Signal Representations, Analysis and Modulations

EE564: Digital Communication and Coding Systems

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Course Topic (from Syllabus)

- Overview of Comm/Coding
- Signal representation and Random Processes
- Optimal demodulation and decoding
- Uncoded modulations, demod, performance
- Classical FEC
- Modern FEC
- Non-AWGN channels (intersymbol interference)
- Practical consideration (PAPR, synchronization, spectral masks, etc.)

“Signaling” Topics

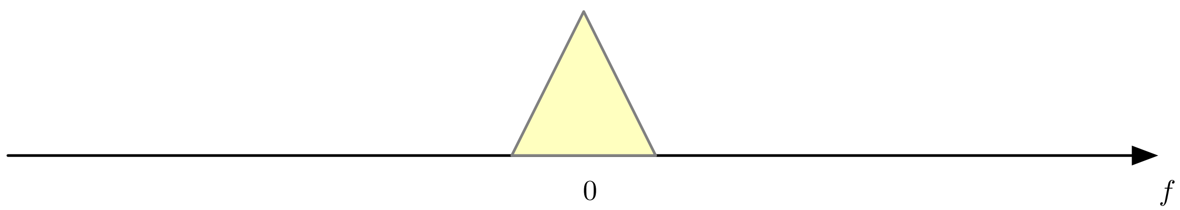
- Complex baseband representation (deterministic)
- Signal space representation and dimensionality (deterministic)
- Common methods of digital modulation
- Summary of some results from 562
- Additive White Gaussian (AWGN) channel
- Complex baseband representation (random processes)
- Power spectral density of common digital modulations

Complex Baseband Representation (Deterministic)

$$x(t) \text{ real} \quad \Longleftrightarrow \quad X(f) \text{ Hermitian Symmetric}$$

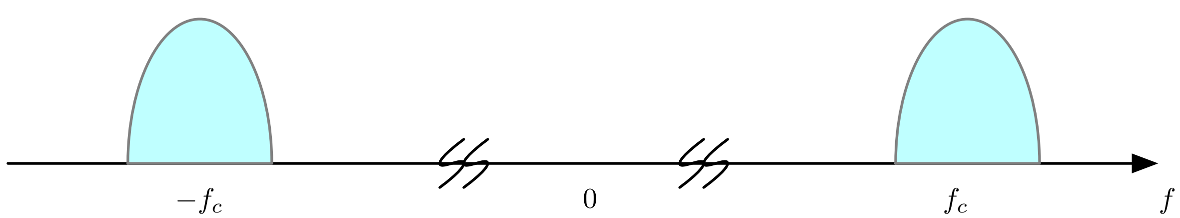
examples of spectra of real-valued time domain signals

baseband signal



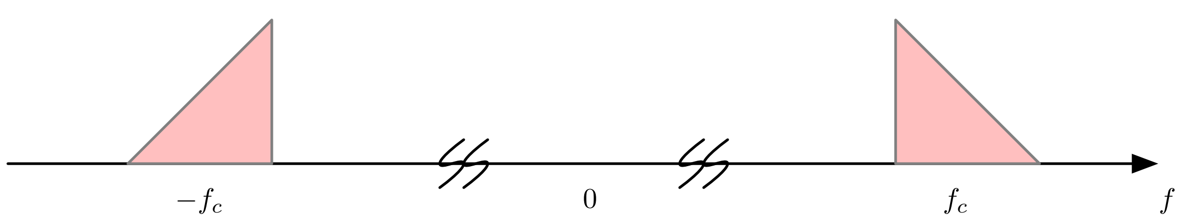
symmetric around 0

passband signal



symmetric around 0, symmetric around f_c

passband signal

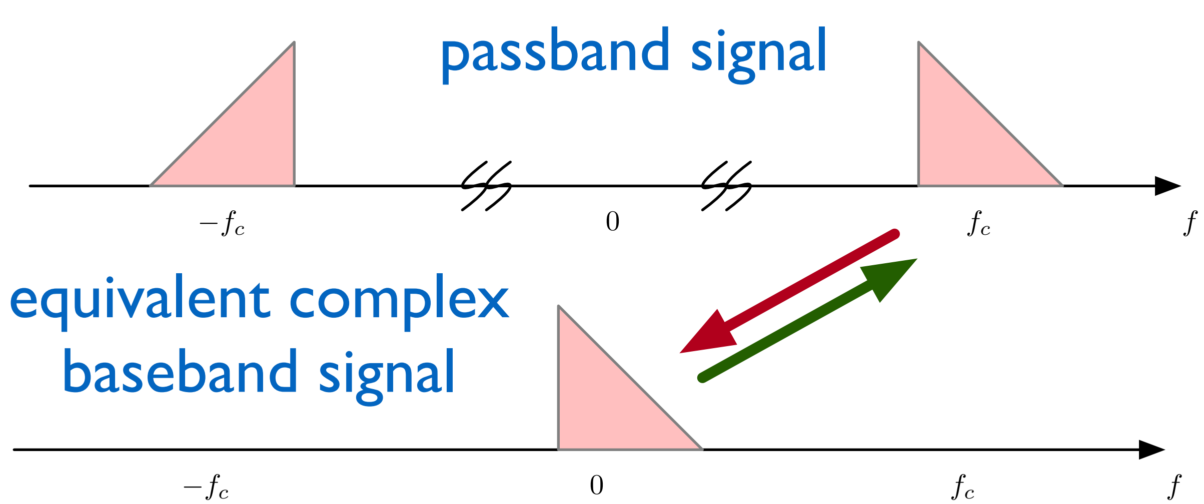


symmetric around 0, **asymmetric** around f_c

Complex Baseband Representation (Deterministic)

It is common and useful to think of an equivalent baseband signal for a given passband signal:

equivalent complex baseband signal



can always get the passband signal back from complex baseband by exploiting Hermitian Symmetry

Why do this?

- Notational shorthand — do not need to keep writing cos(), sin()
- Simulation: don't have to simulate at a sample rate of 2x f_c, just 2x bandwidth fo the signal
- Some signal processors / circuit building blocks support complex arithmetic directly

Complex Baseband Equivalent

passband signal $$s(t) = A(t)\sqrt{2}\cos(2\pi f_c t + \theta(t))$$

$$= [A(t)\cos\theta(t)]\sqrt{2}\cos(2\pi f_c t) - [A(t)\sin\theta(t)]\sqrt{2}\sin(2\pi f_c t)$$

$$= s_I(t)\sqrt{2}\cos(2\pi f_c t) - s_Q(t)\sqrt{2}\sin(2\pi f_c t)$$

$$= \Re\left\{[s_I(t) + j s_Q(t)]\sqrt{2}e^{j2\pi f_c t}\right\}$$

$$= \Re\left\{\bar{s}(t)\sqrt{2}e^{j2\pi f_c t}\right\}$$

equivalent complex baseband signal (aka complex envelope)

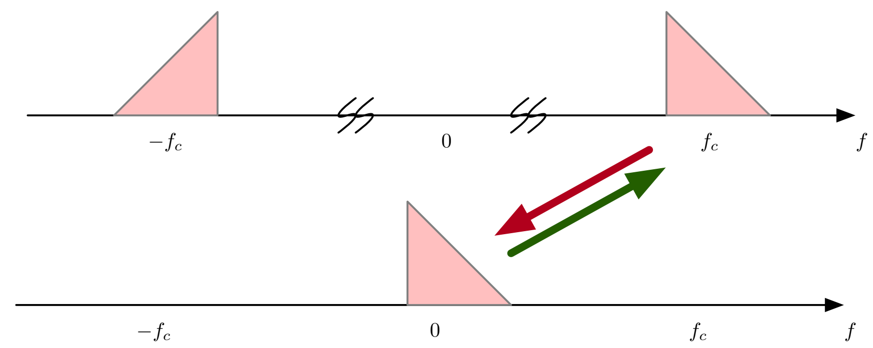
$$\bar{s}(t) = s_I(t) + j s_Q(t) = A(t)e^{j\theta(t)}$$

Narrowband assumption:

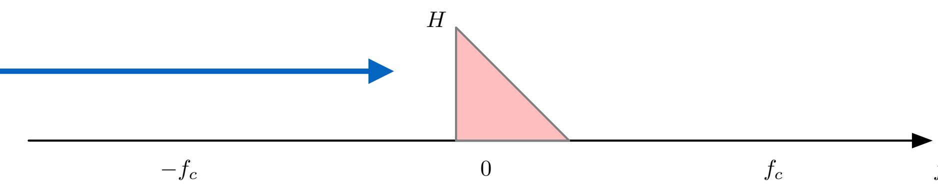
bandwidth of s_i(t) and s_q(t) is much, much smaller than the carrier frequency

Complex Baseband: Spectrum Relationship

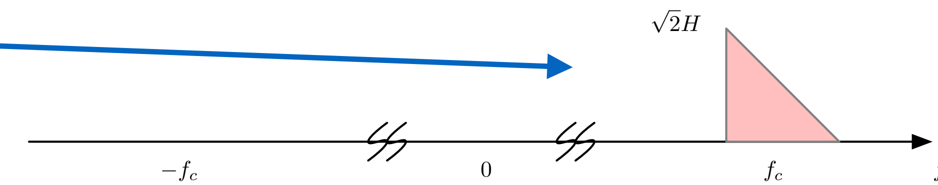
Relation between FT of passband and complex BB signals?



$$\bar{S}(f) = \mathbb{FT}\left\{\bar{s}(t)\right\}$$



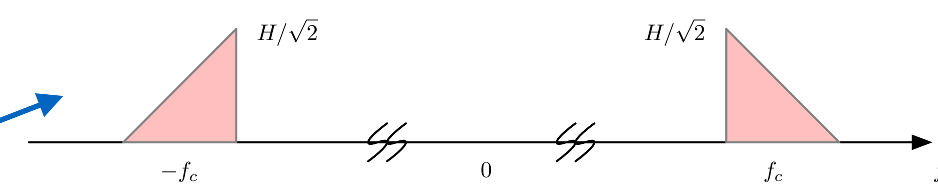
$$\sqrt{2}\bar{S}(f - f_c) = \mathbb{FT}\left\{\bar{s}(t)\sqrt{2}e^{j2\pi f_c t}\right\}$$



$$S(f) = \mathbb{FT}\left\{s(t)\right\} = \mathbb{FT}\left\{\Re\left\{\bar{s}(t)\sqrt{2}e^{j2\pi f_c t}\right\}\right\}$$

$$= \mathbb{HS}\left\{\mathbb{FT}\left\{\bar{s}(t)\sqrt{2}e^{j2\pi f_c t}\right\}\right\}$$

$$= \frac{1}{\sqrt{2}}\left[\bar{S}(f - f_c) + \bar{S}^*(-f - f_c)\right]$$



$$\boxed{S(f) = \frac{1}{\sqrt{2}}\left[\bar{S}(f - f_c) + \bar{S}^*(-f - f_c)\right]}$$

