EE564 Signal Space Handout

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1 Summary

This handout is intended to aid in understanding signals (functions) as points in an abstract space. These concepts are directly analogous to representing vectors in $\mathcal{R}^k$. First, the algebraic properties of these spaces are reviewed by recalling the properties of linear spaces. Second, the topological properties (shapes) of theses spaces are considered through distances, norms and inner products. Finally, the issues of infinite dimensional spaces is considered.

A detailed understanding of these concepts is beyond the scope of the class, but this document should aid in developing a good intuitive understanding. In this class, we will consider digtial communication signals as points in an abstract space, $\mathcal{L}_2[0,T]$ – the space of finite energy signals defined on the time interval $[0,T]$. The most important conclusion from this handout is that $\mathcal{L}_2[0,T]$ is directly analogous to $\mathcal{R}^k$ and nearly every concept you are familiar with from linear algebra carries over.

Figure 1 shows how the concepts covered in this handout are related. A linear space is composed of vectors and scalars and the key notion is that linear combinations of vectors are consistently defined. A metric space is one in which a logically consistent distance measure exists. We are accustomed to working in spaces, like $\mathcal{R}^k$, that have both these algebraic and topological properties. Specifically, inner product spaces are linear spaces with an inner product (generalized dot product). An inner product space has both algebraic and topological properties since an inner product, which allows us to measure properties like angles, implies a norm, which measures length, which allows us to measure distance.

Hilbert spaces are infinite dimensional inner product spaces that have nice limiting properties (*e.g.,* complete inner product spaces). While the concept of an orthonormal basis is limited to finite dimensional inner product spaces, this can be extended to Hilbert spaces in the form of complete orthonormal sets and the generalized Fourier series.

2 Linear Space

2.1 Definitions

A real vector space (or linear space) is a set of "vectors" (or points) together with rules for vector addition and multiplication by scalars (real or complex number for our purposes). The addition and multiplication must produce vectors that are within the space, and they must satisfy eight axioms. We will denote the space and scalar field by

Linear Space: $\mathcal{X}$

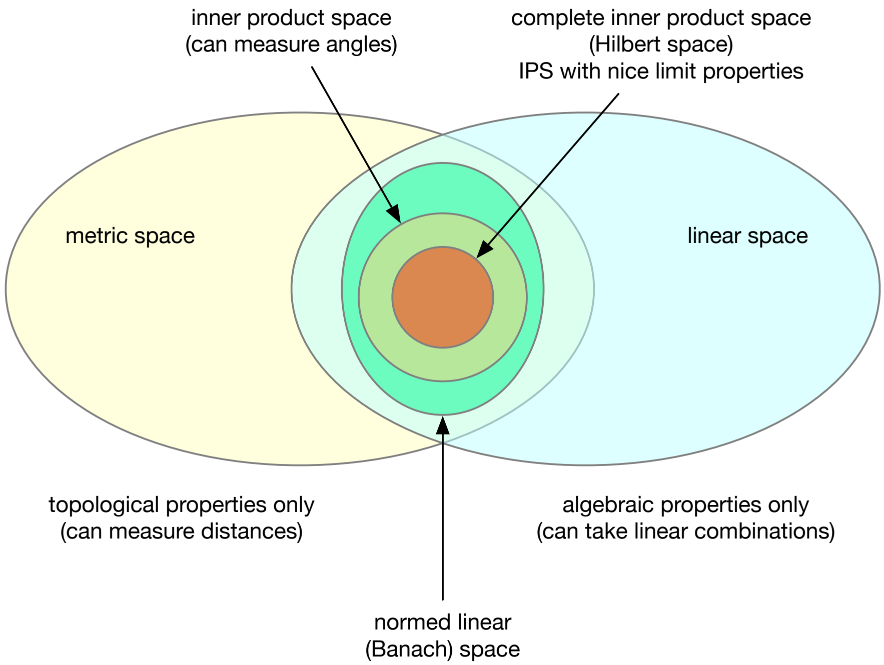

Figure 1: A Venn diagram showing the relationship between different spaces briefly summarized in this document.

Scalar Field: $\mathcal{F}$ ($\mathcal{F} = \mathcal{C}$ or $\mathcal{F} = \mathcal{R}$)

Operations: vector addition "+", scalar multiplication "·".

