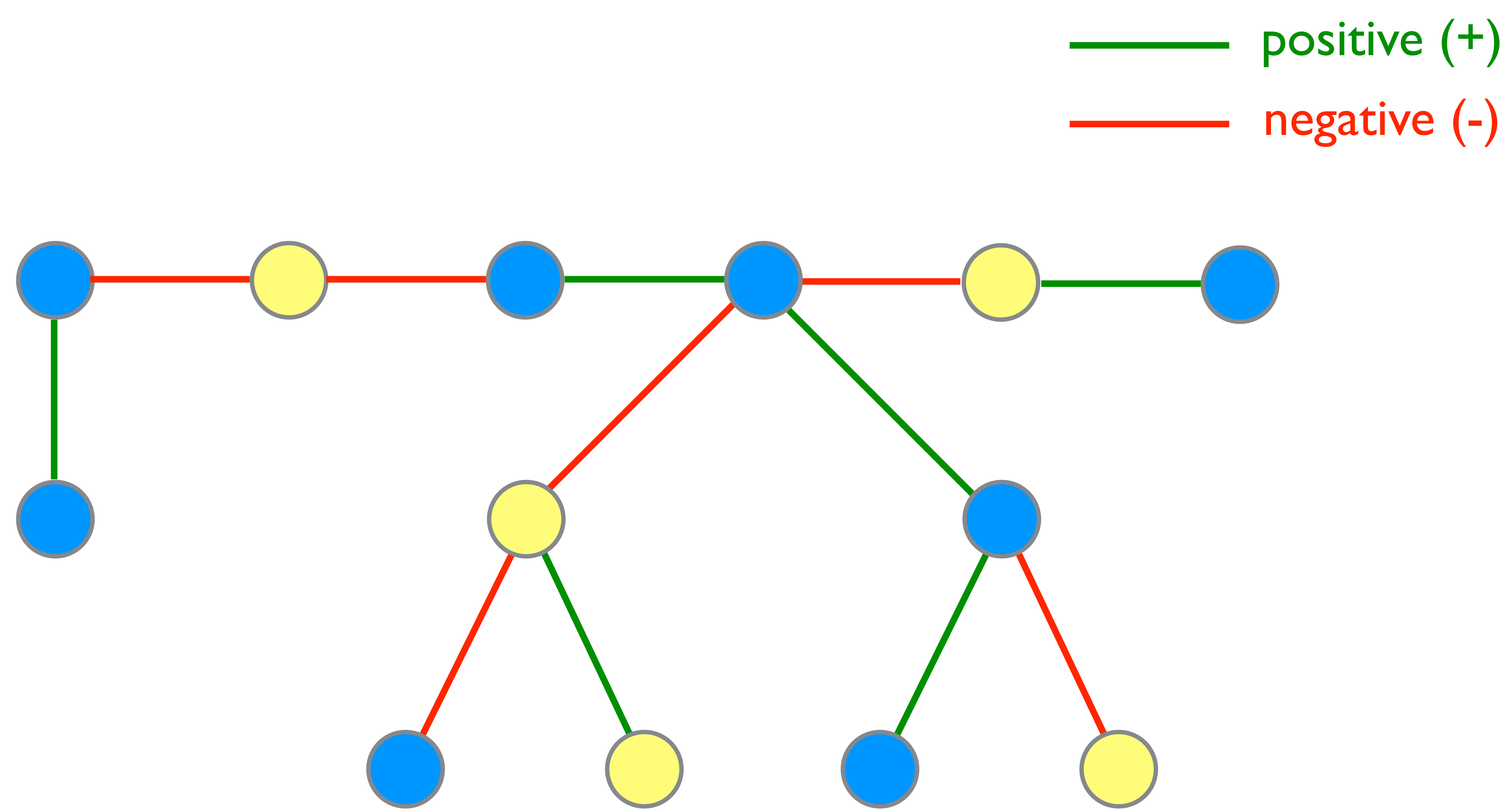
- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks



- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks



- Can we just start labeling a node in one group and change group label when we encounter a negative connection?

