Notes on Narrowband Signals and Complex Base Band Representation

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1 Introduction

In many communication systems, we often have band pass transmitting signals of the following form:

$$s(t) = \sqrt{2}\mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\} = \sqrt{2}\mathrm{Re}\{\tilde{s}(t)\}\cos(2\pi f_c t) - \sqrt{2}\mathrm{Im}\{\tilde{s}(t)\}\sin(2\pi f_c t) \tag{1}$$

Here, f_c corresponds to the carrier frequency, and $\tilde{s}(t)$, complex in general, is called the *equivalent complex base band signal* of $s(t)$. PAM, MPSK, and QAM signals all fit the above expression. Now, we may define the deterministic narrowband signal as follow:

Definition.

Let $s(t)$ be a deterministic waveform of the form in (1), and $S(f)$ be its Fourier Transform. If $S(f)$ is negligibly small except in the frequency range $|f - f_c| < B \ll f_c$, than $s(t)$ is a deterministic narrowband signal.

Of course, the above definition has a heavy "Engineer flavor", since it didn't really specify the term "negligible" quantitatively. However, in most practical cases, this definition is acceptable. We wish to establish some relationships between $s(t)$ and $\tilde{s}(t)$, since it is often more convenient to work with base band signals.

Property 1. *Let the Fourier Transform of $s(t)$ and $\tilde{s}(t)$ be $S(f)$ and $\tilde{S}(f)$ respectively, than the following holds:*

$$S(f) = \frac{1}{\sqrt{2}}[\tilde{S}(f - f_c) + \tilde{S}^*(-f - f_c)] \tag{2}$$

Proof. Exercise. □

The above property establishes the frequency domain relationship between $s(t)$ and $\tilde{s}(t)$.

Property 2. *If $s(t)$ is a deterministic narrowband signal, than the following holds:*

$$E = \int_{-\infty}^{\infty} s^2(t)dt = \int_{-\infty}^{\infty} |\tilde{s}(t)|^2 dt \tag{3}$$

Proof. Exercise. □

The above property indicates that the signal Energy is the same for $s(t)$ and $\tilde{s}(t)$ if $s(t)$ is a narrowband signal. Note that in some references, the factor of $\sqrt{2}$ in (1) is omitted, thus leads to slightly different results in property 2 and 3. However, the reader should be able to switch between two definitions without difficulty.

2 Linear Band Pass System

Given a linear filter system with real impulse response $h(t)$. Since $h(t)$ is real, we have:

$$H^*(-f) = H(f) \tag{4}$$

If we define:

$$\tilde{H}(f - f_c) = \begin{cases} H(f) & f > 0 \\ 0 & f < 0 \end{cases} \tag{5}$$

Then

$$H(f) = \tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c) \tag{6}$$

Or equivalently,

$$h(t) = 2\mathrm{Re}\{\tilde{h}(t)e^{j2\pi f_c t}\} \tag{7}$$

Where $\tilde{h}(t)$ is the Inverse Fourier Transform of $\tilde{H}(f)$, and it is the *equivalent complex base band impulse response* of the linear system. Note that if $h(t)$ is narrowband, than the system is called a narrowband linear system. Now, consider the narrowband signal $s(t)$ as the input to the narrowband linear system with impulse response $h(t)$. Then at the output, we have:

$$r(t) = \int_{-\infty}^{\infty} s(\tau)h(t - \tau)d\tau \tag{8}$$

Or in the frequency domain,

$$R(f) = S(f)H(f) \tag{9}$$

Using (2) and (6), together with the narrowband assumptions of $s(t)$ and $h(t)$, we have (verify!):

$$R(f) = \frac{1}{\sqrt{2}}[\tilde{R}(f - f_c) + \tilde{R}^*(-f - f_c)] \tag{10}$$

Where

$$\tilde{R}(f) = \tilde{S}(f)\tilde{H}(f) \tag{11}$$

Or equivalently,

$$r(t) = \sqrt{2}\mathrm{Re}\{\tilde{r}(t)e^{j2\pi f_c t}\} \tag{12}$$

Where

$$\tilde{r}(t) = \int_{-\infty}^{\infty} \tilde{s}(\tau)\tilde{h}(t - \tau)d\tau \tag{13}$$

Using the previous development, we already have enough tools to deal with a band pass system using its base band equivalent model. Note that we can always switch between the band pass signal and its base band equivalence by using (1) or (12).

3 Narrow Band Stochastic Process

Now, having defined the deterministic narrowband signal, we have a similar definition for a narrowband stochastic process. We consider the random process of the following form:

$$n(u,t) = \sqrt{2}\mathrm{Re}\{\tilde{n}(u,t)e^{j2\pi f_c t}\} = \sqrt{2}\mathrm{Re}\{\tilde{n}(u,t)\}\mathrm{cos}(2\pi f_c t) - \sqrt{2}\mathrm{Im}\{\tilde{n}(u,t)\}\mathrm{sin}2\pi f_c t \tag{14}$$

This form arises in a communication system when the noise process passes through the band pass filter at the Front-End of the receiver. Thus,f_c usually corresponds to the carrier frequency of the system, which is also the central frequency of the band pass filter. $\tilde{n}(u,t)$, complex in general, is called the *equivalent complex base band process* of $n(u,t)$.We define a narrowband stochastic process as follow:

Definition.

Let $n(u,t)$ be a stochastic process of the form in (14), where $n_c(t) \equiv Re\{\tilde{n}(u,t)\}$ and $n_s \equiv Im\{\tilde{n}(u,t)\}$ are zero mean, jointly WSS processes. Let $S_n(f)$ be the PSD of $n(u,t)$. If $S_n(f)$ is negligibly small except in the frequency range $|f - f_c| < B \ll f_c$, than $n(u,t)$ is a narrowband stochastic process.

