Summary of Performance Limits

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There is a basic trade-off between throughput and fidelity in a communications channel – i.e., the faster one would like to send data, the less reliable it can be communicated. There are a number of ways to quantify this trade-off. The most well known measure is Shannon's channel capacity, which defines the maximum information transfer rate that can be achieved with arbitrarily low error probability. The proof of this capacity result is based on averaging over random codes and taking the limit as the encoding block size approaches infinity.

In practice, one can only communicate with finite block sizes and must tolerate some non-zero probability of error. Determining this precise trade-off between block size, error probability, and achievable throughput is quite difficult, especially if one places realistic constraints on the channel. For many years, the coding community used the so-called Channel Cut-off Rate [1, 2] as a measure of this finite-block size trade-off. The cut-off rate is determined by upper-bounding the performance of random coding with the union bound. The cut-off rate was a good tool for gauging the achievable performance of classical codes and to a great extent, the coding community considered it the practical limit for error correction coding. However, after the development of modern turbo-like codes (TLCs), it became clear that the cut-off rate was an artificial measure and that practical codes could perform much better than predicted by the cut-off rate.

Other finite block size performance bounds existed in the literature [3, 4], but were not widely used as a comparison criterion for practical codes because they are difficult to evaluate and, as mentioned above, classical codes were far from these limits. In particular, the sphere-packing bound and the random coding bound are two useful finite block size bounds that approach the channel capacity as the block size increases asymptotically.

In this document, we briefly describe measures of achievable performance for finite block size coding schemes. We suggest numerical methods for evaluating and/or approximating these measures. Finally, we demonstrate a fairly simple measure for gauging the performance limits of finite block size codes. There exist simple TLC constructions that approach these limits over a wide range of operating scenarios (rate, block size, and target error rate) within approximately 1 dB in E_b/N_0. Furthermore, for a particular operating scenario, point designs typically can be found which are within 0.5 dB in E_b/N_0 of these perform limits. Thus, the suggested measure can be used as a guideline for the achievable performance of a link using modern FEC and also as a benchmark for modern codecs.

1 Channel Model and Rate Measures

For the remainder of this document, we will consider the modulation constrained additive white Gaussian noise (AWGN) channel, where each channel use is modeled by

$$\mathbf{z}(u) = \sqrt{\frac{E_s}{N_0}}\mathbf{x}(u) + \mathbf{w}(u) \qquad (D \times 1) \tag{1}$$

where D is the number of dimensions for each channel use. The information-bearing signal $\mathbf{x}(u)$ is distributed over a finite set $S = \{\mathbf{s}_m\}_{m=0}^{M-1}$ with distribution $p_m = \text{Pr}\{\mathbf{x}(u) = \mathbf{s}_m\}$. The noise $\mathbf{w}(u)$ is a zero-mean Gaussian vector, mutually independent over all channel uses. To simplify the later numerical analysis, the observation is assumed to be normalized so that the noise variance in each dimension is 1/2, specifically

$$\mathbb{E}\left\{\|\mathbf{x}(u)\|^2\right\} = \sum_{m=0}^{M-1} p_m \|\mathbf{s}_m\|^2 = 1 \tag{2}$$

$$\mathbb{E}\left\{\mathbf{w}(u)\mathbf{w}^{\mathrm{t}}(u)\right\} = \frac{1}{2}\mathbf{I} \tag{3}$$

This is equivalent to a model where the expected value of squared magnitude of the signal is E_s and the noise has variance $N_0/2$ in each dimension, but this format is preferable for the method of numerical integration used in the following. With this convention, the channel likelihood is

$$p(\mathbf{z}|\mathbf{s}_m) = \frac{1}{\pi^{D/2}} \exp\left(-\left\|\mathbf{z} - \sqrt{\frac{E_s}{N_0}}\mathbf{s}_m\right\|^2\right) \tag{4}$$