Let $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z} \in \mathcal{X}$, denote arbitrary elements of the space, and $a, b \in \mathcal{F}$, denote arbitrary scalars. The following axioms must be satisfied:

(LS1) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ (Commutative of +)

(LS2) $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ (Associative of +)

(LS3) $\exists\, \mathbf{0} \in \mathcal{X} : \boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ (Additive Identity)

(LS4) $\exists\, \boldsymbol{y} \in \mathcal{X} : \boldsymbol{x} + \boldsymbol{y} = \mathbf{0}$ (we denote $\boldsymbol{y}$ by $-\boldsymbol{x}$) (Additive Inverse)

(LS5) $a \cdot (b \cdot \boldsymbol{x}) = (ab) \cdot \boldsymbol{x}$ (Associative of $\cdot$)

(LS6) $1 \cdot \boldsymbol{x} = \boldsymbol{x}$ (Multiplicative Identity)

(LS7) $a \cdot (\boldsymbol{x} + \boldsymbol{y}) = a \cdot \boldsymbol{x} + a \cdot \boldsymbol{y}$ ($\cdot$ Distributes Over +)

(LS8) $(a + b) \cdot \boldsymbol{x} = a \cdot \boldsymbol{x} + b \cdot \boldsymbol{x}$ (Scalar Addition Distributes Over $\cdot$)

Note that, when no confusion can occur, we write $a \cdot \boldsymbol{x}$ as simply $a\boldsymbol{x}$.

2.2 Linear Independence

$\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ..., $\boldsymbol{x}_k \in \mathcal{L}$ are linearly independent if and only if

$$\sum_{i=1}^{k} c_i \boldsymbol{x}_i = \mathbf{0} \iff c_i = 0 \ \forall i = 1, ..., k$$

For example, $\mathcal{L} = \mathcal{R}^2$: $[1\ 0]^T$, $[0\ 1]^T$ are linearly independent and $[1\ 1]^T$, $[2\ 2]^T$ are not.

2.3 Subspace

A subspace of $\mathcal{L}$ is a linear space contained by $\mathcal{L}$ (e.g. subset of $\mathcal{L}$ with the properties in 2.1.

The subspace spanned by $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ..., $\boldsymbol{x}_k \in \mathcal{L}$ is the set of all linear combinations such that

$$\sum_{i=1}^{k} c_i \boldsymbol{x}_i \quad c_i \in \mathcal{C}$$

$\mathcal{R}^2$ is a subspace spanned by $\mathbf{e}_1 = [1\ 0\ 0]^T$, $\mathbf{e}_2 = [0\ 1\ 0]^T$.

2.4 Dimension and Basis

If $\mathcal{L}$ is spanned by a <u>finite</u> number of elements, it is called finite dimensional.

If $\mathcal{L}$=span$\{\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ..., $\boldsymbol{x}_k\}$ and $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ..., $\boldsymbol{x}_k$ are linearly independent, then the <u>**dimension**</u> of $\mathcal{L}$ is k and $\{\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ..., $\boldsymbol{x}_k\}$ is a <u>**basis**</u> for $\mathcal{L}$, i.e.

$$\boldsymbol{y} \in \mathcal{L} \implies \boldsymbol{y} = \sum_{i=1}^{k} Y_i \boldsymbol{x}_i$$

For example, in $\mathcal{L} = \mathcal{R}^3$, we can express $\boldsymbol{y} = Y_1\boldsymbol{e}_1 + Y_2\boldsymbol{e}_2 + Y_3\boldsymbol{e}_3$ using any basis. Here are some examples:

(1): $\mathbf{e}_1 = [1\ 0\ 0]^T$, $\mathbf{e}_2 = [0\ 1\ 0]^T$, $\mathbf{e}_3 = [0\ 0\ 1]^T$ (the standard basis)

(2): $\mathbf{e}_1 = [1\ 1\ 1]^T$, $\mathbf{e}_2 = [1\ -1\ 0]^T$, $\mathbf{e}_3 = [1\ 0\ 1]^T$ (a non-orthonormal basis)

2.5 Infinite Dimensional Spaces

(Example 1) l_2: {all sequences $\{x_i\}_{i=\infty}^{\infty}$: $x_i \in \mathcal{R}$, $\sum_i {x_i}^2 < \infty$}.