Structural Balance in Incomplete Networks

- Consider a test for balance in a single-cycle graph

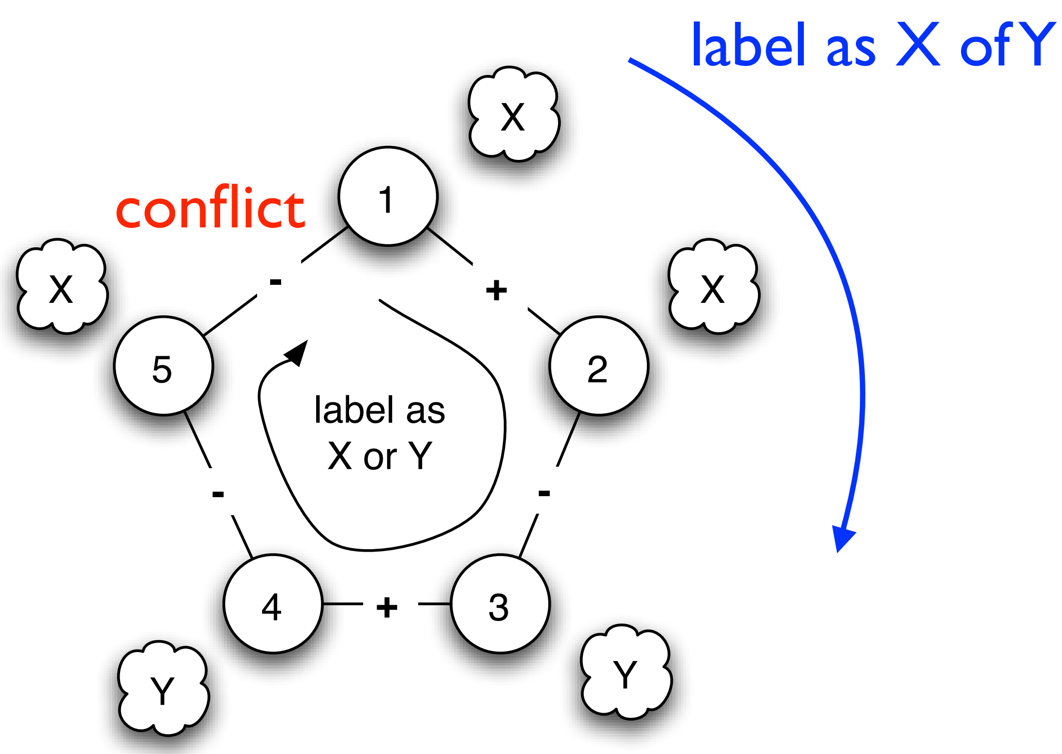


Figure 5.10: If a signed graph contains a cycle with an odd number of negative edges, then it is not balanced. Indeed, if we pick one of the nodes and try to place it in X, then following the set of friend/enemy relations around the cycle will produce a conflict by the time we get to the starting node.

Easley & Kleinberg

problem: odd number of negative (-) edges on this cycle

This graph is not bipartite

Structural Balance in Incomplete Networks

- *A signed graph (possibly incomplete) is structurally balanced if and only if it contains no cycles with an odd number of negative edges*

- Proof outline:
  - Create a reduced graph with components containing only positive nodes reduced to a "super node"
    - *This reduced graph has only negative edges*
  - Original graph is balanced iff reduced graph is bipartite
    - Use BFS to test if the reduced graph is bipartite

Proof: Balance in Incomplete Networks

- Try to obtain a "balanced division" — i.e., a partition of nodes into groups X & Y such that all are friends inside X and inside Y, but all edges between X and Y are negative
- First, find the connected components if only positive edges are considered
  - These are candidates for groups of X and Y labels (i.e., groups of friends) — "positive blobs"
    - If there is a negative edge in any of these positive blobs, the network is not balanced (see next slide)