The throughput or rate of the signaling format in (1) can be characterized in several ways. Assume that the channel in (1) is used q times in an attempt to send k information bits. The rate of this system is then $\eta_{\mathrm{b/sym}} = k/q$ information bits per M-ary symbol. One way to achieve this is to use an (n, k) binary code with rate $r = k/n$ and then map the n coded bits onto $q = \frac{n}{\log_2(M)}$ symbols. It follows that

$$\eta_{\mathrm{b/sym}} = k/q = r \log_2(M) \tag{5}$$

Whether the mapping from k input bits to q M-ary symbols is achieved by a binary code with a modulation mapper or directly through a coded modulation design, the following is maintained for the same information bit rate and transmit power

$$E_s = \eta_{\mathrm{b/sym}} E_b \tag{6}$$

Since the number of dimensions D in (1) is tied to the bandwidth used on the channel, it is also useful to consider a rate normalized to the number of dimensions per channel symbol. Normalizing to two dimensions is particularly useful, so we introduce the rate $\eta_{\mathrm{b/2d}}$ measured in information bits per two dimensions

$$\eta_{\mathrm{b/2d}} = \frac{2}{D}\eta_{\mathrm{b/sym}} = \frac{2k}{Dq} \tag{7}$$

Finally, under ideal conditions one can convey two dimensions per second per Hz of bandwidth. Thus we consider the maximum throughput in bits per second per Hz of bandwidth (bps/Hz) as

$$\eta_{\mathrm{bps/Hz}} = \eta_{\mathrm{b/2d}} \qquad \text{(ideal)} \tag{8}$$

In practice, the value of $\eta_{\mathrm{bps/Hz}}$ will be less than $\eta_{\mathrm{b/2d}}$. For example, if root raised cosine (rrc) pulse shaping is used, $\eta_{\mathrm{bps/Hz}} = \eta_{\mathrm{b/2d}}/(1+\beta)$, where β is the excess bandwidth of the rrc pulse.

2 Capacity and Symmetric Information Rate

The mutual information rate of the channel in (1) is [3, 5]

$$\begin{aligned} \mathrm{I}(\mathbf{z}(u);\mathbf{x}(u)) &= \sum_{m=0}^{M-1} p_m \int_{R^D} p(\mathbf{z}|\mathbf{s}_m)\log_2\left(\frac{p(\mathbf{z}|\mathbf{s}_m)}{p(\mathbf{z})}\right)d\mathbf{z} && \text{(9a)}\\ &= \sum_{m=0}^{M-1} p_m \int_{R^D} p(\mathbf{z}|\mathbf{s}_m)\log_2\left(\frac{p(\mathbf{z}|\mathbf{s}_m)}{\sum_{n=0}^{M-1} p(\mathbf{z}|\mathbf{s}_n)p_n}\right)d\mathbf{z} && \text{(9b)}\\ &= \sum_{m=0}^{M-1} p_m J_m && \text{(9c)} \end{aligned}$$

The units for $\mathrm{I}(\mathbf{z}(u);\mathbf{x}(u))$ are bits of information per (D-dimensional) channel use.

The *constrained capacity* is the maximum of $\mathrm{I}(\mathbf{z}(u);\mathbf{x}(u))$ over all distributions on $\mathbf{x}(u)$. For some signal sets this maximum can be shown to occur at the uniform distribution $p_m = 1/M$ (*e.g.,* 2^m-PSK constellations). However, for other signal formats (*e.g.,* 64-QAM) the maximizing distribution is not uniform. Given an efficient method for evaluating the integral in (9), it is possible to find the distribution on S that maximizes the mutual information. This is tedious, however, and it is common to work with the *symmetric information rate (SIR)* which is the mutual information under the uniform distribution $p_m = 1/M$. Note that the SIR is a lower bound on the capacity. The difference between the capacity and the SIR, known as the shaping gain, is expected to increase as M increases and for many modulations used in practice is negligible. In fact, the SIR is often erroneously referred to as the constrained capacity in the literature.