Complex Baseband: Spectrum Relationship

There are other conventions for complex BB — most vary by factors of 2 or sqrt(2)

$$s(t) = \Re\left\{\bar{s}(t)\boxed{\sqrt{2}}e^{j2\pi f_c t}\right\}$$

$\sqrt{2}\cos(2\pi f_c t)$ and $\sqrt{2}\sin(2\pi f_c t)$ are unit power sinusoids

The convention adopted is "inner product preserving"

$$\boxed{\int_I x(t)y(t)dt = \Re\left\{\int_I \bar{x}(t)\bar{y}^*(t)dt\right\}}$$

Signal energy is the same whether computed on the passband signal or the equivalent complex BB signal

$$\boxed{E_s = \int_I s^2(t)dt = \Re\left\{\int_I |\bar{s}(t)|^2 dt\right\}}$$

Complex Baseband: Spectrum Relationship

The convention adopted is "inner product preserving"

$$\boxed{\int_I x(t)y(t)dt = \Re\left\{\int_I \bar{x}(t)\bar{y}^*(t)dt\right\}}$$

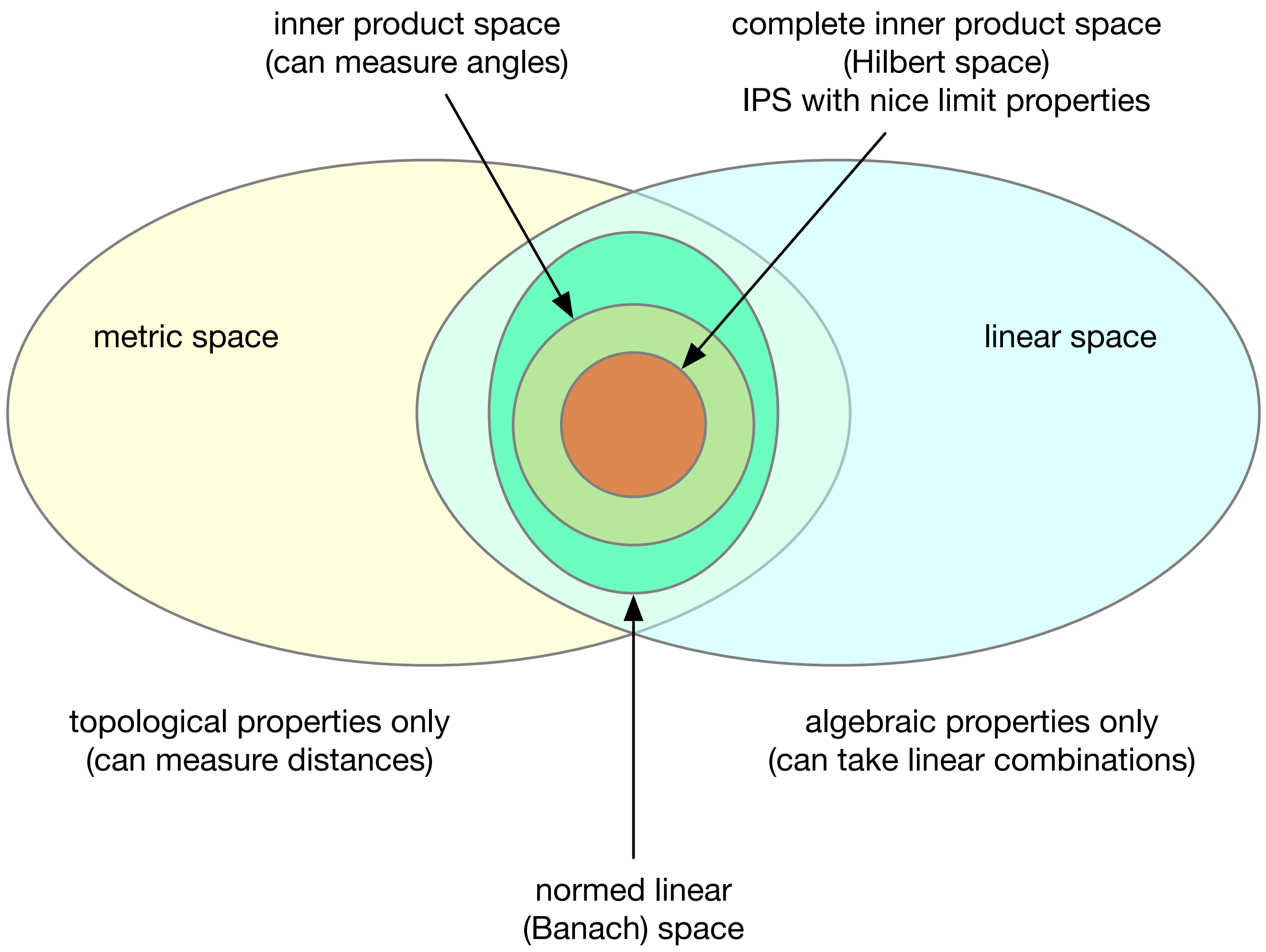
$$\Re\{z\}\Re\{w\} = \frac{\Re\{zw * + zw\}}{2}$$

$$\int_I x(t)y(t)dt = \int_I \Re\left\{\bar{x}(t)\sqrt{2}e^{j2\pi f_c t}\right\}\Re\left\{\bar{y}(t)\sqrt{2}e^{j2\pi f_c t}\right\}dt$$

$$= \Re\left\{\int_I \bar{x}(t)\bar{y}^*(t)dt\right\} + \underbrace{\Re\left\{\int_I \bar{x}(t)\bar{y}(t)e^{j2\pi(2f_c)t}dt\right\}}_{\approx 0 \text{ by narrowband assumption}}$$

Signal Space Representation

see the “spaces” handout



- signals as vectors in an abstract space — generalization of vectors
- linear independence, dimension, and basis
- Grahm-Schmidt for orthonormal basis and finding dimension
- Finding expansion in terms of ortho-normal basis and generalized Fourier series

PSK Modulations

passband signals

$$s_m(t) = \sqrt{E_s} p(t) \sqrt{2} \cos\left(2\pi f_c t + \frac{2\pi}{M} m\right) \qquad m = 0, 1, \ldots M-1$$

orthonormal basis

$$\phi_1(t) = p(t)\sqrt{2}\cos(2\pi f_c t)$$

$$\phi_2(t) = -p(t)\sqrt{2}\sin(2\pi f_c t)$$

signal vectors

$$\mathbf{s}_m = \sqrt{E_s} \begin{pmatrix} \cos\left(\frac{2\pi}{M} m\right) \\ \sin\left(\frac{2\pi}{M} m\right) \end{pmatrix} \qquad m = 0, 1, \ldots M-1$$

Note: dimension is D=2 for M>2, and D=1 for M=2

$$\int p^2(t) dt = 1$$

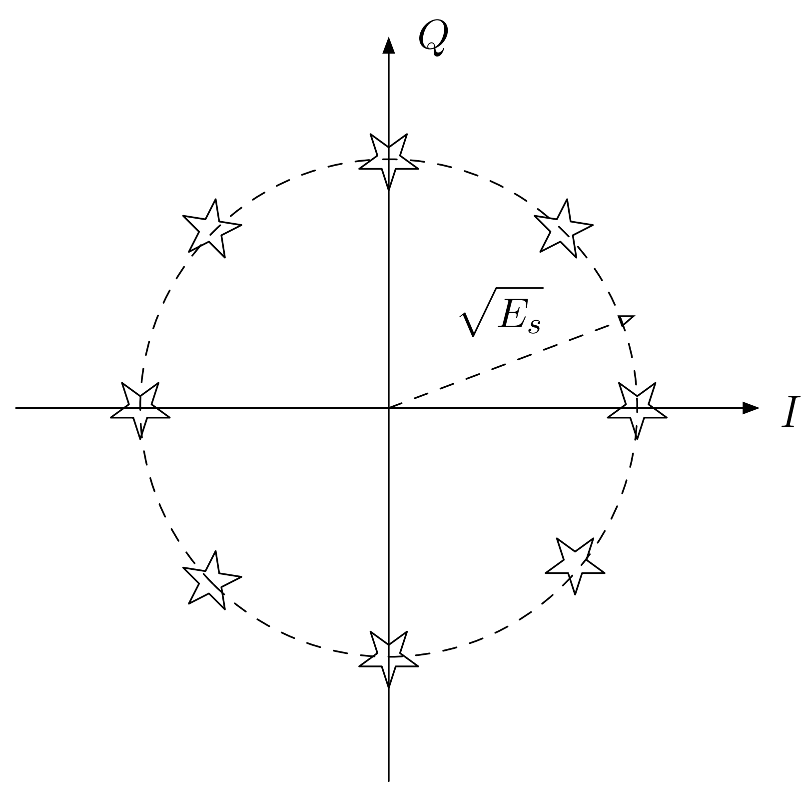
PSK Modulations

complex BB $$\bar{s}_m(t) = \sqrt{E_s} p(t) e^{j\frac{2\pi}{M}m} \qquad m = 0, 1, \ldots M-1$$

orthonormal basis $$\bar{\phi}_1(t) = p(t)$$

signal vectors $$\bar{s}_m = \sqrt{E_s} e^{j\frac{2\pi}{M}m} \qquad m = 0, 1, \ldots M-1$$

Note: complex dimension is D=1 and this is real if M=2



8PSK

Pulse Amplitude Modulation (PAM)

passband signals

$$s_m(t) = d\left(\frac{2m+1-M}{2}\right)p(t)\sqrt{2}\cos\left(2\pi f_c t\right) \qquad m = 0, 1, \ldots M-1$$

orthonormal basis

$$\phi_1(t) = p(t)\sqrt{2}\cos(2\pi f_c t)$$

signal vectors

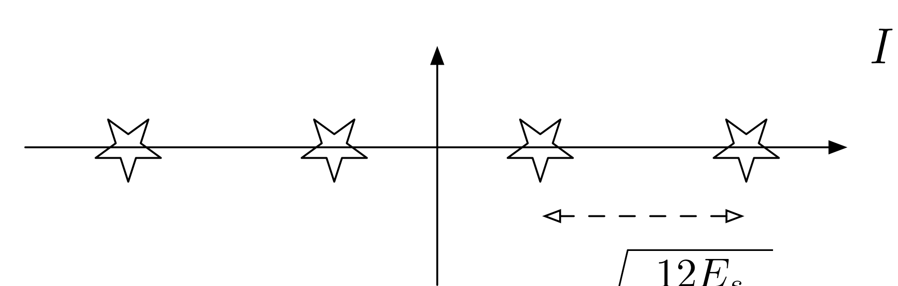
$$s_m = d\left(\frac{2m+1-M}{2}\right) \qquad m = 0, 1, \ldots M-1$$

$$\sum_{m=0}^{M-1}(2m+1-M)^2 = \frac{M(M^2-1)}{3}$$

$$E_s = \frac{d^2(M^2-1)}{12}$$

$$d^2 = \frac{12E_s}{M^2-1}$$

4PAM



$$d = \sqrt{\frac{12E_s}{M^2-1}}$$

Note: dimension is D=1, (BPSK is special case of M=2)

$$\int p^2(t)dt = 1$$

Pulse Amplitude Modulation (PAM)

complex BB
$$\bar{s}_m(t) = d\left(\frac{2m+1-M}{2}\right)p(t) \qquad m = 0, 1, \ldots M-1$$