(Example 2) $\mathcal{L}_2[0, T]$ See 3.6

2.6 Distance

A distance function (also referred to as a metric) measures the distance between points of a (not necessarily Linear) space. The distance between two points is denoted $d(\boldsymbol{x}, \boldsymbol{y})$; it must satisfy the following properties

(D1) $d(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ (Positivity)

(D2) $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$ (Symmetry)

(D3) $d(\boldsymbol{x}, \boldsymbol{y}) \leq d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y})$ (Triangle Inequality)

(D4) $d(\boldsymbol{x}, \boldsymbol{y}) = 0$ implies $\boldsymbol{x} = \boldsymbol{y}$ (Strict Positivity).

2.7 Norm

A norm allows the measurement of "length" in the linear space. A norm maps an element of the linear space into a non-negative scalar (it's length), denoted by $\|\boldsymbol{x}\|$. A norm must satisfy the following four properties

(N1) $\|\boldsymbol{x}\| \geq 0$ (Positivity)

(N2) $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (Subadditivity)

(N3) $\|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\|$ (Positive Homogeneity)

(N4) $\|\boldsymbol{x}\| = 0$ implies $\boldsymbol{x} = \boldsymbol{0}$ (Positive Definite).

2.8 Any Norm Implies a Distance

In a normed linear space $(X, \|\cdot\|)$, the metric is defined in terms of the given norm by $d(\boldsymbol{x}, y) = \|\boldsymbol{x} - y\|$. These follow easily from the properties of the norm.

(D1) $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| \geq 0$ by (N1).

(D2) $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| = |-1| \|\boldsymbol{y} - \boldsymbol{x}\| = d(\boldsymbol{y}, \boldsymbol{x})$ by (N3) with $\alpha = -1$.

(D3) This follows from (N2)

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{y}) &= \|\boldsymbol{x} - \boldsymbol{y}\| && (1) \\ &= \|(\boldsymbol{x} - \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{y})\| && (2) \\ &\leq \|\boldsymbol{x} - \boldsymbol{z}\| + \|\boldsymbol{z} - \boldsymbol{y}\| && (3) \\ &= d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y}). && (4) \end{aligned}$$

(D4) $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| = 0$ implies that $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{0}$ by (N4), so $\boldsymbol{x} = \boldsymbol{y}$.

3 Hilbert Space

Hilbert space is a complete inner product space.

3.1 Cauchy Sequence

A sequence is said to be Cauchy if and only if

$$\lim_{n,m \to \infty} \|x_n - x_m\|^2 = 0 \qquad (5)$$

3.2 Complete Metric Space

A metric space $(\mathcal{X}, d)$ is complete if each Cauchy sequence in $(\mathcal{X}, d)$ is a convergent sequence in $(\mathcal{X}, d)$. $\Leftrightarrow \exists\, \boldsymbol{x} \in\ \mathcal{X}$ s.t. $\lim_{n\to\infty} \|\boldsymbol{x} - \boldsymbol{x}_n\|^2 = 0$

3.3 Inner Product

An *inner product* maps two elements of a linear space into a scalar, denoted by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. An inner product allows us to measure "angles" between elements of the space. An inner product must satisfy the following properties ($\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are arbitrary elements of the linear space and α is a complex scalar (i.e. $\mathcal{F} = \mathcal{C}$))

(IP1) $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = (\langle \boldsymbol{y}, \boldsymbol{x} \rangle)^*$ (Hermitian Symmetry)

(IP2) $\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$ (Additivity)

(IP3) $\langle \alpha\boldsymbol{x}, \boldsymbol{y} \rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ (Homogeneity in first argument)

(IP4) if $\boldsymbol{x} \neq \boldsymbol{0}$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ (Positivity).

3.4 Cauchy-Schwartz Inequality

The Cauchy-Schwartz inequality is that for any $\boldsymbol{z}$ and $\boldsymbol{x}$ in an inner product space

$$|\langle \boldsymbol{z}, \boldsymbol{x} \rangle|^2 \leq \langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{z}, \boldsymbol{z} \rangle.$$

with equality if and only if $\boldsymbol{x} = c\boldsymbol{z}$ for some scalar c.