Proof: Balance in Incomplete Networks

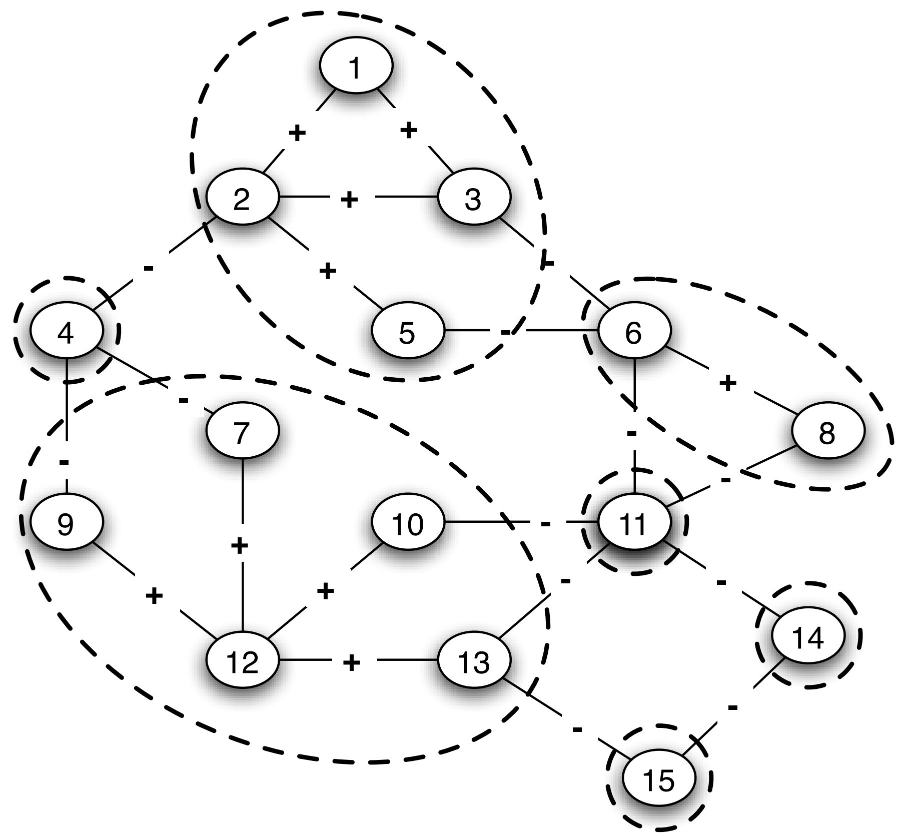


Figure 5.11: To determine if a signed graph is balanced, the first step is to consider only the positive edges, find the connected components using just these edges, and declare each of these components to be a *supernode*. In any balanced division of the graph into X and Y, all nodes in the same supernode will have to go into the same set.
Easley & Kleinberg

- Creating positive blobs by considering connected components when negative edges are removed

Structural Balance in Incomplete Networks

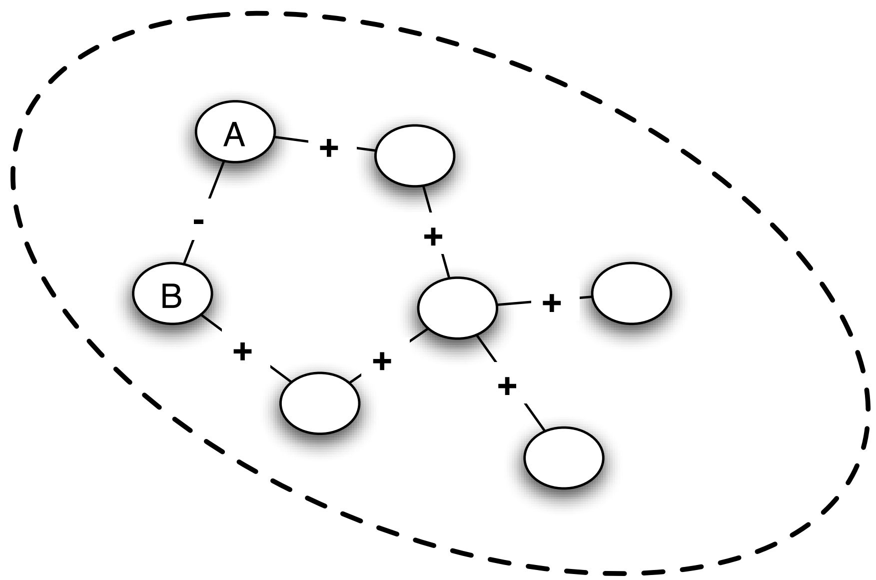


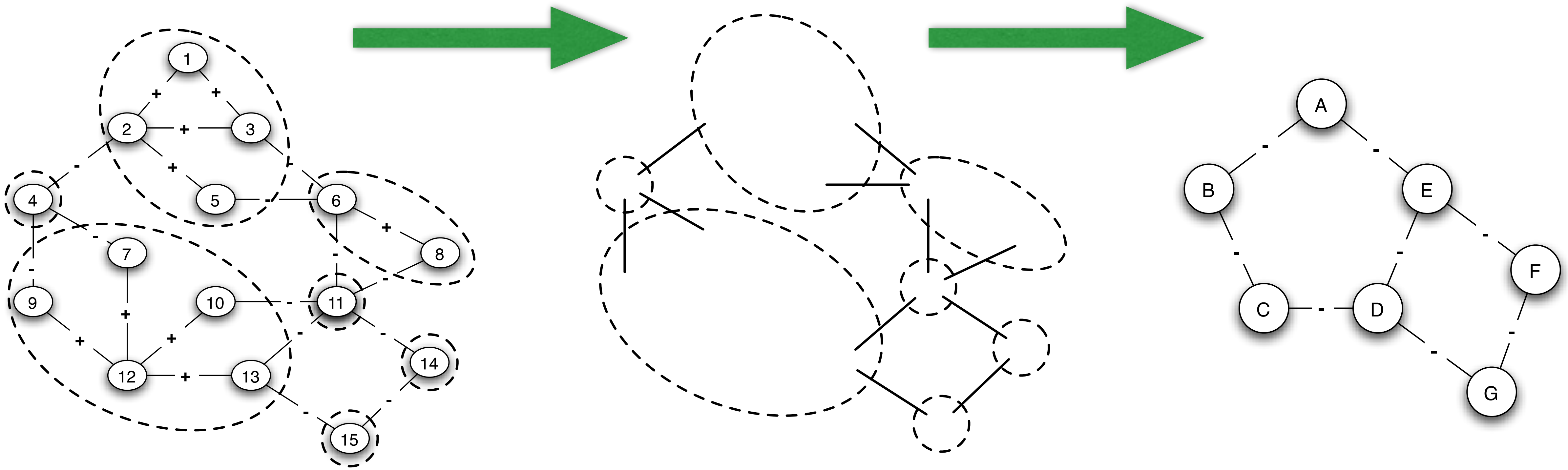
Figure 5.12: Suppose a negative edge connects two nodes A and B that belong to the same supernode. Since there is also a path consisting entirely of positive edges that connects A and B through the inside of the supernode, putting this negative edge together with the all-positive path produces a cycle with an odd number of negative edges.