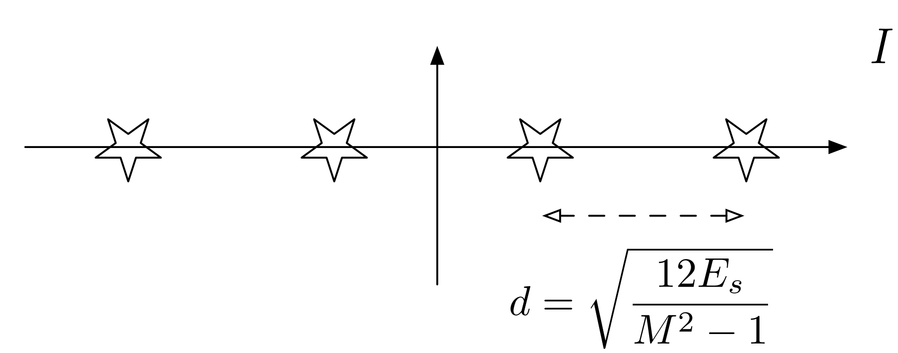
orthonormal basis
$$\phi_1(t) = p(t)$$

signal vectors
$$\bar{s}_m = d\left(\frac{2m+1-M}{2}\right) \qquad m = 0, 1, \ldots M-1$$

$$E_s = \frac{d^2(M^2-1)}{12}$$

$$d^2 = \frac{12E_s}{M^2-1}$$

4PAM



Note: complex dimension is D=1 (real valued)

$$\int p^2(t)dt = 1$$

Quadrature Amplitude Modulation (QAM)

passband signals

$$s_m(t) = d\left(\frac{2m_I + 1 - M}{2}\right) p(t)\sqrt{2}\cos\left(2\pi f_c t\right)$$

$$- d\left(\frac{2m_Q + 1 - M}{2}\right) p(t)\sqrt{2}\sin\left(2\pi f_c t\right) \qquad m = 0, 1, \ldots M - 1$$

orthonormal basis

$$\phi_1(t) = p(t)\sqrt{2}\cos(2\pi f_c t)$$

$$\phi_2(t) = -p(t)\sqrt{2}\sin(2\pi f_c t)$$

signal vectors

$$\mathbf{s}_m = \frac{d}{2}\begin{pmatrix} (2m_I + 1 - M_p) \\ (2m_Q + 1 - M_p) \end{pmatrix} \qquad m = 0, 1, \ldots M - 1$$

$$M = M_p^2 \in \{4, 16, 64, 256 \ldots\}$$

Note: dimension D=2

$$\int p^2(t)dt = 1$$

Quadrature Amplitude Modulation (QAM)

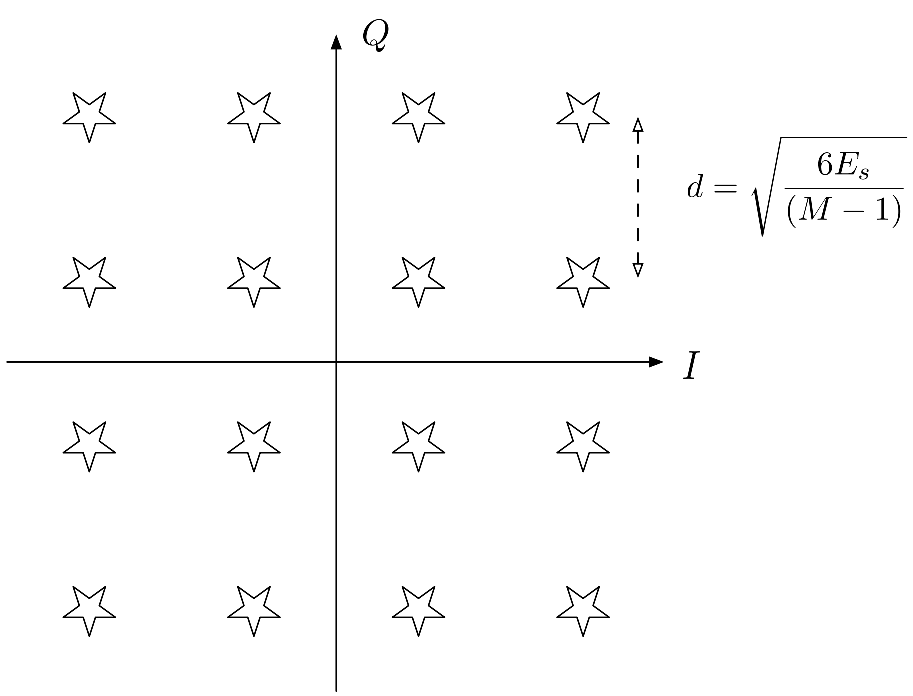
complex BB $$\bar{s}_m(t) = \frac{d}{2}\left[(2m_I + 1 - M_p) + j(2m_Q + 1 - M_p)\right] p(t) \qquad m = 0, 1, \ldots M-1$$

orthonormal basis $$\phi_1(t) = p(t)$$

signal vectors $$\bar{s}_m = \frac{d}{2}\left[(2m_I + 1 - M_p) + j(2m_Q + 1 - M_p)\right] \qquad m = 0, 1, \ldots M-1$$

$$M = M_p^2 \in \{4, 16, 64, 256\ldots\}$$

16QAM



Note: dimension D=2

$$\int p^2(t)dt = 1$$

Quadrature Amplitude Modulation (QAM)

$$E_s = \mathbb{E}\left\{\|\mathbf{x}(u)\|^2\right\}$$

$$= \mathbb{E}\left\{\left\| \begin{matrix} x_I(u) \\ x_Q(u) \end{matrix} \right\|^2\right\}$$

$$= \mathbb{E}\left\{x_I^2(u)\right\} + \mathbb{E}\left\{x_Q(u)^2\right\}$$

$$= 2\mathbb{E}\left\{x_I^2(u)\right\}$$

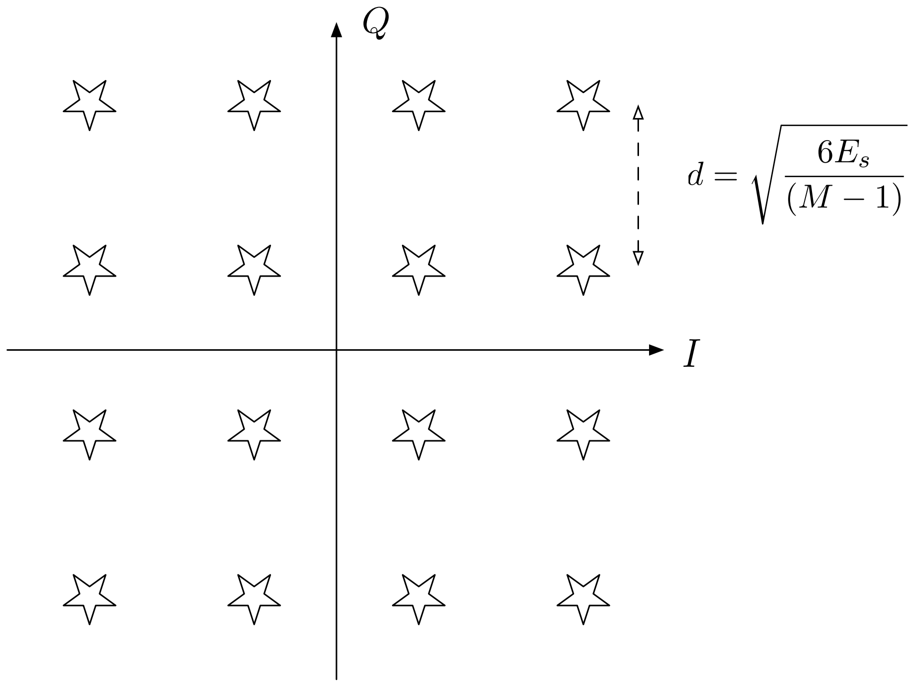
$$= \frac{2d^2(M_p^2 - 1)}{12}$$

$$= \frac{d^2(M-1)}{6}$$

$$d^2 = \frac{6E_s}{(M-1)}$$

QAM has PAM in I and PAM in Q and energy is the sum of the energies of these

16QAM



General QASK

- Two dimensional real vector model
- One dimensional complex model
- same basis signals as PSK, QAM
- Any constellation points