2.1 Form for Numerical Evaluation

Let us manipulate the expression in (9) to a form suitable for numerical evaluation as described in Section 4. Substituting for the conditional density in the expression for I_m, we obtain

$$\begin{aligned} J_m &= \frac{1}{\pi^{D/2}\ln(2)}\int_{R^D} e^{-\|\mathbf{z}\|^2} g_m(\mathbf{z})d\mathbf{z} && \text{(10)}\\ g_m(\mathbf{z}) &= -\ln\left(\sum_{n=0}^{M-1} e^{-\beta_{m,n}(\mathbf{z})}\right) && \text{(11)}\\ &= \min_n{}^{*}\beta_{m,n}(\mathbf{z}) && \text{(12)}\\ \beta_{m,n}(\mathbf{z}) &= -\ln(p_m) + \sqrt{\frac{E_s}{N_0}}(\mathbf{s}_m-\mathbf{s}_n)^{\mathrm{t}}\left[2\mathbf{z} + \sqrt{\frac{E_s}{N_0}}(\mathbf{s}_m-\mathbf{s}_n)\right] && \text{(13)} \end{aligned}$$

In the above we have used the definition

$$\min{}^*(x_1, x_2 \dots x_n) \triangleq -\ln(e^{-x_1} + e^{-x_2} + \cdots e^{-x_n}) \quad (14)$$

and the resulting facts

$$\begin{aligned} \min{}^*(x,y) &= \min(x,y) - \ln\left(1 + e^{-|x-y|}\right) && \text{(15)}\\ \min{}^*(x,y,z) &= \min{}^*(\min{}^*(x,y),z) && \text{(16)} \end{aligned}$$

3 (Symmetric) Random Coding Bound

The random coding bound (RCB) is an upper bound on the average probability of codeword error under maximum likelihood (ML) decoding, $\bar{P}_{\text{cw}}$. The probability law for selecting codewords is assumed to be separable so that each coordinate is selected independently. In other words, for the transmitted codeword, the value of $\mathbf{x}(u)$ for each channel use of the form (1) is selected independently with signal $\mathbf{s}_m$ sent with probability p_m. With this assumption, the RCB is

$$\bar{P}_{\text{cw}} \leq \exp\left(-qE_r(\eta_{\text{b/sym}})\right) \tag{17}$$

where $E_r(\eta_{\text{b/sym}})$ is the *random coding exponent* and is given by

$$E_r(\eta_{\text{b/sym}}) = \max_{0\leq\rho\leq 1} \max_{\mathbf{p}} \left[E_0(\rho, \mathbf{p}, \eta_{\text{b/sym}}) - \rho \ln(2)\eta_{\text{b/sym}}\right] \tag{18}$$

and the *Gallager function* is

$$E_0(\rho, \mathbf{p}, \eta_{\text{b/sym}}) = \int_{R^D} \left[\sum_{m=0}^{M-1} p_m \left\{p(\mathbf{z}|\mathbf{s}_m)\right\}^{\frac{1}{1+\rho}}\right]^{1+\rho} d\mathbf{z} \tag{19}$$

where $\mathbf{p}$ is the $(M \times 1)$ vector with m^{th} component p_m and the maximum is over all valid probability mass functions.

The importance of the RCB derives from the fact that for $\eta_{\text{b/sym}}$ less than the capacity, the random coding exponent is positive, implying that the average probability of error using random coding decays exponentially with block length given that the attempted transmission rate is below the capacity. This also implies the channel coding theorem since as we let $q \to \infty$, the error probability will tend to zero for rates below the capacity.

Once again, maximization over all input distribution functions is tedious. For the same reasons discussed with regard to the capacity and SIR discussed in Section 1, it is reasonable to consider the *symmetric random coding bound (SRCB)*, which is the bound obtained with (17) when instead of maximizing over $\mathbf{p}$ in (18), we use $p_m = 1/M$. Note that this still provides a valid upper-bound on $\bar{P}_{\text{cw}}$, although it will be slightly looser if the uniform distribution does not maximize the quantity in (18).