Easley & Kleinberg

- Positive blob cannot have a negative edge. A negative edge makes it impossible to label all nodes in the blob the same. It creates a cycle with odd number of negative edges.
  - *If negative edge in positive blob, then network is unbalanced*

Proof: Balance in Incomplete Networks

- If no negative edges in positive blobs, create a reduced graph with
  - *Positive blobs become "super nodes" (all mutual friends, so all labeled the same)*
  - *All edges are negative*

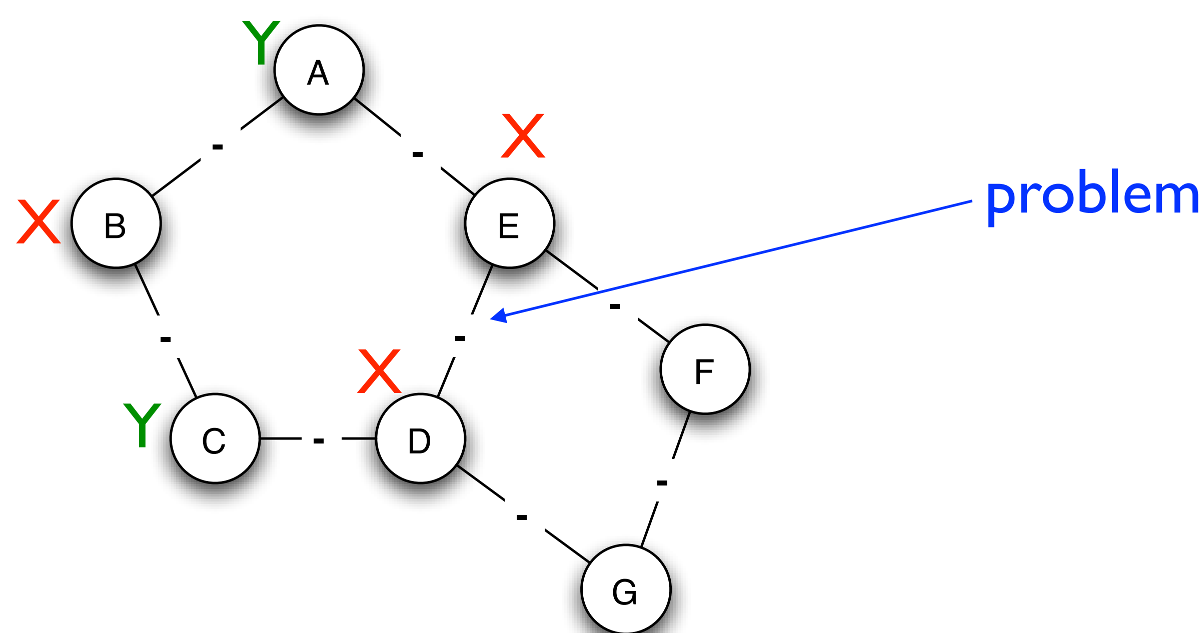


reduced graph

Easley & Kleinberg

Proof: Balance in Incomplete Networks

- Problem now is whether super nodes can be labeled X and Y so that neighbors differ. This is possible:
    - iff no odd length cycles in reduced graph
    - iff no cycles in original graph with odd number of negative edges
    - iff reduced graph has a bipartite labeling