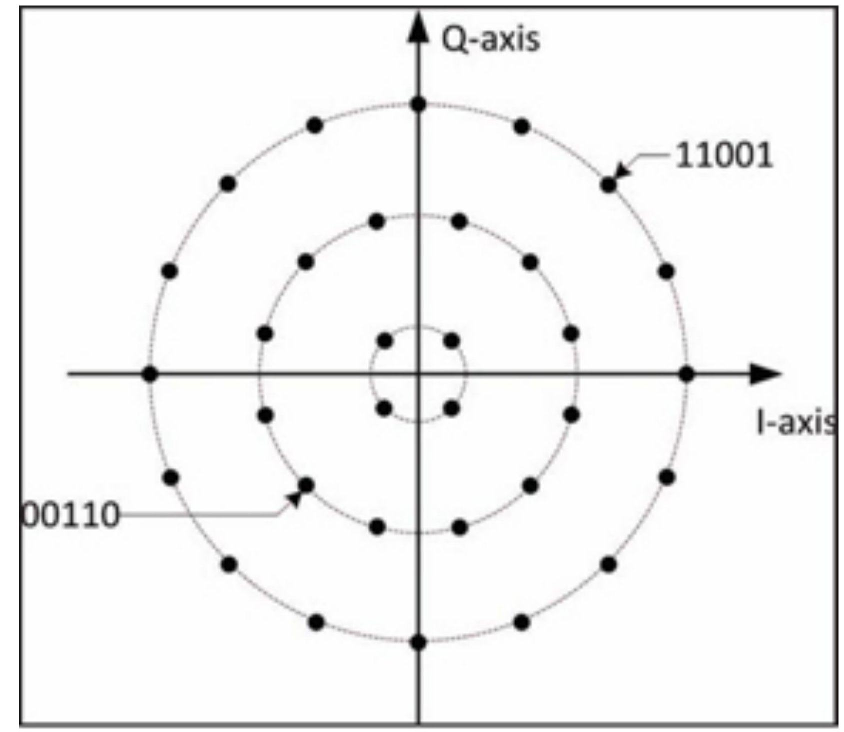


Figure 4. 32-symbol APSK Modulation Scheme Used for Satellite Video

32 QASK
(aka 32 QAM)

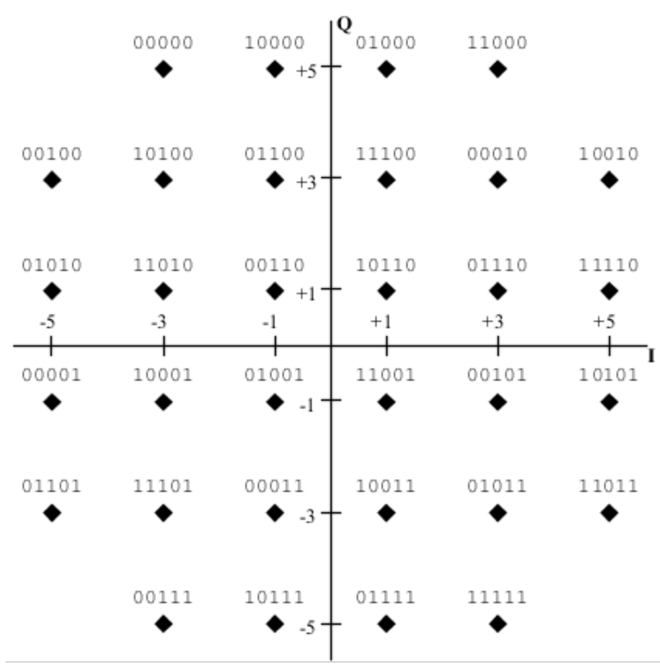
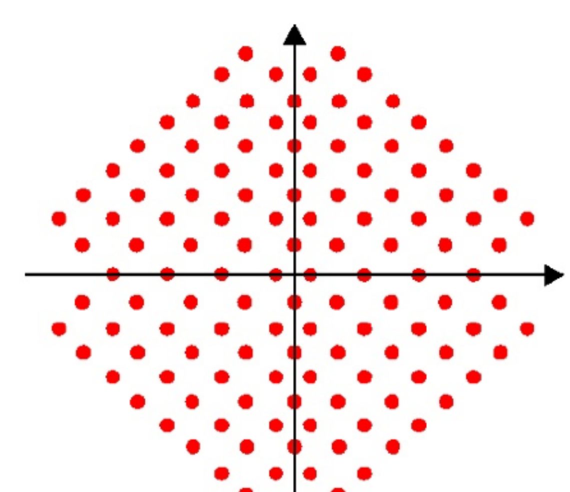


Figure 6-29

V.33 Constellation



WCB/McGraw-Hill

Orthogonal Modulations

- All signals are orthogonal
  - Phase coherent demodulation:

$$\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle = \Re\left\{ \langle \bar{\boldsymbol{s}}_m, \bar{\boldsymbol{s}}_n \rangle \right\} = E_s \delta[m-n]$$

$$\phi_i(t) = \frac{s_i(t)}{\sqrt{E_s}}$$

$$\mathbf{s}_m^{\mathrm{t}} = \sqrt{E_s} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 \dots & 0 \end{pmatrix}$$

Note: dimension D=M

Orthogonal Modulations

- All signals are orthogonal
  - Phase noncoherent demodulation:

$$\langle \bar{\boldsymbol{s}}_m, \bar{\boldsymbol{s}}_n \rangle = E_s \delta[m-n]$$

- Orthogonal in the noncoherent sense implies orthogonal in the coherent sense
  - Meaning: signals are orthogonal even under an arbitrary phase rotation

Frequency Shift Keying (FSK)

passband signals

$$s_m(t) = \sqrt{E_s} p(t) \sqrt{2} \cos\left(2\pi \left[f_c + f_m\right] t\right) \qquad m = 0, 1, \ldots M-1$$

complex BB

$$\bar{s}_m(t) = \sqrt{E_s} p(t) \exp\left(j 2\pi f_m t\right)$$

$$f_m = \frac{2m + 1 - M}{2} \Delta$$

$$\Delta = \text{tone separation}$$

orthonormal basis: use Gram-Schmidt

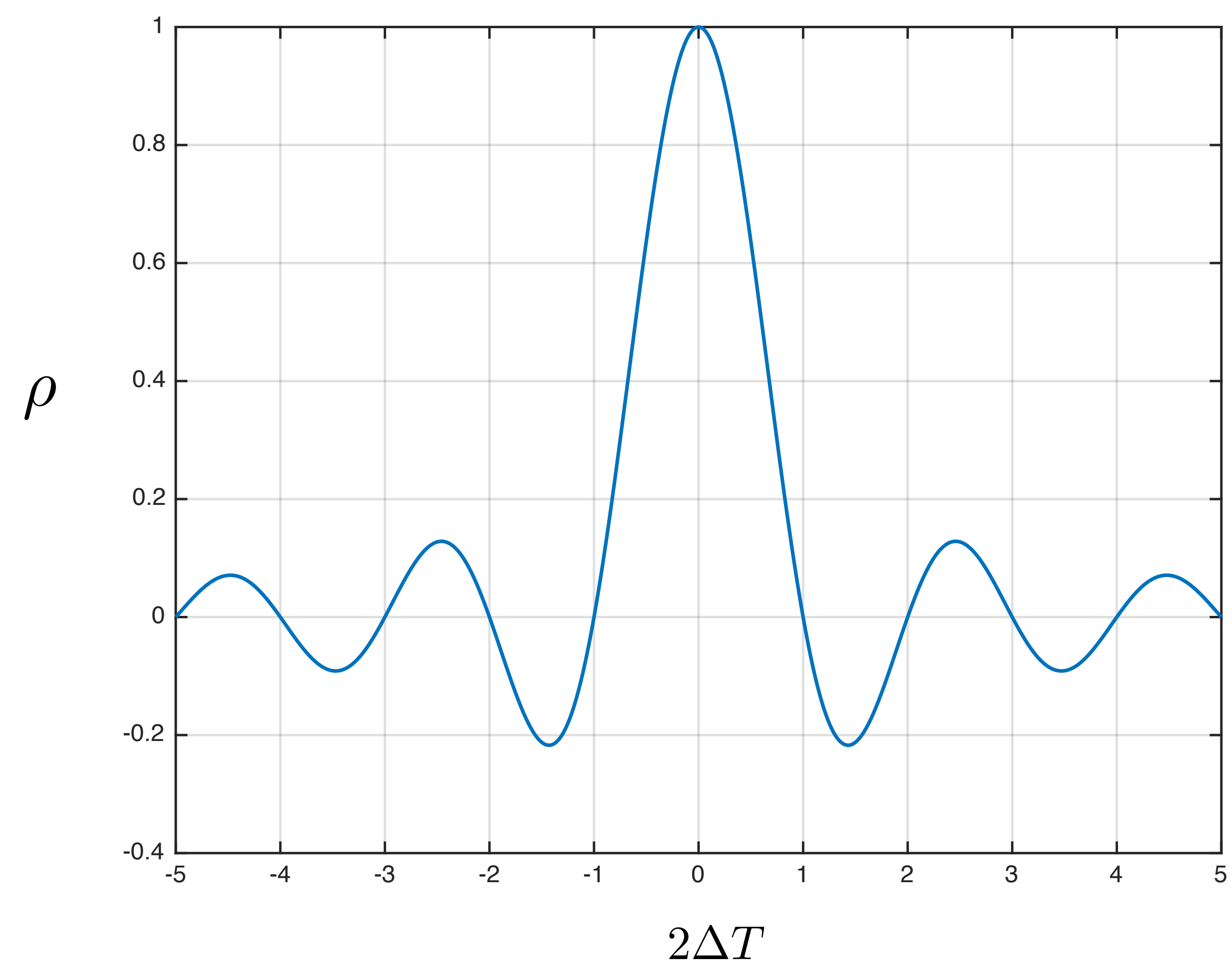
$$\int p^2(t) dt = 1$$

Orthogonal Frequency Shift Keying (FSK)

$$\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle = \frac{E_s}{T} \int_0^T \cos\left(2\pi \left[f_c + \frac{2m+1-M}{2}\Delta\right] t\right) \cos\left(2\pi \left[f_c + \frac{2n+1-M}{2}\Delta\right] t\right) dt$$

$$\langle \bar{\boldsymbol{s}}_m, \bar{\boldsymbol{s}}_n \rangle = \frac{E_s}{T} \int_0^T \exp\left(j2\pi \left[\frac{2m+1-M}{2}\Delta\right] t\right) \exp\left(-j2\pi \left[\frac{2n+1-M}{2}\Delta\right] t\right) dt$$

$$\begin{aligned} \langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle &= \Re\left\{\langle \bar{\boldsymbol{s}}_m, \bar{\boldsymbol{s}}_n \rangle\right\} \\ &= E_s \mathrm{sinc}((m-n)2\Delta T) \end{aligned}$$



$$p(t) = \begin{cases} 1/\sqrt{T} & t \in [0, T] \\ 0 & \text{else} \end{cases}$$

Orthogonal Frequency Shift Keying (FSK)

$$p(t) = \begin{cases} 1/\sqrt{T} & t \in [0, T] \\ 0 & \text{else} \end{cases}$$

- With rectangular pulse shape
  - Minimum tone spacing for orthogonal signals

$$\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle = \Re\left\{ \langle \bar{\boldsymbol{s}}_m, \bar{\boldsymbol{s}}_n \rangle \right\} = 0 \quad \Longrightarrow \quad \Delta_{\min} = \frac{1}{2T}$$

- With rectangular pulse shape
  - Minimum tone spacing for orthogonal signals (phase noncoherent)