3.1 Form for Numerical Evaluation

The most difficult part of evaluating the RCB is the integral in (19), so we focus on computing the Gallager function. Also, since the Gallager function is positive, we may work in the log-domain for additional efficiency and numerical stability. Using the density in (4), the expression in (19) we have

$$-\ln\left(E_0(\rho, \mathbf{p}, \eta_{\text{b/sym}})\right) = \frac{D}{2}\ln(\pi) - \ln J \tag{20}$$

$$J = \int_{R^D} e^{-\|\mathbf{z}\|^2} \left[\sum_{m=0}^{M-1} e^{-\alpha_m(\mathbf{z})}\right]^{1+\rho} d\mathbf{z} \tag{21}$$

$$\alpha_m(\mathbf{z}) = \ln(p_m) - \frac{\sqrt{\frac{E_s}{N_0}}\mathbf{s}_m^{\text{t}}\left(2\mathbf{z} - \sqrt{\frac{E_s}{N_0}}\mathbf{s}_m\right)}{1+\rho} \tag{22}$$

Again, using log-domain equivalent operations, we have

$$\sum_{m=0}^{M-1} e^{-\alpha_m(\mathbf{z})} = e^{-A(\mathbf{z})} \tag{23}$$

$$A(\mathbf{z}) = \min_m{}^* \alpha_m(\mathbf{z}) \tag{24}$$

Using this relation, we obtain the following form for the integral J in (21)

$$J = \int_{R^D} e^{-\|\mathbf{z}\|^2} \exp\left[-(1+\rho)A(\mathbf{z})\right] d\mathbf{z} \tag{25}$$

3.2 Basic Facts Used in Deriving the RCB

While the bound appears complex and we do not present a proof of the bound here, it follows from two basic bounds [3]. These are a pairwise error probability bound for ML decoding and a generalization of the union bound. For the pairwise error probability, consider a two way decision at the ML receiver between $\mathbf{s}_m$ and $\mathbf{s}_i$. Given that that $\mathbf{s}_m$ was transmitted, the conditional pairwise error probability is the probability that $\mathbf{z}(u)$ fall outside the decision region $Z_{PW}(m)$, as defined by the ML rule

$$\begin{aligned}
P_{PW}(i|m) &= \text{Pr}\left\{\mathbf{z}(u) \notin Z_{PW}(m) | \mathbf{x}(u) = \mathbf{s}_m\right\} && \text{(26a)}\\
&= \int_{Z^c_{PW}(m)} p(\mathbf{z}|\mathbf{s}_m) d\mathbf{z} && \text{(26b)}\\
&\leq \int_{Z^c_{PW}(m)} [p(\mathbf{z}|\mathbf{s}_m)]^{1-s}[p(\mathbf{z}|\mathbf{s}_i)]^s d\mathbf{z} \qquad 0 < s < 1 && \text{(26c)}\\
&\leq \int_{R^D} [p(\mathbf{z}|\mathbf{s}_m)]^{1-s}[p(\mathbf{z}|\mathbf{s}_i)]^s d\mathbf{z} \qquad 0 < s < 1 && \text{(26d)}
\end{aligned}$$

where the inequality in (26c) follows from the fact that for any $\mathbf{z} \in Z^c_{PW}(m)$, $p(\mathbf{z}|\mathbf{s}_i) \geq p(\mathbf{z}|\mathbf{s}_m)$ since this is the region where $\mathbf{s}_i$ is more likely than $\mathbf{s}_m$. The inequality in (26d) follows since the integrand is nonnegative and $Z^c_{PW}(m) \subseteq R^D$.