Now, we shall establish some properties of the process described above. First we obtain the autocorrelation function of $n(u,t)$. We will omit the variable in the following work, but the reader should keep in mind that these are stochastic processes. Using (14), we have:

$$\begin{aligned}
\mathrm{E}[n(t)n(t+\tau)] =& 2\{\mathrm{E}[n_c(t)n_c(t+\tau)]\mathrm{cos}(2\pi f_c t)\mathrm{cos}(2\pi f_c(t+\tau)) \\
&- \mathrm{E}[n_c(t)n_c(t+\tau)]\mathrm{cos}(2\pi f_c t)\mathrm{sin}(2\pi f_c(t+\tau)) \\
&- \mathrm{E}[n_s(t)n_c(t+\tau)]\mathrm{sin}(2\pi f_c t)\mathrm{cos}(2\pi f_c(t+\tau)) \\
&+ \mathrm{E}[n_s(t)n_c(t+\tau)]\mathrm{sin}(2\pi f_c t)\mathrm{sin}(2\pi f_c(t+\tau))\}
\end{aligned} \tag{15}$$

We can see that $n(t)$ is WSS if and only if

$$\mathrm{E}[n_c(t)n_c(t+\tau)] = \mathrm{E}[n_s(t)n_s(t+\tau)] \tag{16}$$

and

$$\mathrm{E}[n_c(t)n_s(t+\tau)] = -\mathrm{E}[n_s(t)n_c(t+\tau)] \tag{17}$$

The detail is left to the reader as an exercise. Now we assume $n(t)$ is a WSS process, thus (16) and (17) are satisfied, and the autocorrelation function of $n(t)$ becomes:

$$R_n(\tau) = 2[R_{n_c}(\tau)\mathrm{cos}(2\pi f_c\tau) + R_{n_c n_s}(\tau)\mathrm{sin}(2\pi f_c\tau)] \tag{18}$$

where

$$R_{n_c}(\tau) = \mathrm{E}[n_c(t)n_c(t+\tau)] = R_{n_s}(\tau) \tag{19}$$

and

$$R_{n_c n_s}(\tau) = \mathrm{E}[n_c(t+\tau)n_s(t)] = -R_{n_s n_c}(\tau) \tag{20}$$

We now find the autocorrelation function of the equivalent complex base band process $\tilde{n}(t)$. Since $\tilde{n}(t) = \text{Re}\{\tilde{n}(t)\} + j\text{Im}\{\tilde{n}(t)\} = n_c(t) + jn_s(t)$, using (16) and (17), we have:

$$\tilde{R}_n(\tau) = \text{E}[\tilde{n}(t+\tau)\tilde{n}^*(t)] = 2[R_{n_c}(\tau) - jR_{n_c n_s}(\tau)] \tag{21}$$

Thus, we can express (18) using (21):

$$R_n(\tau) = \text{Re}\{\tilde{R}_n(\tau)e^{j2\pi f_c t}\} \tag{22}$$

Or equivalently,

$$S_n(f) = \frac{1}{2}[\tilde{S}_n(f - f_c) + \tilde{S}_n^*(-f - f_c)] \tag{23}$$

where $\tilde{S}_n(f)$ is the Fourier Transform of $\tilde{R}_n(\tau)$.

The equations (22) and (23) establish the relationships between WSS process $\tilde{n}(t)$ and its equivalent complex base band process . Note that by (20) and the fact that $R_{n_c n_s}(\tau) = R_{n_s n_c}(-\tau)$, we have $R_{n_c n_s}(0) = R_{n_s n_c}(0) = 0$. Thus, we have the following relationship.

$$\text{E}[|n(t)|^2] = R_n(0) = \text{Re}\{\tilde{R}_n(0)\} = \tilde{R}_n(0) = \text{E}[|\tilde{n}(t)|^2] = 2R_{n_c}(0) \tag{24}$$

The above equation tells us that both $n(t)$ and $\tilde{n}(t)$ has the same average power, and the average power of $\tilde{n}(t)$ is equally distributed between the process $n_c(t)$ and $n_s(t)$.

In many practical cases, $\tilde{S}_n(f)$ is even (This is due to the design of the band pass filter at the receiver, which is usually symmetric about its central frequency f_c). By the property of Fourier Transform, we know that $\tilde{R}_n(\tau)$ is real. But by (21), this means $\tilde{R}_n(\tau) = 2R_{n_c}(\tau)$, and $R_{n_c n_s}(\tau) = 0 \ \forall\tau$. We conclude that $n_c(t)$ and $n_s(t)$ are uncorrelated. If they are also jointly gaussian, then they are independent processes. In summary, given a real band pass WSS process $n(t)$ and its PSD $S_n(f)$, we can write the process in the form of (14), where $\tilde{n}(t) = n_c(t) + jn_s(t)$ has PSD $\tilde{S}_n(f)$ satisfying (23), $n_c(t)$ and $n_s(t)$, satisfies (19) and (20). We must point out here that in general, the equality in (14) is valid up to second moment description of the process, while for higher order moments, (14) may no longer be true. However, if the process is gaussian, as is usually the case in our communication systems, then (14) is strictly valid. This is because a gaussian process is completely described by its second moment properties.

4 Conclusions

In this note, we introduce the concept of narrowband signals (stochastic processes) and their equivalent complex base band representations. The tools in the previous sections enable us to transform a band pass signal (stochastic process) into its complex base band representation. This not only simplifies our problem, but also indicates that the carrier frequency is really not a factor when we analyze the performance of the system. In order to become more familiar with these tools, the reader should try to do the performance analysis of some modulations (e.g. MPSK, QAM, PAM) over AWGN channel, using both the pass band model and the complex base band equivalent model. You will see that they both reach the same conclusion.

References

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