Proof: By (IP4), we have that for any scalar α

$$\langle \boldsymbol{z} - \alpha\boldsymbol{x}, \boldsymbol{z} - \alpha\boldsymbol{x} \rangle \geq 0, \tag{6}$$

and in fact equality holds if and only if $\boldsymbol{z} - \alpha\boldsymbol{x} = \boldsymbol{0}$ (by (IP3) and (IP4)). If you minimize the above expression with respect to α, you will find that

$$\alpha_{\text{opt}} = \frac{\langle \boldsymbol{z}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \tag{7}$$

is the value of α which minimizes the above expression. This minimization can be carried out in several ways; see the remark at the end of this problem - this becomes *very important* when we get to estimation theory. Substituting this value of α yields

$$\langle \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x}, \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x} \rangle = \langle \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x}, \boldsymbol{z} \rangle - \underbrace{\langle \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x}, \alpha_{\text{opt}}\boldsymbol{x} \rangle}_{=0}., \tag{8}$$

The second term is zero since

$$\begin{aligned} \langle \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x}, \alpha_{\text{opt}}\boldsymbol{x} \rangle &= \alpha^*_{\text{opt}} \left(\langle \boldsymbol{z}, \boldsymbol{x} \rangle - \alpha_{\text{opt}} \langle \boldsymbol{x}, \boldsymbol{x} \rangle \right) & (9) \\ &= \alpha^*_{\text{opt}} \left(\langle \boldsymbol{z}, \boldsymbol{x} \rangle - \frac{\langle \boldsymbol{z}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \langle \boldsymbol{x}, \boldsymbol{x} \rangle \right) & (10) \\ &= 0. & (11) \end{aligned}$$

To get this, we used (IP1) thru (IP3) and the fact that $\langle \boldsymbol{y}, \alpha\boldsymbol{x}\rangle = \alpha^*\langle \boldsymbol{y}, \boldsymbol{x}\rangle$, which follows by using (IP1) and (IP3) together. Therefore we have

$$\begin{aligned}
\langle \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x}, \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x}\rangle &= \langle \boldsymbol{z} - \alpha_{\text{opt}}\boldsymbol{x}, \boldsymbol{z}\rangle & (12)\\
&= \langle \boldsymbol{z}, \boldsymbol{z}\rangle - \alpha_{\text{opt}}\langle \boldsymbol{x}, \boldsymbol{z}\rangle & (13)\\
&= \langle \boldsymbol{z}, \boldsymbol{z}\rangle - \frac{\langle \boldsymbol{z}, \boldsymbol{x}\rangle}{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}\langle \boldsymbol{x}, \boldsymbol{z}\rangle & (14)\\
&= \langle \boldsymbol{z}, \boldsymbol{z}\rangle - \frac{|\langle \boldsymbol{z}, \boldsymbol{x}\rangle|^2}{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}. & (15)
\end{aligned}$$

Since this must be non-negative $\Rightarrow$

$$\langle \boldsymbol{z}, \boldsymbol{z}\rangle - \frac{|\langle \boldsymbol{z}, \boldsymbol{x}\rangle|^2}{\langle \boldsymbol{x}, \boldsymbol{x}\rangle} \geq 0 \tag{16}$$

or

$$|\langle \boldsymbol{z}, \boldsymbol{x}\rangle|^2 \leq \langle \boldsymbol{x}, \boldsymbol{x}\rangle\langle \boldsymbol{z}, \boldsymbol{z}\rangle. \tag{17}$$

Notice that if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$, the above method does not work, but in this case the inequality is trivial. Also notice that equality holds if only if $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{z} = \alpha_{\text{opt}}\boldsymbol{x}$.