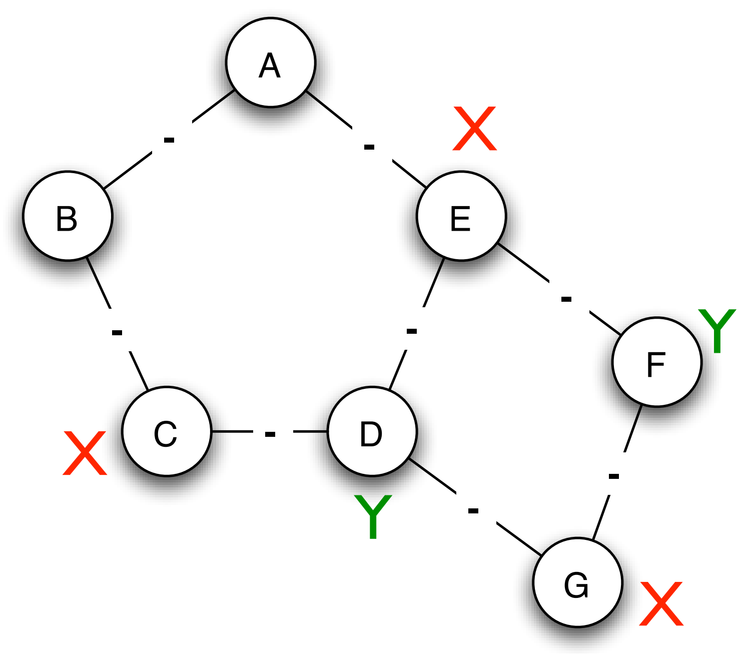


Easley & Kleinberg

reduced graph

Proof: Balance in Incomplete Networks

- Test for bipartite labeling via BFS in the reduced graph



reduced graph

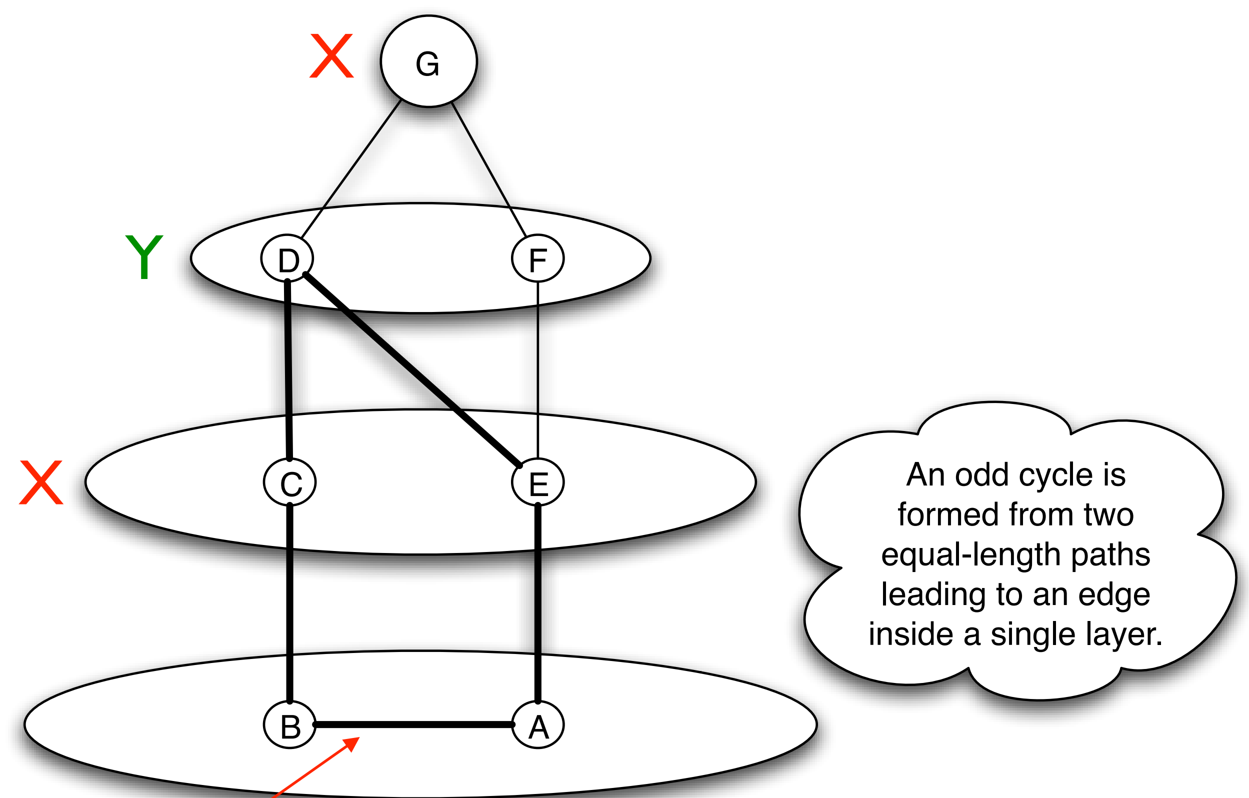


Figure 5.16: When we perform a breadth-first search of the reduced graph, there is either an edge connecting two nodes in the same layer or there isn't. If there isn't, then we can produce the desired division into X and Y by putting alternate layers in different sets. If there is such an edge (such as the edge joining A and B in the figure), then we can take two paths of the same length leading to the two ends of the edge, which together with the edge itself forms an odd cycle.