The second bound used to develop the RCB is the generalized union bound given by

$$P\left(\bigcup_i A_i\right) \leq \left[\sum_i P(A_i)\right]^\rho \qquad 0 \leq \rho \leq 1 \tag{27}$$

which is the standard union bound for $\rho = 1$. This bound follows from the fact that

$$P\left(\bigcup_i A_i\right) \leq \min\left\{1, \sum_i P(A_i)\right\} \leq \left[\sum_i P(A_i)\right]^\rho \qquad 0 \leq \rho \leq 1 \tag{28}$$

where the first inequality is a trivial extension of the union bound and the second inequality follows from that fact that $x^\rho \geq x$ when $x \in [0, 1]$ and $\rho \in (0, 1]$.

4 Numerical Evaluation of Performance Limits

A good numerical algorithm for evaluating the integrals in (10) and (25) is the Gauss-Hermite method, which for a one-dimensional integral is

$$\int_{-\infty}^{\infty} e^{-z^2} f(z) dz \approx \sum_{i=1}^{I-1} w_i f(z_i) \tag{29}$$

where the points z_i and the coefficients w_i are determined by the properties of the Hermite polynomials. The greater the the degree of the approximation I, the more accurate the result. This method was applied to compute the SIR in [6]. Tables of the values of the coefficients $\{w_i\}$ and the points $\{z_i\}$ for various values of I are available. The online resource in [7] is particularly convenient.

For integrals over R^D with $D > 1$ the approximation is

$$\int_{R^D} e^{-\|\mathbf{z}\|^2} f(\mathbf{z}) d\mathbf{z} \approx \sum_{i_1, i_2, \ldots i_D} w_{i_1} w_{i_2} \cdots w_{i_D} f(z_{i_1}, z_{i_1}, \ldots z_{i_D}) \tag{30}$$

Since the approximation sum has I^D terms, this becomes impractical for large D. Most cases of practical interest are for $D = 1$ or $D = 2$, in which case this method is very fast.

If the function $f(z)$ in (29) is positive for all values of z, then we can use log-domain equivalent operations. Specifically, define $f(z) = \exp(-m(z))$, then

$$-\ln\left[\int_{-\infty}^{\infty} e^{-z^2} f(z) dz\right] \approx -\ln\left[\sum_{i=0}^{I-1} w_i \exp(-m(z_i))\right] = \min_i{}^* \left(-\ln(w_i) + m(z_i)\right) \tag{31}$$

which also uses the fact that the coefficients w_i are positive. Thus, in the case where $f(z)$ is positive, the Gauss-Hermite approximation can be carried out with greater efficiency and stability in the log domain. The multi-dimensional case in (30) can be evaluated similarly in the case when the function $f(\mathbf{z})$ is positive.

4.1 Evaluating the SIR

The Gauss-Hermite approximation can be used to compute the integral J_m in (10). For example, for $D = 1$, we have

$$J_m \approx \frac{1}{\ln(2)\sqrt{\pi}} \sum_{i=0}^{I-1} w_i \min_n{}^* \beta_{m,n}(z_i) \tag{32}$$

where the min-star operation is over $n = 0, \ldots M - 1$. The mutual information can then be computed using (9c).

Pseudo-code for evaluating the $D = 2$ SIR is given in Algorithm 1.

4.2 Evaluating the SRCB

In evaluating the RCB, one needs to repeatedly evaluate the integral J in (25). Since the implied function in the Gauss-Hermite approximation is $f(\mathbf{z}) = \exp[-(1 + \rho)A(\mathbf{z})]$, which is positive for all

Algorithm 1: Symmetric Information Rate with $D = 2$

input : $\sqrt{E_s/N_0}$, the Gauss-Hermite coefficents and points $w[i]$, $z[i]$, indexed from $i = 0 \ldots (I-1)$ and the signals $s_m[d]$, for $d = 0, 1$, normalized to unit norm

output: The SIR in bits per channel use

```
SIR ← 0
for m ← 0 to (M − 1)
    J_m ← 0
    for i_0 ← 0 to (I − 1)
        for i_1 ← 0 to (I − 1)
            X ← ∞
            for n ← 0 to (M − 1)
                t_0 ← sqrt(E_s/N_0)(s_m[0] − s_n[0])  // temp variable
                t_1 ← sqrt(E_s/N_0)(s_m[1] − s_n[1])  // temp variable
                β_{m,n} ← ln(M) + t_0(t_0 + 2z[i_0]) + t_1(t_1 + 2z[i_1])
                // above line uses p_m = 1/M
                X ← min*(X, β_{m,n})
            end
            J_m ← J_m + w[i_0]w[i_1]X
        end
    end
    SIR ← SIR + J_m/M
end
SIR ← SIR /(π ln(2))
return SIR
```