3.5 Any Inner Product Defines a Norm

(N1) From (IP1) we know that $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$ is real, and (IP4) assures us that it is non-negative. Therefore

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle} \geq 0. \tag{18}$$

(N2) Following the hint we start with the square of the norm of the sum

$$\begin{aligned}
\|\boldsymbol{x} + \boldsymbol{y}\|^2 &= \langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y}\rangle & (19)\\
&= \langle \boldsymbol{x}, \boldsymbol{x}\rangle + \langle \boldsymbol{y}, \boldsymbol{y}\rangle + 2\Re\left\{\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right\} & (20)\\
&= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\Re\left\{\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right\} & (21)\\
&\leq \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\left|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right| & (22)\\
&\leq \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| & (23)\\
&= \left(\|\boldsymbol{x}\| + \|\boldsymbol{y}\|\right)^2. & (24)
\end{aligned}$$

Since both sides are non-negative, taking square roots yields the sub-additivity property.

The only significant steps in the above sequence were suggested by the hint (i.e. first $\Re\left\{\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right\} \leq |\langle \boldsymbol{x}, \boldsymbol{y}\rangle|$, then the Cauchy-Schwartz inequality).

(N3) As noted in part (a), a scalar factors out of the second argument with a conjugate $\Rightarrow$

$$\|\alpha\boldsymbol{x}\|^2 = \langle \alpha\boldsymbol{x}, \alpha\boldsymbol{x}\rangle = \alpha^*\langle \alpha\boldsymbol{x}, \boldsymbol{x}\rangle = |\alpha|^2\langle \boldsymbol{x}, \boldsymbol{x}\rangle. \tag{25}$$

Taking square roots of both sides (both sides are non-negative) yields the result.

(N4) If $\boldsymbol{x} \neq \boldsymbol{0}$, then by (IP4), $\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0$. Applying (IP3) with $\alpha = 0$ implies that $\langle \boldsymbol{0}, \boldsymbol{0}\rangle = 0$. So $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0}$.

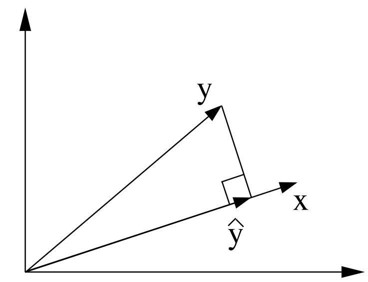

Figure 2: $\hat{\boldsymbol{y}}$ is the projection of $\boldsymbol{y}$ on $\boldsymbol{x}$

3.6 Hilbert Space Example: $\mathcal{L}_2[0, T]$

The space of all functions which are square-integrable (finite energy) on the interval $[0, T]$ is denoted by $\mathcal{L}_2[0, T]$. We can consider this space to be either over the complex field, or specialize to real scalars. A point in this space $\boldsymbol{x}$ represents a function $\boldsymbol{x} = \{x(t) : 0 \leq t \leq T, \int_0^T |x(t)|^2 dt < \infty\}$. Scalar operations are carried out in the obvious manner: $\boldsymbol{z} = \alpha\boldsymbol{x} + \beta\boldsymbol{y}$ is shorthand for $z(t) = \alpha x(t) + \beta y(t)$ for all $t \in [0, T]$. It is straightforward to show that the following is a valid inner product:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \int_0^T x(t) y^*(t) dt \tag{26}$$

The implied norm is thus the RMS value

$$\|\boldsymbol{x}\|^2 = \int_0^T |x(t)|^2 dt \tag{27}$$

and the distance is

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\int_0^T |x(t) - y(t)|^2 dt} \tag{28}$$

Note that, by definition $\boldsymbol{x} = \boldsymbol{y}$ in this space if $d(\boldsymbol{x}, \boldsymbol{y}) = 0$, or if the energy in the difference between the two functions is zero – i.e., this is different than pointwise equality.

3.7 Closest Point Theorem (Special case of Hilbert Space Projection Theorem)

The solution of $\min_{\alpha \in \mathcal{C}} \|\boldsymbol{y} - \alpha\boldsymbol{x}\|^2$ in a Hilbert space is

$$\alpha_{\text{opt}} = \frac{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$$

The closest point to $\hat{\boldsymbol{y}}$ in span$\{\boldsymbol{x}\}$ is $\hat{\boldsymbol{y}} = \frac{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}\boldsymbol{x}$ as in Fig 2. $\langle \boldsymbol{y} - \hat{\boldsymbol{y}}, \alpha\boldsymbol{x} \rangle = \mathbf{0}\ \forall \boldsymbol{x}$