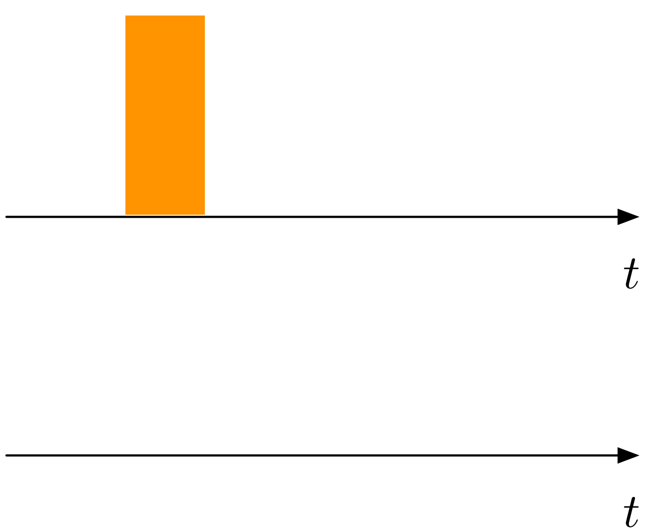
$$\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle = 0 \qquad \Longleftrightarrow \Delta_{\min} = \frac{1}{T}$$

$$\Longleftrightarrow \cos\left(2\pi f_c t + \frac{2\pi}{M} m t\right) \perp \cos\left(2\pi f_c t + \frac{2\pi}{M} n t + \phi\right) \quad \forall\ \phi,\ m \neq n$$

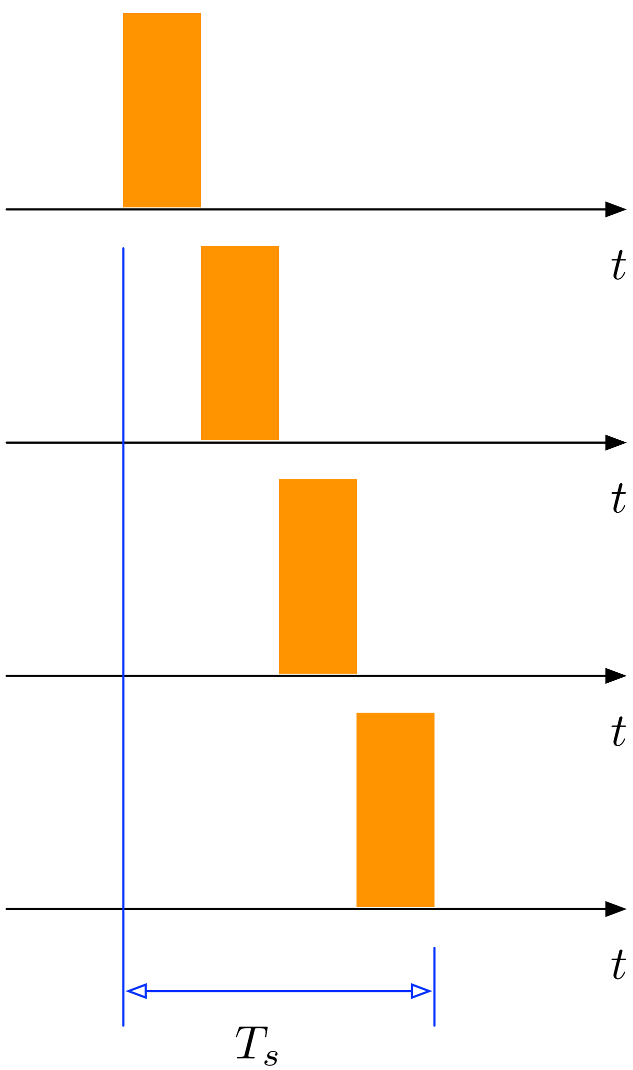
Orthogonal Modulations

- Many other methods for orthogonal modulation

On-off Keying (OOK, M=2)



Orthogonal Pulse Position Modulation (PPM), M=4



Orthogonal Modulations

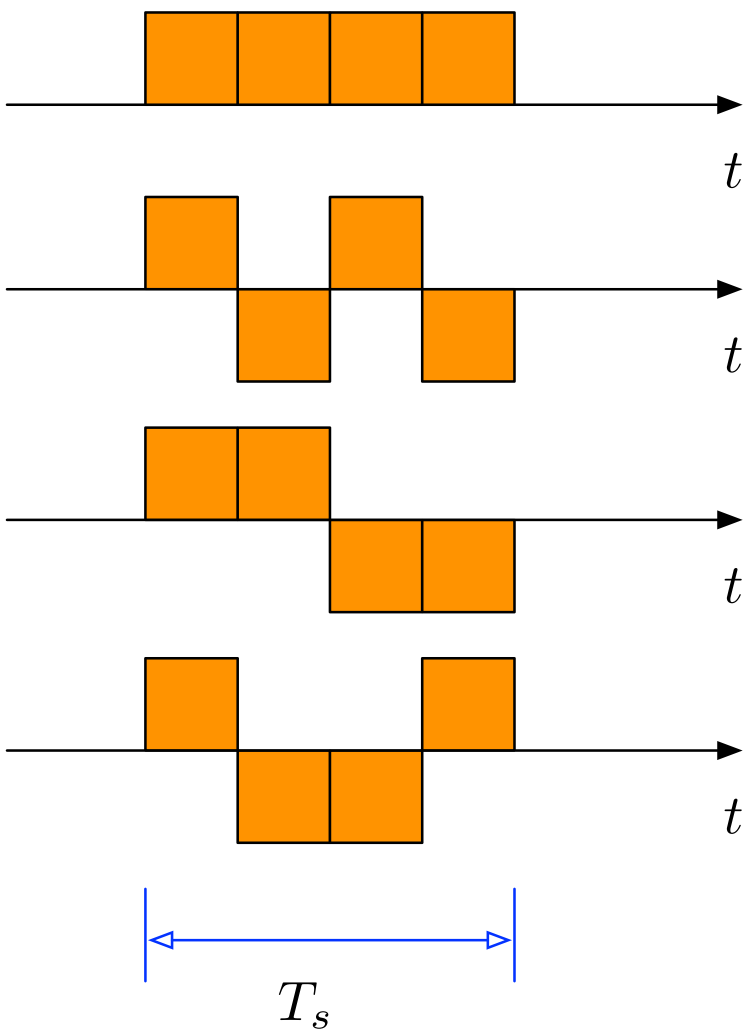
- Walsh-Hadamard sequences for orthogonal pulses

$$H_0 = +$$

$$H_2 = \begin{bmatrix} +H_0 & +H_0 \\ +H_0 & -H_0 \end{bmatrix} = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$$

$$H_4 = \begin{bmatrix} +H_2 & +H_2 \\ +H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$H_{2i} = \begin{bmatrix} +H_i & +H_i \\ +H_i & -H_i \end{bmatrix}$$



Orthogonal Modulations

- Note that in contrast to QASK modulations, the dimension of the signal set grows with M for orthogonal modulations

MQASK: $$\frac{\log_2(M)}{2} \text{ bits/dimension}$$

M Orthogonal: $$\frac{\log_2(M)}{M} \text{ bits/dimension}$$

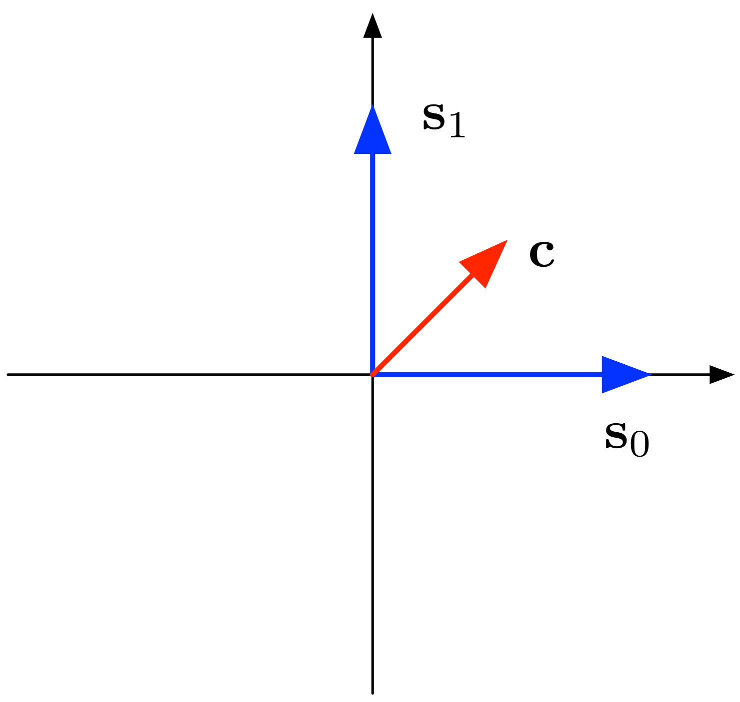
- Rate (bits/dimension) for orthogonal goes to 0 as M increases
  - Increasing M
    - Worse performance (SNR loss) for QASK
    - Better performance (SNR gain) for orthogonal

Orthogonal-like Modulations

- The signal set centroid

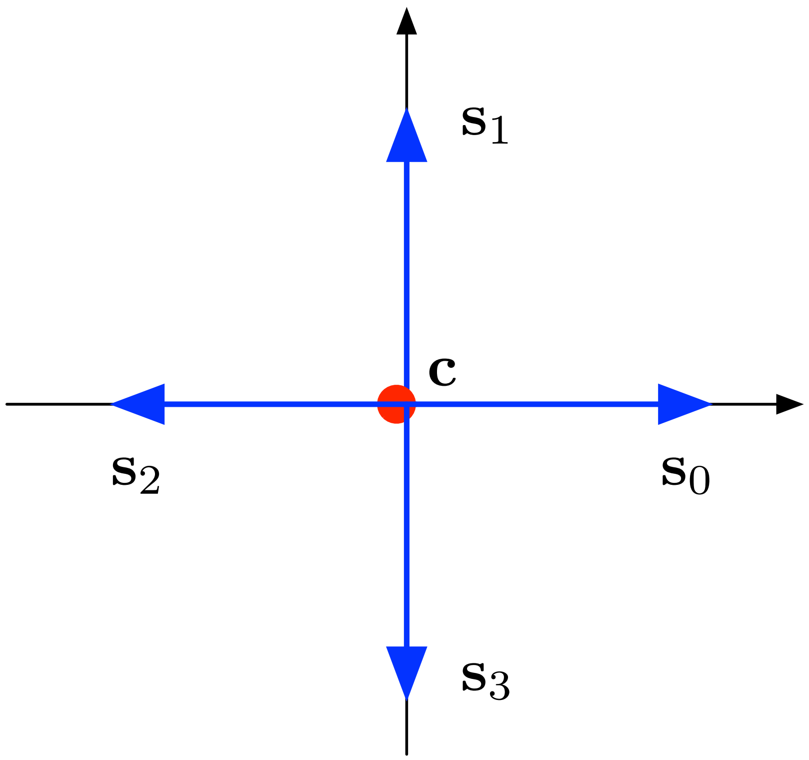
$$\mathbf{c} = \frac{1}{M} \sum_{m=0}^{M-1} \mathbf{s}_m$$

- A non-zero centroid can convey no information, it is just sending the information signal + some constant offset
  - It is desired to have a zero centroid
  - Orthogonal signal sets have nonzero centroids