$\mathbf{z}$, we can use the log-domain expression of the form in (31) so that, for $D = 1$, we have

$$-\ln J \approx \sum_{i=0}^{I-1} w_i e^{-(1+\rho)A(\mathbf{z})} \tag{33}$$

$$= \min_i{}^{*} \left[-\ln(w_i) + (1+\rho)A(\mathbf{z})\right] \tag{34}$$

This result can then be substituted into (20) to evaluate the Gallager function. This can then be used in (18) and maximized over all ρ using standard numerical methods.

Pseudo-code for evaluating the quantity in (18) for a fixed value of ρ is given in Table 2.

Algorithm 2: Symmetric Random Coding Exponent with $D = 2$, and fixed ρ

input : $\sqrt{E_s/N_0}$, $\eta_{\text{b/sym}}$, the neg-log of Gauss-Hermite coefficents $v[i] = -\ln(w[i])$ and points $z[i]$, indexed from $i = 0 \ldots (I-1)$, the signals $s_m[d]$, for $d = 0, 1$, normalized to unit norm, and a value of $\rho \in (0, 1)$

output: The random coding exponent: $E_r(\eta_{\text{b/sym}}; \rho)$

$E_0 \leftarrow \infty$
for $i_0 \leftarrow 0$ **to** $(I-1)$
  for $i_1 \leftarrow 0$ **to** $(I-1)$
    $A \leftarrow \infty$
    for $m \leftarrow 0$ **to** $(M-1)$
      $t_0 \leftarrow \sqrt{E_s/N_0}\, s_m[0]$ `// temp variable`
      $t_1 \leftarrow \sqrt{E_s/N_0}\, s_m[1]$ `// temp variable`
      $\alpha_m \leftarrow \ln(M) - \left[t_0(2z[i_0] - t_0) + t_1(2z[i_1] - t_1)\right]/(1+\rho)$
      `// above line uses` $p_m = 1/M$
      $A \leftarrow \min^*(A, \beta_{m,n})$
    end
    $E_0 \leftarrow \min^*(E_0, (1+\rho)A + v[i_0] + v[i_1])$ `// computing` $-\ln J$
  end
end
$E_0 \leftarrow E_0 + \ln(\pi)$
$E_r \leftarrow E_0 - \rho \ln(2)\eta_{\text{b/sym}}$
return E_r

5 An Approximate Symmetric Sphere Packing Bound

Finite block size performance bounds were considered in [8]. In particular, the *sphere packing bound (SPB)* is considered for the modulation unconstrained AWGN channel. The SPB is a lower bound on the codeword error probability of any code. In [8] they consider a version of the SPB that is normalized to the minimum value of E_b/N_0 required by the capacity of the average power limited, bandwidth-limited AWGN channel

$$\left(\frac{E_b}{N_o}\right)_{\min} = \frac{2^{\eta_{\text{bps/Hz}}} - 1}{\eta_{\text{bps/Hz}}} \tag{35}$$