Easley & Kleinberg

- link between same nodes at same depth in BFS implies:
  - odd length cycle in reduced graph
  - odd number of negative edges in original graph
  - unbalanced original graph

Generalizations of Structural Balance

- Approximately balanced (complete) networks
- Structural balance for incomplete graphs
  - *Definition and algorithmic test*
- Weak balance (complete networks)
  - *Only prohibit 2 + per triangle*
  - *Yields multiparty, mutually antagonistic structure*

Weakly Balanced Networks

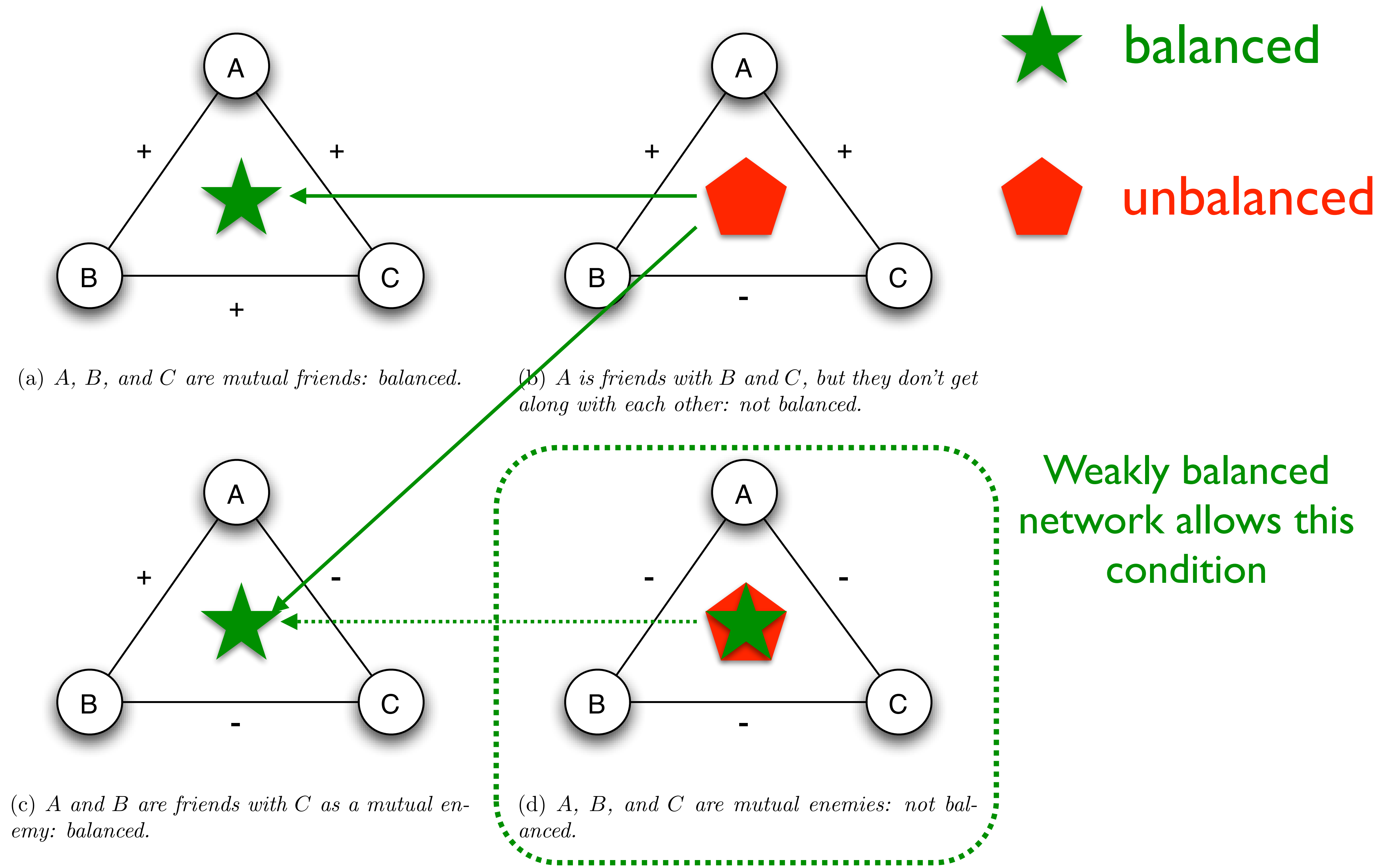


Figure 5.1: Structural balance: Each labeled triangle must have 1 or 3 positive edges.