Bi-Orthogonal Modulation

- Start with an M-ary orthogonal signal set
  - Add in -s for every signal in the original signal set
  - Results in 2M signals in M dimensions, each energy E

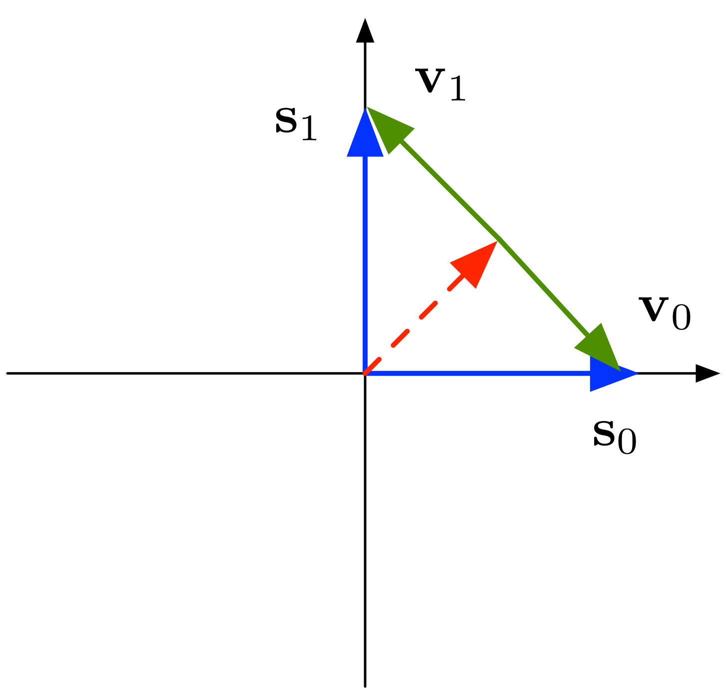


Simplex Modulation

- Start with an M-ary orthogonal signal set
  - subtract the centroid from all signals
  - Results in M signals in M-1 dimensions
    - Energy reduction because centroid is removed

$$\mathbf{v}_m = \mathbf{s_m} - \frac{1}{M}\sum_{m=0}^{M-1}\mathbf{s}_m$$

$$E_v = \frac{M-1}{M}E_s$$



Other Modulations

- Continuous Phase Modulation (CPM)
  - Used for saturated power amplified channels
  - Waveform with memory
    - Optimal demod is Viterbi algorithm or Forward-Backward algorithm
- Orthogonal Frequency Division Multiplexing
  - Using many parallel frequency channels and put a QASK signal train on each
    - FFT/IFFT based processing
    - For channels with frequency selective gain

“Signaling” Topics

- Complex baseband representation (deterministic)
- Signal space representation and dimensionality (deterministic)
- Common methods of digital modulation
- Summary of some results from 562
- Additive White Gaussian (AWGN) channel
- Complex baseband representation (random processes)
- Power spectral density of common digital modulations

Key Ideas from EE562

- Random process is a generalization of one random variable or two random variables to a random vector, random sequence, or random waveform
- Complete statistical descriptions vs Second moment descriptions
- Stationarity and Wide-Sense Stationarity
- Gaussian processes and linear processing
- Power Spectral Density (PSD) — frequency domain view
- Linearity of the expectation operator

$$\mathbb{E}\left\{\mathsf{L}(\boldsymbol{x}(u))\right\} = \mathsf{L}(\mathbb{E}\left\{\boldsymbol{x}(u)\right\})$$

(real) Random Vectors

EE503 review

random vector

$$\mathbf{x}(u) = \begin{bmatrix} x(u,1) \\ x(u,2) \\ \vdots \\ x(u,n) \end{bmatrix} \qquad (n \times 1)$$

Complete statistical description

$$f_{\mathbf{x}(u)}(\mathbf{x}) = f_{x(u,1),x(u,2),\cdots x(u,n)}(x_1, x_2, \cdots x_n) \qquad \text{(pdf or cdf or pmf)}$$

Second Moment Description

$$\mathbf{m}_{\mathbf{x}} = \mathbb{E}\left\{\mathbf{x}(u)\right\}$$

mean vector

$$\mathbf{R}_{\mathbf{x}} = \mathbb{E}\left\{\mathbf{x}(u)\mathbf{x}^{\mathrm{t}}(u)\right\}$$

correlation matrix

$$[\mathbf{R}_{\mathbf{x}}]_{i,j} = \mathbb{E}\left\{x_i(u)x_j(u)\right\}$$

$$\mathbf{K}_{\mathbf{x}} = \mathbb{E}\left\{(\mathbf{x}(u) - \mathbf{m}_{\mathbf{x}})(\mathbf{x}(u) - \mathbf{m}_{\mathbf{x}})^{\mathrm{t}})\right\}$$

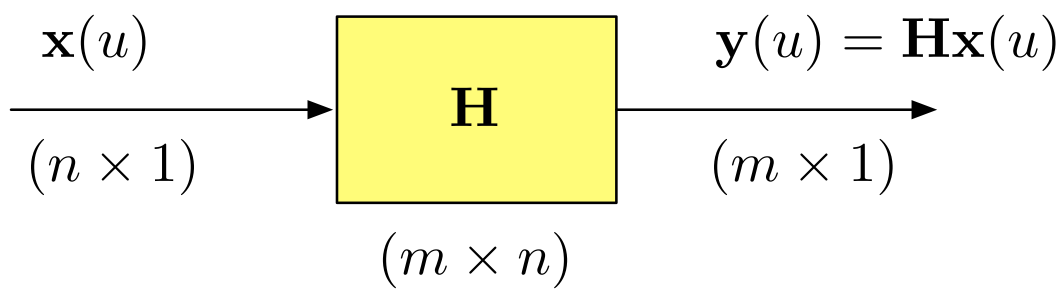
covariance matrix

$$= \mathbf{R}_{\mathbf{x}} - \mathbf{m}_{\mathbf{x}}\mathbf{m}_{\mathbf{x}}{}^{\mathrm{t}}$$

$$[\mathbf{K}_{\mathbf{x}}]_{i,j} = \operatorname{cov}\left[x_i(u), x_j(u)\right]$$

(real) Random Vectors

EE503 review



$$\mathbf{m}_{\mathbf{y}} = \mathbf{H}\mathbf{m}_{\mathbf{x}}$$

$$\mathbf{R}_{\mathbf{y}} = \mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathrm{t}}$$

$$\mathbf{K}_{\mathbf{y}} = \mathbf{H}\mathbf{K}_{\mathbf{x}}\mathbf{H}^{\mathrm{t}}$$

Special case

$$y(u) = \mathbf{b}^{\mathrm{t}}\mathbf{x}(u) \qquad (1 \times 1)$$

$$m_y = \mathbf{b}^{\mathrm{t}}\mathbf{m}_{\mathbf{x}}$$

$$\mathbb{E}\left\{y^2(u)\right\} = \mathbf{b}^{\mathrm{t}}\mathbf{R}_{\mathbf{x}}\mathbf{b}$$

$$\sigma_y^2 = \mathbf{b}^{\mathrm{t}}\mathbf{K}_{\mathbf{x}}\mathbf{b}$$

(real) Random Waveforms

random waveform

$$x(u,t) \quad t \in (-\infty, \infty)$$

Complete statistical description

$$\mathbf{v}(u;\mathbf{t}_n) = \begin{bmatrix} x(u,t_1) & x(u,t_2) & \cdots & x(u,t_n) \end{bmatrix}^{\mathrm{t}}$$

$$\mathbf{t}_n = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}^{\mathrm{t}}$$

$$f_{\mathbf{v}(u;\mathbf{t}_n)}(\mathbf{v}) = f_{x(u,t_1),x(u,t_2),\cdots x(u,t_n)}(v_1, v_2, \cdots v_n) \qquad \text{(pdf or cdf or pmf)}$$

$$\forall\ n \in \mathbb{Z}, \quad \forall\ \mathbf{t}_n$$

Second Moment Description

$$m_x(t) = \mathbb{E}\left\{x(u,t)\right\}$$ *mean function*

$$R_x(t_1,t_2) = \mathbb{E}\left\{x(u,t_1)x(u,t_2)\right\}$$ *correlation function*

$$K_x(t_1,t_2) = \mathbb{E}\left\{[x(u,t_1) - m_x(t_1)][x(u,t_2) - m_x(t_2)]\right\}$$ *covariance function*

$$= R_x(t_1,t_2) - m_x(t_1)m_x(t_2)$$

Stationary Random Waveforms

Complete statistical description does not change with shifts

$$\mathbf{v}(u;\mathbf{t}_n) = \begin{bmatrix} x(u,t_1) \\ x(u,t_2) \\ \vdots \\ x(u,t_n) \end{bmatrix} \qquad \mathbf{v}(u;\mathbf{t}_n + \tau\mathbf{1}) = \begin{bmatrix} x(u,t_1+\tau) \\ x(u,t_2+\tau) \\ \vdots \\ x(u,t_n+\tau) \end{bmatrix}$$

$$f_{\mathbf{v}(u;\mathbf{t}_n)}(\mathbf{v}) = f_{\mathbf{v}(u;\mathbf{t}_n+\tau\mathbf{1})}(\mathbf{v}) \quad \forall\, n \in \mathbb{Z}, \quad \forall\, \mathbf{t}_n, \quad \forall\, \tau \in \mathbb{R}$$

(sometimes called strictly stationary)