For a given operational scenario – *i.e.,* a given choice for $\eta_{\text{bps/Hz}}$, input block size k, and required codeword error probability P_{CW} – the SPB also will provide a minimum value of E_b/N_0. In other words, the lower bound on P_{CW} can be made equal to the desired codeword error probability if the value of E_b/N_0 is greater than this specific value. This minimum value of E_b/N_0 will be greater than that given by the capacity expression in (35). In [8], a normalized version of the SPB is suggested by considering the difference between the required E_b/N_0 value provided by the SPB and the minimum value of E_b/N_0 according to (35). With the assumption that one can achieve two dimensions per second per Hz (*i.e.,* $\eta_{\text{bps/Hz}} = \eta_{\text{b/2d}}$), this difference in dB is

$$\Delta_{\text{dB}} = \sqrt{\frac{20\eta_{\text{b/2d}}\left(2^{\eta_{\text{b/2d}}}+1\right)\left[10\log_{10}(1/P_{\text{CW}})\right]}{k\ln(10)\left(2^{\eta_{\text{b/2d}}}-1\right)}} \tag{36}$$

Thus, this *sphere packing bound approximation (SPBA)* is

$$\left(\frac{E_b}{N_0}\right)_{\text{min,SPB, (dB)}} \approx 10\log_{10}\left[\frac{2^{\eta_{\text{b/2d}}}-1}{\eta_{\text{b/2d}}}\right] + \Delta_{\text{dB}} \tag{37}$$

The SPBA in (37) is compared to accurate evaluation of the SPB in [8] for relatively low rates – *i.e.,* for $\eta_{\text{b/2d}} < 1$ corresponding to less than one half a bit per dimension. For these cases it was found to be an accurate approximation for block lengths of $k \gtrsim 512$. As the block size get smaller, the approximation is observed to be conservative – *i.e.,* over-estimating the required value of E_b/N_0. Although it is not pointed out, the approximation is also also a reasonably accurate predictor of the values presented in Fig. 5 of [8] which considers larger information rates. For example, at $k = 1024$ and a codeword error probability of 10^{-4}, the approximation in (37) predicts that the E_b/N_0 should be at least $8.6 + 1.4 = 10$ dB for $\eta_{\text{b/2d}} = 2 * 2.65 = 5.3$. By comparison, the accurate evaluation of the SPB in Fig. 5 of [8] yields a required E_b/N_0 of 9.8 dB. So, again, the approximation is slightly conservative.

5.1 Application to Modulation Constrained AWGN Channels

We consider applying the finite block size penalty in (36) to modulation constrained AWGN channels. This is done by replacing the reference minimum value of E_b/N_0 predicted by the modulation unconstrained capacity in (35) by the corresponding value predicted by the modulation constrained AWGN channel capacity, or the SIR approximation thereof. More precisely, there is a minimum value of E_s/N_0 for which the SIR is greater than a desired $\eta_{\text{b/sym}}$. If we take this minimum value of E_s/N_0 and consider a system operating at this point, then we have via (6)

$$\left(\frac{E_b}{N_o}\right)_{\text{min,SIR}} = \frac{1}{\text{SIR}}\left(\frac{E_s}{N_o}\right)_{\text{min,SIR}} \tag{38}$$

We then obtain an estimate of the minimum value of E_b/N_0 to achieve a given finite block size operational scenario by adding Δ_{dB} in (36) to the expression in (38) expressed in dB. We refer to the resulting predicted performance trade-off for finite block sizes as the SIR-SPBA (symmetric information rate, SPB approximation). Specifically, we have

$$\left(\frac{E_b}{N_0}\right)_{\text{min,SIR-SPBA, (dB)}} \approx \left(\frac{E_b}{N_0}\right)_{\text{min,SIR, (dB)}} + \Delta_{\text{dB}} \tag{39}$$

where Δ_{dB} is as in (36).

6 Comparisons and Conclusions

The main conclusion of this document is that the SIR-SPBA can be used as a relatively simple gauge for finite block size performance. It is considerably more simple to evaluate than the RCB and for most operational scenarios of practical interest, it is very to close to the RCB.

The following is to be added to this document

- Discussion of the floor in the RCB
- Plots showing the similar E_b/N_0 requirements for the SIR-SPBA and the SRCB
- Discussion of the critical rate and implications for when the RCB and SPB will differ most
- Additional references and reading.

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