3.8 Gram-Schmidt Procedure

Gram-Schmidt procedure converts any basis $\{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_k\}$ into an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_k\}$ such that $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_K(i - j)$

$$\begin{aligned}
\boldsymbol{y}_1 &= \boldsymbol{x}_1 \Longrightarrow \boldsymbol{e}_1 = \frac{\boldsymbol{y}_1}{\|\boldsymbol{y}_1\|} \\
\boldsymbol{y}_2 &= \boldsymbol{x}_2 - \langle \boldsymbol{x}_2, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 \Longrightarrow \boldsymbol{e}_2 = \frac{\boldsymbol{y}_2}{\|\boldsymbol{y}_2\|} \\
\boldsymbol{y}_3 &= \boldsymbol{x}_3 - \langle \boldsymbol{x}_3, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 - \langle \boldsymbol{x}_3, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2 \Longrightarrow \boldsymbol{e}_3 = \frac{\boldsymbol{y}_3}{\|\boldsymbol{y}_3\|} \\
\boldsymbol{y}_k &= \boldsymbol{x}_k - \sum_{i=1}^{k-1} \langle \boldsymbol{x}_k, \boldsymbol{e}_i \rangle \boldsymbol{e}_i \Longrightarrow \boldsymbol{e}_k = \frac{\boldsymbol{y}_k}{\|\boldsymbol{y}_k\|}
\end{aligned}$$

3.8.1 Example

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{y}_1 = \mathbf{x}_1 \Longrightarrow \mathbf{e}_1 = \tfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \mathbf{e}_1^{\mathrm{t}} \mathbf{x}_2 \Longrightarrow \mathbf{e}_2 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{2}{\sqrt{3}} \mathbf{e}_1 - \tfrac{1}{\sqrt{2}} \mathbf{e}_2 \Longrightarrow \mathbf{e}_3 = \tfrac{1}{6} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

3.9 Representation Using Orthonormal Basis

Given a basis $\{\boldsymbol{e}_j\}_{j=1}^k$, any element of the linear space, $\boldsymbol{x}$, can be represented in terms of this basis

$$\boldsymbol{x} = \sum_{j=1}^{k} X_i \mathbf{e}_j \tag{29}$$

The coefficients $\{X_j\}$ are determined by taking the inner product of both sides with $\boldsymbol{e}_i$

$$\langle \boldsymbol{x}, \mathbf{e}_i \rangle = \left\langle \textstyle\sum_{j=1}^{k} X_j \mathbf{e}_j, \mathbf{e}_i \right\rangle = \sum_{j=1}^{k} X_j \langle \mathbf{e}_j, \mathbf{e}_i \rangle \qquad i = 1, 2, \ldots k \tag{30}$$

This can be written in matrix form as $\mathbf{G}\underline{X} = \mathbf{b}$ where the (i, j) element of $\mathbf{G}$ is $\langle \mathbf{e}_j, \mathbf{e}_i \rangle$ and the i^{th} element of the $(k \times 1)$ vector $\mathbf{b}$ is $\langle \boldsymbol{x}, \mathbf{e}_i \rangle$.

When an *orthonormal basis* is used, this simplifies dramatically since $\mathbf{G}$ is the identity and we obtain

$$\boldsymbol{x} = \sum_{i=1}^{k} x_i \mathbf{e}_i \qquad x_i = \langle \mathbf{x}, \mathbf{e}_i \rangle$$

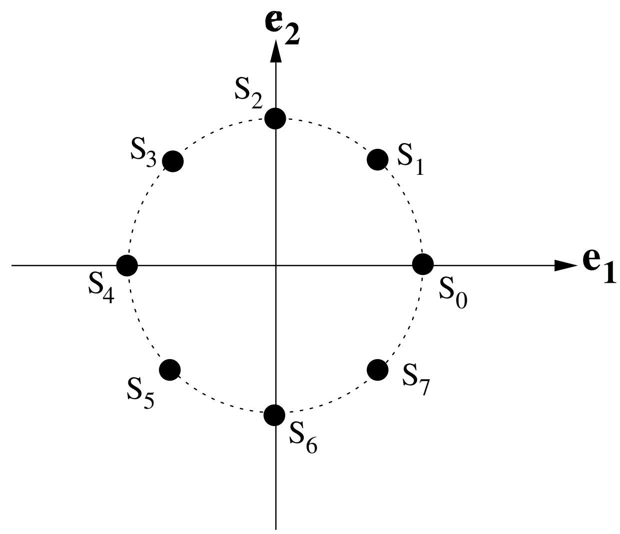

Figure 3: 8-PSK Vector Representation

3.9.1 Example: PSK Signaling

$$s_m(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \frac{m}{M}2\pi); \qquad t \in [0,T], i = 0,1,...M-1, M \geq 2$$