Easley & Kleinberg

Weakly balanced: only disallow triangles with 2 +'s and 1 -

Weakly Balanced Complete Networks

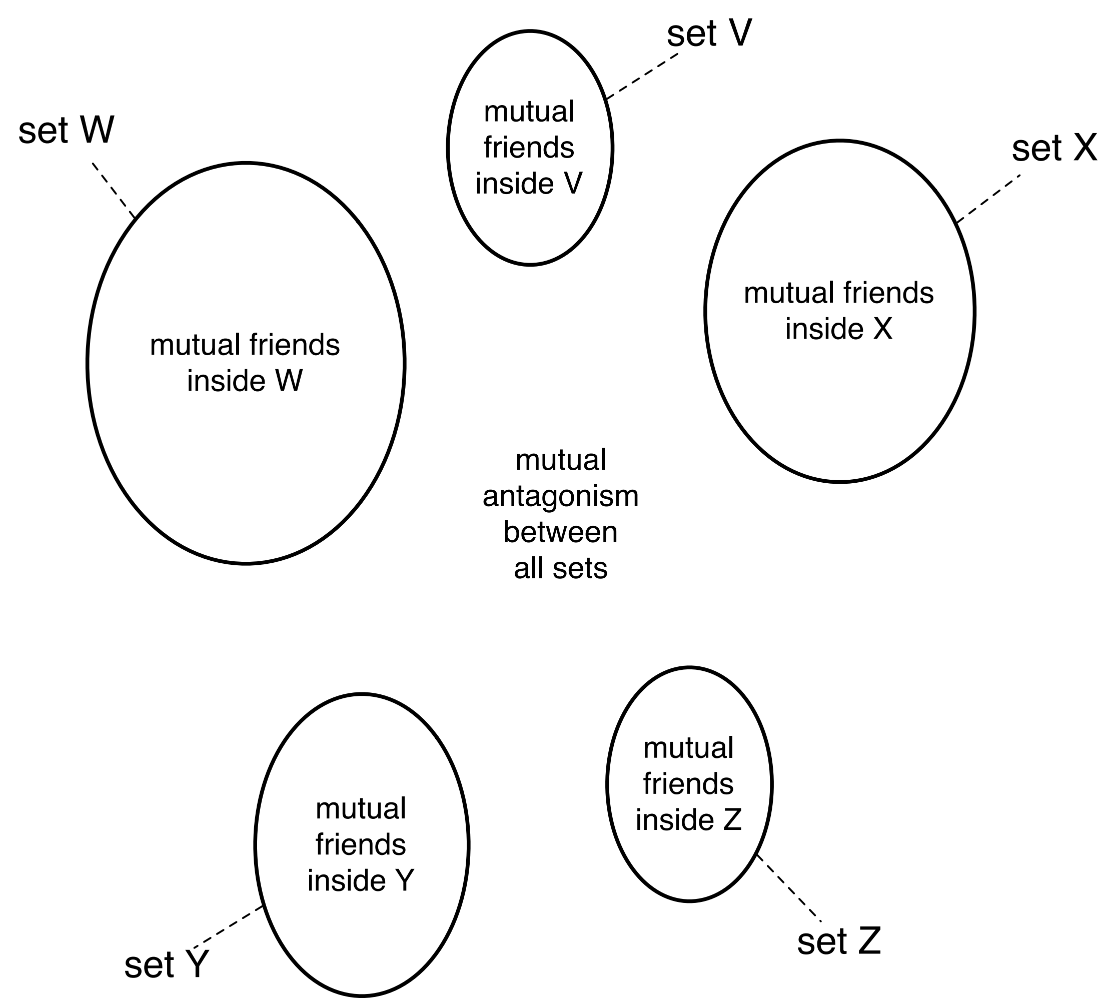


Figure 5.6: A complete graph is weakly balanced precisely when it can be divided into multiple sets of mutual friends, with complete mutual antagonism between each pair of sets. Easley & Kleinberg

- Balance Theorem: any complete weakly balanced network has the above structure — pure multi-party polarization. Other option is that everybody is friends