Wide Sense Stationary (WSS) Random Waveforms

Second moment description does not change with shifts

$$m_x(t) = m_x(t+\tau)$$

$$R_x(t_1, t_2) = R_x(t_1+\tau, t_2+\tau)$$

$$\Downarrow$$

$$m_x(t) = m_x$$

$$R_x(t_1, t_2) = R_x(t_1 - t_2)$$

$$R_x(t+\tau, t) = R_x(\tau)$$

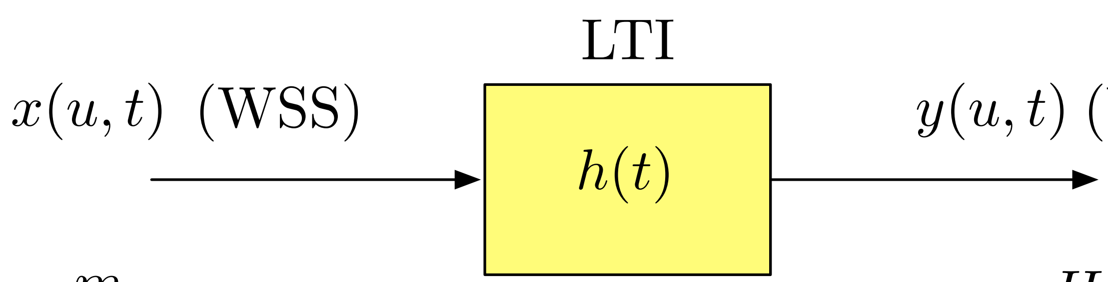
$$m_x = \mathbb{E}\left\{x(u,t)\right\}$$ *mean*

$$R_x(\tau) = \mathbb{E}\left\{x(u,t+\tau)x(u,t)\right\}$$ *correlation function*

$$K_x(\tau) = \mathbb{E}\left\{(x(u,t+\tau) - m_x)(x(u,t) - m_x)\right\}$$

$$= R_x(\tau) - m_x^2$$ *covariance function*

WSS (real) Random Processes and LTI Systems



m_x

$R_x(\tau)$

$$m_y = m_x H(0) = m_x \int_{-\infty}^{\infty} h(t)dt$$

$$R_y(\tau) = h(\tau) * R_x(\tau) * h(-\tau)$$

Output is also WSS and second moments only function input second moments

Complex Random Vectors

$$\mathbf{z}(u) = \mathbf{x}(u) + j\mathbf{y}(u)$$

$$\mathbf{m_z} = \mathbb{E}\left\{\mathbf{z}(u)\right\} = \mathbf{m_x} + j\mathbf{m_y}$$

mean vector

$$\mathbf{K_z} = \mathbb{E}\left\{(\mathbf{z}(u) - \mathbf{m_z})(\mathbf{z}(u) - \mathbf{m_z})^{\dagger}\right\}$$

covariance matrix

$$= \mathbf{K_x} + \mathbf{K_y} + j\left(\mathbf{K_{yx}} - \mathbf{K_{xy}}\right)$$

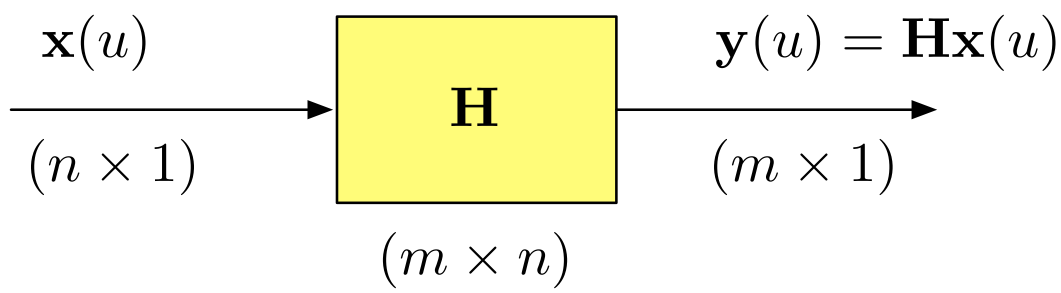
$$\widetilde{\mathbf{K}}_{\mathbf{z}} = \mathbb{E}\left\{(\mathbf{z}(u) - \mathbf{m_z})(\mathbf{z}(u) - \mathbf{m_z})^{\mathrm{t}}\right\}$$

``pseudo-covariance" matrix

$$= \mathbf{K_x} - \mathbf{K_y} + j\left(\mathbf{K_{yx}} + \mathbf{K_{xy}}\right)$$

Circular complex: $\widetilde{\mathbf{K}}_{\mathbf{z}} = \mathbf{O}$

Complex Random Vectors



$$\mathbf{m}_{\mathbf{y}} = \mathbf{H}\mathbf{m}_{\mathbf{x}}$$

$$\mathbf{K}_{\mathbf{y}} = \mathbf{H}\mathbf{K}_{\mathbf{x}}\mathbf{H}^{\dagger}$$

$$\widetilde{\mathbf{K}}_{\mathbf{y}} = \mathbf{H}\widetilde{\mathbf{K}}_{\mathbf{x}}\mathbf{H}^{\mathrm{t}}$$

Special case

$$y(u) = \mathbf{b}^{\dagger}\mathbf{x}(u) \qquad (1 \times 1)$$

$$m_y = \mathbf{b}^{\dagger}\mathbf{m}_{\mathbf{x}} \qquad (1 \times 1)$$

$$\sigma_y^2 = \mathbb{E}\left\{|y(u) - m_y|^2\right\}$$

$$= \mathbf{b}^{\dagger}\mathbf{K}_{\mathbf{x}}\mathbf{b}$$

$$\mathbb{E}\left\{[y(u) - m_y]^2\right\} = \mathbf{b}^{\mathrm{t}}\widetilde{\mathbf{K}}_{\mathbf{x}}\mathbf{b}$$

(circular in implies circular out)

Complex Random Waveforms

Complete statistical description = joint complete description of real and imaginary parts

Second Moment Description

$$m_z(t) = \mathbb{E}\left\{z(u,t)\right\}$$

mean function

$$R_z(t_1,t_2) = \mathbb{E}\left\{z(u,t_1)z^*(u,t_2)\right\}$$

correlation function

$$K_z(t_1,t_2) = \mathbb{E}\left\{[z(u,t_1) - m_z(t_1)][z(u,t_2) - m_z(t_2)]^*\right\}$$

$$= R_z(t_1,t_2) - m_z(t_1)m_z^*(t_2)$$

covariance function

$$\widetilde{R}_z(t_1,t_2) = \mathbb{E}\left\{z(u,t_1)z(u,t_2)\right\}$$

pseudo-correlation function

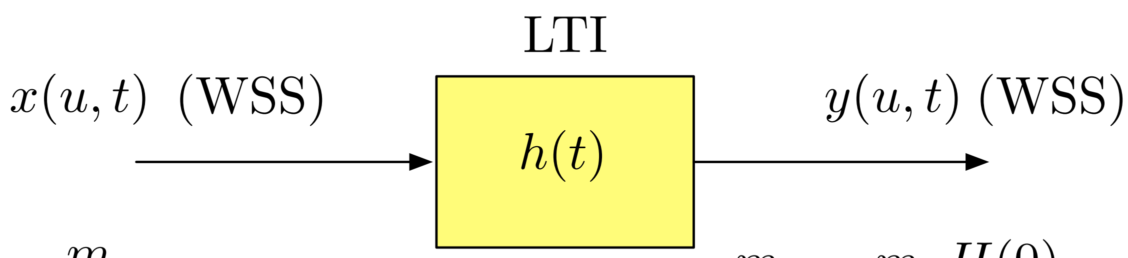
$$\widetilde{K}_z(t_1,t_2) = \mathbb{E}\left\{[z(u,t_1) - z_x(t_1)][z(u,t_2) - m_z(t_2)]\right\}$$

$$= \widetilde{R}_z(t_1,t_2) - m_z(t_1)m_z(t_2)$$

pseudo-covariance function

circular $\widetilde{K}_z(t_1,t_2) = 0$

WSS Random Processes and LTI Systems



m_x

$R_x(\tau)$

$\widetilde{R}_x(\tau)$

$$m_y = m_x H(0) = m_x \int_{-\infty}^{\infty} h(t)dt$$

$$R_y(\tau) = h(\tau) * R_x(\tau) * h^*(-\tau)$$

$$\widetilde{R}_y(\tau) = h(\tau) * \widetilde{R}_x(\tau) * h(-\tau)$$

Output is also WSS and second moments only
function input second moments

(circular in implies circular out)

Gaussian Random Processes

Gaussian Random Vector (real)

$$f_{\mathbf{x}(u)}(\mathbf{x}) = \mathcal{N}_n\left(\mathbf{x}; \mathbf{m}_\mathbf{x}; \mathbf{K}_\mathbf{x}\right) = \frac{1}{\sqrt{(2\pi)^n|\mathbf{K}_\mathbf{x}|}} \exp\left(\frac{-(\mathbf{x}-\mathbf{m}_\mathbf{x})^{\mathrm{t}}\mathbf{K}_\mathbf{x}^{-1}(\mathbf{x}-\mathbf{m}_\mathbf{x})}{2}\right)$$

$$\mathcal{N}_n\left(\mathbf{x}; \mathbf{m}_\mathbf{x}; \frac{N_0}{2}\mathbf{I}\right) = \frac{1}{\sqrt{(\pi N_0)^n}} \exp\left(\frac{-\|\mathbf{x}-\mathbf{m}_\mathbf{x}\|^2}{N_0}\right)$$

- Gaussian Random Process is one in which all finite random vectors drawn from the process are Gaussian
  - Stationarity iff Wide-Sense Stationarity
- Any linear processing of a Gaussian process yields a Gaussian process (dot products, convolution, matrix multiplication, etc.)
- Uncorrelated implies independent

Complex Circular Random Process

Circular Complex Gaussian Random Vector

$$f_{\mathbf{z}(u)}(\mathbf{z}) = f_{\mathbf{x}(u),\mathbf{y}(u)}(\mathbf{x},\mathbf{y}) = \frac{1}{\pi^n |\mathbf{K}_\mathbf{z}|} \exp\left(-(\mathbf{z}-\mathbf{m}_\mathbf{z})^\dagger \mathbf{K}_\mathbf{z}^{-1}(\mathbf{z}-\mathbf{m}_\mathbf{z})\right) \triangleq \mathcal{N}_n^{cc}(\mathbf{z};\mathbf{m}_\mathbf{z};\mathbf{K}_\mathbf{z})$$

$$\mathcal{N}_n^{cc}(\mathbf{x};\mathbf{m}_\mathbf{z};N_0\mathbf{I}) = \frac{1}{(\pi N_0)^n} \exp\left(\frac{-\|\mathbf{z}-\mathbf{m}_\mathbf{z}\|^2}{N_0}\right)$$

- CCG Random Process is one in which all finite random vectors drawn from the process are CCG
- Stationarity iff Wide-Sense Stationarity
- Any linear processing of a CCG process yields a CCG process (dot products, convolution, matrix multiplication, etc.)
- Uncorrelated implies independent

Power Spectral Density

General Case
$$S_x(f) = \lim_{T\to\infty} \mathbb{E}\left\{ \frac{1}{2T} \left| \int_{-T}^{+T} x(u,t) e^{-2\pi f t} dt \right|^2 \right\}$$

WSS Case
$$S_x(f) = \mathbb{FT}\{R_x(\tau)\} = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f \tau} d\tau$$

$x(u,t)$ (WSS) $\longrightarrow$ LTI $\boxed{h(t)}$ $\longrightarrow$ $y(u,t)$ (WSS)

m_x

$R_x(\tau)$

$$m_y = m_x H(0) = m_x \int_{-\infty}^{\infty} h(t) dt$$

$$R_y(\tau) = h(\tau) * R_x(\tau) * h^*(-\tau)$$

$S_x(f)$

$$S_y(f) = |H(f)|^2 S_x(f)$$

$$\text{Power in } x(u,t) \text{ in } f \in B = \int_B S_x(f) df$$

$$\text{Total power in } x(u,t) = \int_{-\infty}^{\infty} S_x(f) df = R_x(0) = \mathbb{E}\left\{ |x(u,t)|^2 \right\}$$

