Complex Circular Random Variables and Random Vectors

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A complex random variable $z(u)$ is one with real part $x(u)$ and imaginary part $y(u)$ so that

$$z(u) = x(u) + jy(u) \tag{1}$$

To describe $z(u)$, you need the joint description of $x(u)$ and $y(u)$. For example, the pdf of $z(u)$ is the joint-pdf of $x(u)$ and $y(u)$. The second moment description of $z(u)$ is the joint second moment description of $x(u)$ and $y(u)$ –*i.e.,* it includes the means of $x(u)$ and $y(u)$, the variances of $x(u)$ and $y(u)$, and the covariance between $x(u)$ and $y(u)$.

An alternative way to keep track of these second moments is to have

$$m_z = \mathbb{E}\left\{z(u)\right\} = m_x + jm_y \tag{2}$$

$$\sigma_z^2 = \mathbb{E}\left\{|z(u) - m_z|^2\right\} = \sigma_x^2 + \sigma_y^2 \tag{3}$$

$$\tilde{\sigma}_z^2 = \mathbb{E}\left\{(z(u) - m_z)^2\right\} = \sigma_x^2 - \sigma_y^2 + 2j\text{cov}\left[x(u), y(u)\right] \tag{4}$$

You can verify that $(m_z, \sigma_z^2, \tilde{\sigma}_z^2)$ determines $(m_x, m_y, \sigma_x^2, \sigma_y^2, \text{cov}\left[x(u), y(u)\right]]$ –*i.e.,* that these descriptions can be converted back and forth.

A *circular complex random variable* is one with $\tilde{\sigma}_z^2 = 0$ – also, it is typically the case that $m_z = 0$ and this is sometimes included in the definition. Note that, in order for $\tilde{\sigma}_z^2 = 0$ to hold, $x(u)$ and $y(u)$ must be uncorrelated and have the same variance. If $z(u)$ is complex circular with zero mean, then the random variable $z_\theta(u) = e^{j\theta}z(u)$ will have the same second moments as $z(u)$. This is where the term circular comes from: a rotation of this random variable in the complex plane does not change its second moment description.

A complex circular Gaussian random variable is one where the real and imaginary parts are independent Gaussians with the same variance. A circular complex Gaussian (CCG) random variable $z(u)$ has pdf given by

$$f_{z(u)}(z) = f_{x(u),y(u)}(x, y) = \frac{1}{\pi N_0} \exp\left(\frac{-(x - m_x)^2 - (y - m_y)^2}{N_0}\right) \tag{5}$$

$$= \frac{1}{\pi N_0} \exp\left(\frac{-|z - m_z|^2}{N_0}\right) \tag{6}$$

$$\triangleq \mathcal{N}^{cc}(z; m_z; N_0) \tag{7}$$

where $\sigma_x^2 = \sigma_y^2 = N_0/2$, $\sigma_z^2 = N_0$, and $z = x + jy$. Note that when a complex random variable is circular, it allows us to simplify the notation for the second moments and, in the case of complex circular Gaussian (CCG), we can write the pdf in compact form with a complex variable argument. This is the primary reason for considering complex random variables – *i.e.,* simplified notation.

The above extends to random vectors, random sequences, and random waveforms. Specifically, a complex random vector is a vector with each component being a complex random variable – or the random vector is of the form $\mathbf{z}(u) = \mathbf{x}(u) + j\mathbf{y}(u)$ where $\mathbf{x}(u)$ and $\mathbf{y}(u)$ are real random vectors. The second moments are the joint second moments for $\mathbf{x}(u)$ and $\mathbf{y}(u)$ –*i.e.,* covariance matrices of $\mathbf{x}(u)$ and $\mathbf{y}(u)$, the mean vectors of $\mathbf{x}(u)$ and $\mathbf{y}(u)$, and covariance matrix between $\mathbf{x}(u)$ and $\mathbf{y}(u)$. Alternatively, an equivalent second moment description is

$$\mathbf{m}_{\mathbf{z}} = \mathbb{E}\left\{\mathbf{z}(u)\right\} = \mathbf{m}_{\mathbf{x}} + j\mathbf{m}_{\mathbf{y}} \tag{8}$$

$$\mathbf{K}_{\mathbf{z}} = \mathbb{E}\left\{(\mathbf{z}(u) - \mathbf{m}_{\mathbf{z}})(\mathbf{z}(u) - \mathbf{m}_{\mathbf{z}})^{\dagger}\right\} \tag{9}$$

$$= \mathbf{K}_{\mathbf{x}} + \mathbf{K}_{\mathbf{y}} + j\left(\mathbf{K}_{\mathbf{yx}} - \mathbf{K}_{\mathbf{xy}}\right) \tag{10}$$

$$\widetilde{\mathbf{K}}_{\mathbf{z}} = \mathbb{E}\left\{(\mathbf{z}(u) - \mathbf{m}_{\mathbf{z}})(\mathbf{z}(u) - \mathbf{m}_{\mathbf{z}})^{\mathrm{t}}\right\} \tag{11}$$

$$= \mathbf{K}_{\mathbf{x}} - \mathbf{K}_{\mathbf{y}} + j\left(\mathbf{K}_{\mathbf{yx}} + \mathbf{K}_{\mathbf{xy}}\right) \tag{12}$$

where $\widetilde{\mathbf{K}}_{\mathbf{z}}$ is called the pseudo-covariance of $\mathbf{z}(u)$.

A circular random vector is one in which $\widetilde{\mathbf{K}}_{\mathbf{z}} = \mathbf{O}$. A common case is $\mathbf{K}_{\mathbf{x}} = \mathbf{K}_{\mathbf{y}}$ and $\mathbf{x}(u)$ and $\mathbf{y}(u)$ are uncorrelated.[1] Similar to the scalar case, we can express the pdf of a CCG in compact form using a complex argument

$$f_{\mathbf{z}(u)}(\mathbf{z}) = f_{\mathbf{x}(u),\mathbf{y}(u)}(\mathbf{x}, \mathbf{y}) \tag{13}$$

$$= \frac{1}{\pi^n |\mathbf{K}_{\mathbf{z}}|} \exp\left(-(\mathbf{z} - \mathbf{m}_{\mathbf{z}})^{\dagger}\mathbf{K}_{\mathbf{z}}^{-1}(\mathbf{z} - \mathbf{m}_{\mathbf{z}})\right) \tag{14}$$

$$\stackrel{\Delta}{=} \mathcal{N}_n^{cc}(\mathbf{z}; \mathbf{m}_{\mathbf{z}}; \mathbf{K}_{\mathbf{z}}) \tag{15}$$

A derivation of this compact from can be found in Scholtz's 562 Notes, section 7.2.3.

[1] This is not required for the pseudo covariance matrix to be zero.