Proof of Balance Theorem

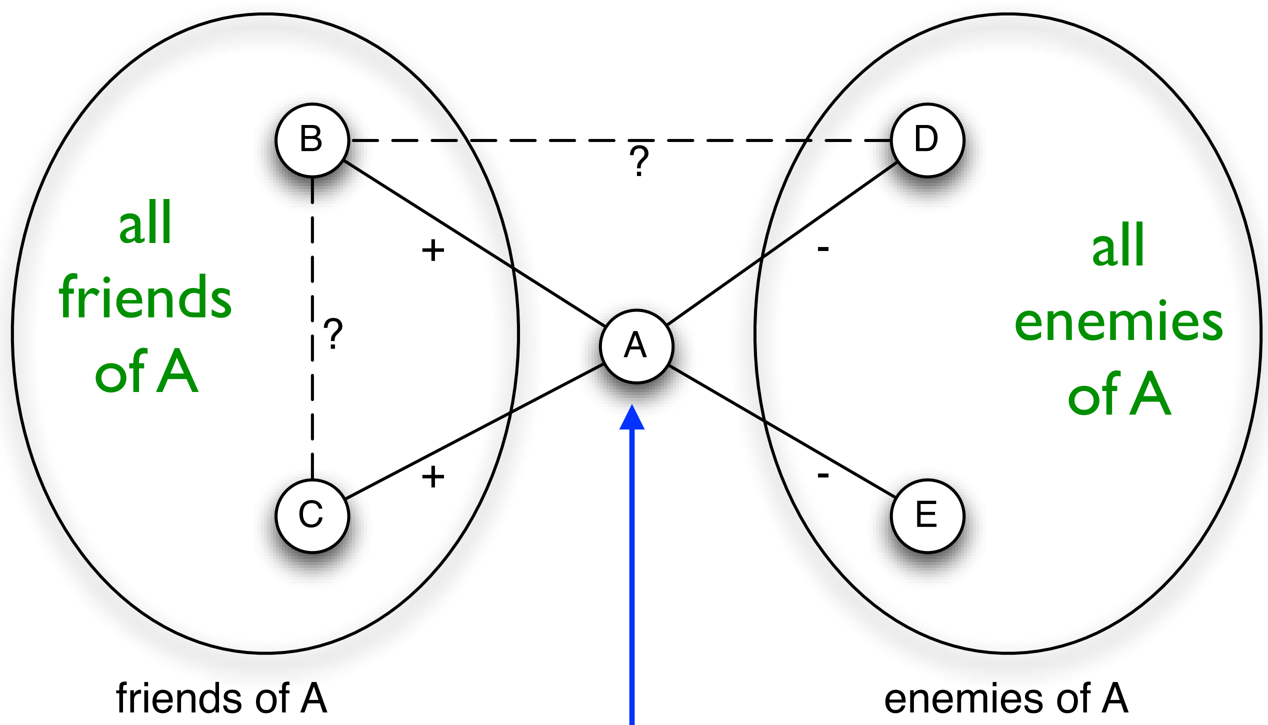


Figure 5.7: A schematic illustration of our analysis of weakly balanced networks. (There may be other nodes not illustrated here.)

Easley & Kleinberg

node A selected arbitrarily

Proof of Balance Theorem

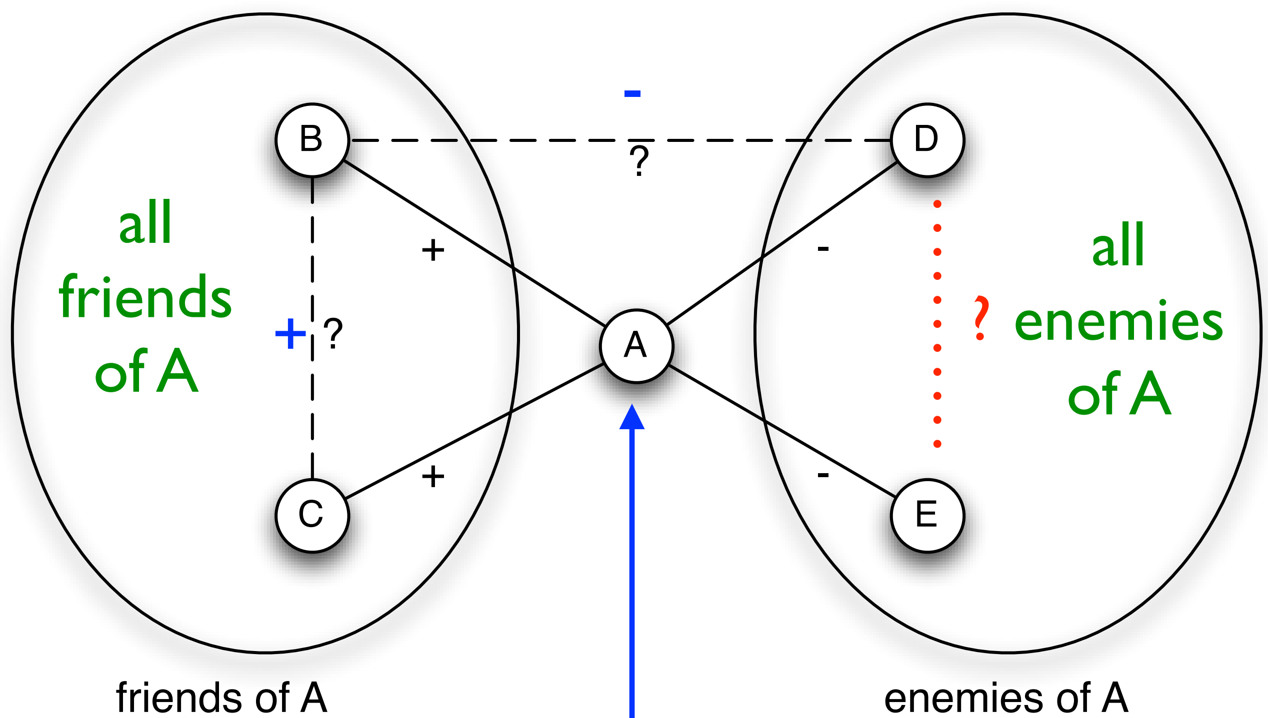


Figure 5.7: A schematic illustration of our analysis of weakly balanced networks. (There may be other nodes not illustrated here.)

Easley & Kleinberg

node A selected arbitrarily

- Apply the same reasoning to the enemies of A to get the result