The signal set $\boldsymbol{s}_m$ are not linearly independent elements of $\mathcal{L}_2[0,t]$. In fact, $\mathcal{L}_2[0,t] \supset S = \text{span}\{s_i\}_{i=0}^{M-1}$, then $\dim(S) = \begin{cases} 2 & \text{if } M \neq 2 \\ 1 & \text{if } M = 2 \end{cases}$
Using an orthonormal basis:

$$\begin{aligned} e_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ e_2(t) &= \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{aligned}$$

We have

$$\begin{aligned} s_m(t) &= S_m(1)e_1(t) + S_m(2)e_2(t) \\ S_m(1) &= \int_0^T s_m(t)e_1(t)dt \\ S_m(2) &= \int_0^T s_m(t)e_2(t)dt \\ \boldsymbol{s}_m &= \begin{bmatrix} s_m(1) \\ s_m(2) \end{bmatrix} = \sqrt{E}\begin{bmatrix} \cos(\frac{m}{M}2\pi) \\ \sin(\frac{m}{M}2\pi) \end{bmatrix} \end{aligned}$$

3.10 Complete Orthonormal Set (CONS)

In an infinite dimensional Hilbert space, we use a "complete orthonormal set" (CONS) in place of a basis.

Definition: A CONS for a Hilbert space $\mathcal{X}$ is a set $\boldsymbol{x}_i$ such that:

(1) $\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle = \delta_K(i-j)$

(2) $\boldsymbol{x} \in \mathcal{X}$ and $\langle \boldsymbol{x}, \boldsymbol{x}_i \rangle = 0 \ \forall i \Longrightarrow \boldsymbol{x} = \boldsymbol{0}$

3.11 Generalized Fourier Series

Let $\boldsymbol{\phi_i}$ be a CONS for a Hilbert Space $\mathcal{H}$. then

(1) for each $\boldsymbol{x} \in \mathcal{H}$: $\boldsymbol{x} = \sum_i X_i \boldsymbol{\phi}_i$ where
(Fourier Series) $X_i = \langle \boldsymbol{x}, \boldsymbol{\phi}_i \rangle$

(2) Parseval's Equality:
$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_i X_i Y_i^*$
Special case: $\|\boldsymbol{x}\|^2 = \sum_i |x_i|^2$

3.11.1 Example: Hilbert Space $\mathcal{L}_2[0,T]$

In $\mathcal{L}_2[0,T]$,

$$\begin{aligned} x(t) &= \sum_i X_i \phi_i(t) \\ X_i &= \int_0^T x(t)\phi_i^*(t)dt \\ \int_0^T |x(t)|^2 dt &= \sum_i |x_i|^2 \end{aligned}$$

If we choose complex exponential as CONS, we have

$$\begin{aligned} \phi_k(t) &= \frac{1}{\sqrt{T}} \exp\left(\frac{j2\pi k}{T} t\right); \ k = 0, \pm 1, \pm 2... \\ X_k &= \mathcal{FS}\{x(t)\} = \frac{1}{\sqrt{T}} \int_0^T x(t) \exp\left(\frac{-j2\pi k}{T} t\right) dt \\ x(t) &= \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{\infty} X_k \exp\left(\frac{j2\pi k}{T} t\right) \end{aligned}$$

If we choose sine and cosine function as CONS, we have the trigonometric Fourier Series

$$\begin{aligned} \phi_{2k}(t) &= \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi}{T} kt\right) \\ \phi_{2k+1}(t) &= \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi}{T} kt\right) \\ k &= 0, 1, 2... \end{aligned}$$

As a conclusion, in general, we can choose the coordinate system for $\mathcal{L}_2[0,T]$ and represent our signals in this coordinate system!