“Signaling” Topics

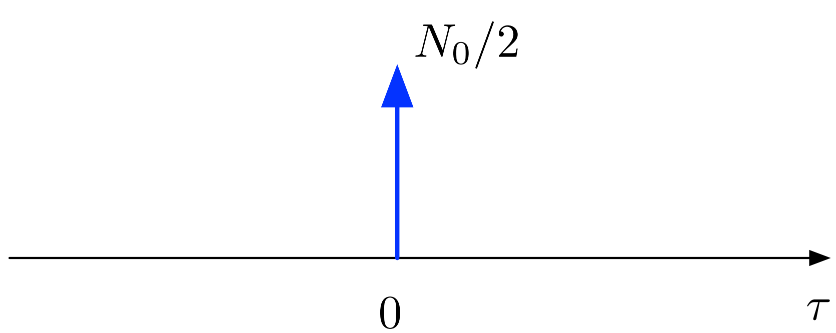
- Complex baseband representation (deterministic)
- Signal space representation and dimensionality (deterministic)
- Common methods of digital modulation
- Summary of some results from 562
- Additive White Gaussian (AWGN) channel
- Complex baseband representation (random processes)
- Power spectral density of common digital modulations

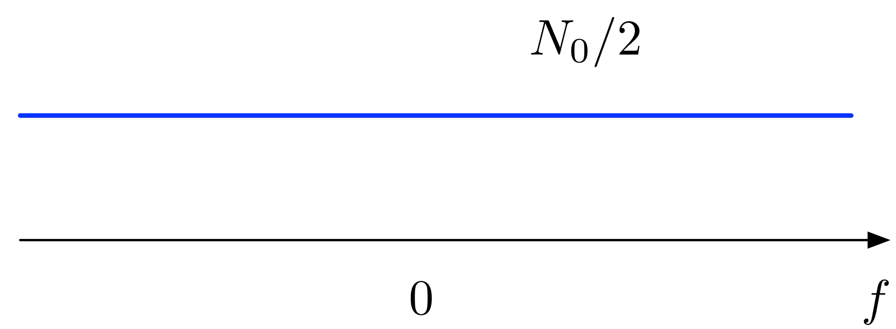
Additive White Gaussian Noise (AWGN)

(real) random process — idealized model (infinite power)

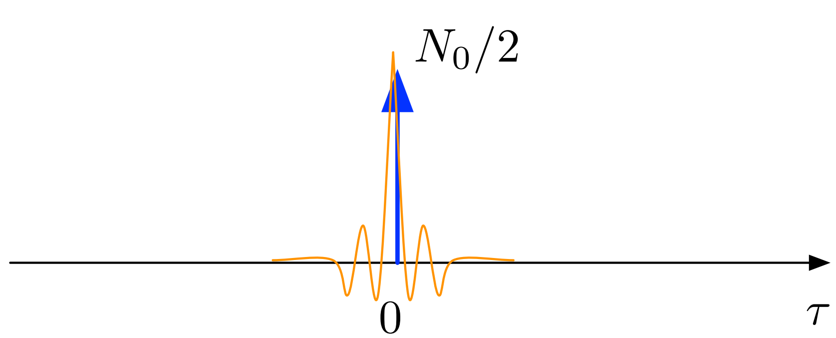
$$R_n(\tau) = \mathbb{E}\left\{n(u,t+\tau)n(u,t)\right\} = \frac{N_0}{2}\delta(\tau)$$

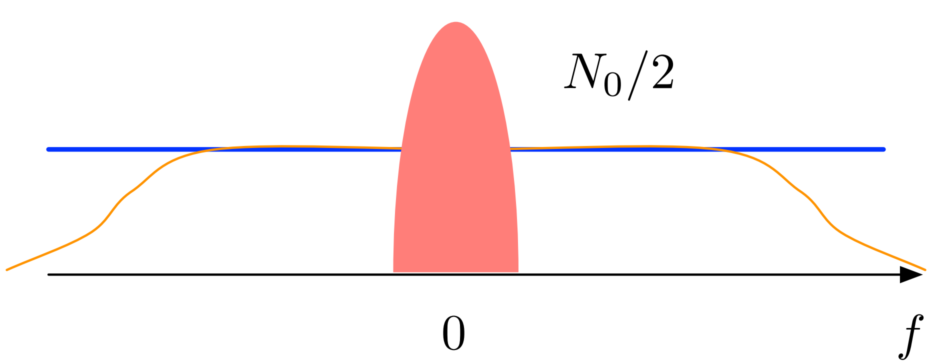
$$S_n(f) = \frac{N_0}{2}$$





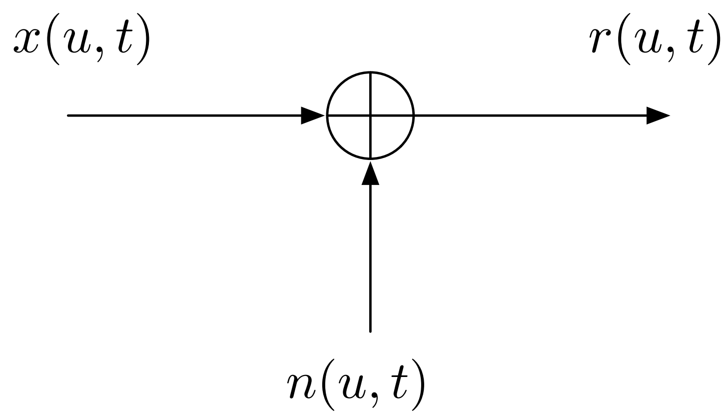
relation to real (finite power) noise



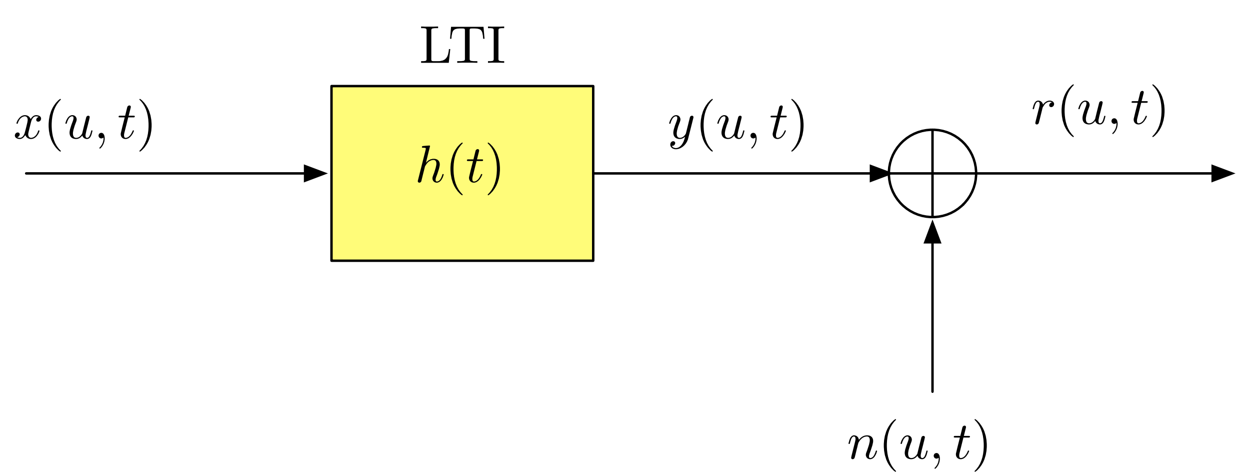


- Simplifies calculations when true PSD is flat over the signal bandwidth

AWGN Channel & ISI-AWGN Channel



AWGN



AWGN - Intersymbol Interference (ISI) Channel

“Signaling” Topics

- Complex baseband representation (deterministic)
- Signal space representation and dimensionality (deterministic)
- Common methods of digital modulation
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- Complex baseband representation (random processes)
- Power spectral density of common digital modulations

Narrowband Random Process

Narrowband process

$$x(u,t) = \Re\left\{\bar{x}(u,t)\sqrt{2}e^{j2\pi f t}\right\}$$

$$= x_I(u,t)\sqrt{2}\cos(2\pi f t) - x_Q(u,t)\sqrt{2}\sin(2\pi f t)$$

Complex BB Equivalent

$$\bar{x}(u,t) = x_I(u,t) + jx_Q(u,t)$$

- Example: random data on I and Q channels of a QASK modulation stream

Complex Baseband Random Process

$$x(u,t) \text{ is WSS } \iff \bar{x}(u,t) \text{ is circular and WSS}$$

$$\iff \widetilde{R}_{\bar{x}}(t_1,t_2) = 0,\ R_{\bar{x}}(t_1,t_2) = R_{\bar{x}}(t_1 - t_2)$$

$$\iff R_{x_I}(\tau) = R_{x_Q}(\tau) \text{ AND } R_{x_I x_Q}(\tau) = R_{x_I x_Q}(-\tau)$$

- If this x(u,t) is WSS:

$$R_x(\tau) = \Re\left\{R_{\bar{x}}(\tau)e^{j2\pi f\tau}\right\}$$

$$\boxed{S_x(f) = \frac{1}{2}S_{\bar{x}}(f - f_c) + \frac{1}{2}S_{\bar{x}}^*(-f - f_c)}$$

Complex BB Equivalent of AWGN

$$\bar{n}(u,t) = n_I(u,t) + jn_Q(u,t)$$

$$R_{n_I}(\tau) = R_{n_I}(\tau)$$

$$= \frac{N_0}{2}\delta(\tau)$$

$$R_{n_i n_Q}(\tau) = 0$$

$$S_{n_I}(f) = S_{n_I}(f)$$

$$= \frac{N_0}{2}$$

$$R_{\bar{n}}(\tau) = N_0\delta(\tau)$$

$$S_{\bar{n}}(f) = N_0$$

The in-phase and quadrature components are real AWGN processes (independent)

This can be viewed as the limit of a narrowband "white" Gaussian noise model as the bandwidth goes to infinity

“Signaling” Topics

- Complex baseband representation (deterministic)
- Signal space representation and dimensionality (deterministic)
- Common methods of digital modulation
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- Complex baseband representation (random processes)
- Power spectral density of common digital modulations

QASK PSD

$$\overline{x}(u,t) = \sum_k \overline{X}_k(u) p(t - kT - \alpha(u))$$

$$\overline{X}_k(u) \sim \text{iid and uniform over (zero-centroid) QASK constellation}$$

$$\alpha(u) \sim \text{uniform on } [0, T] \text{ (to make WSS)}$$

$$\alpha(u), \{X_k(u)\} \sim \text{independent}$$

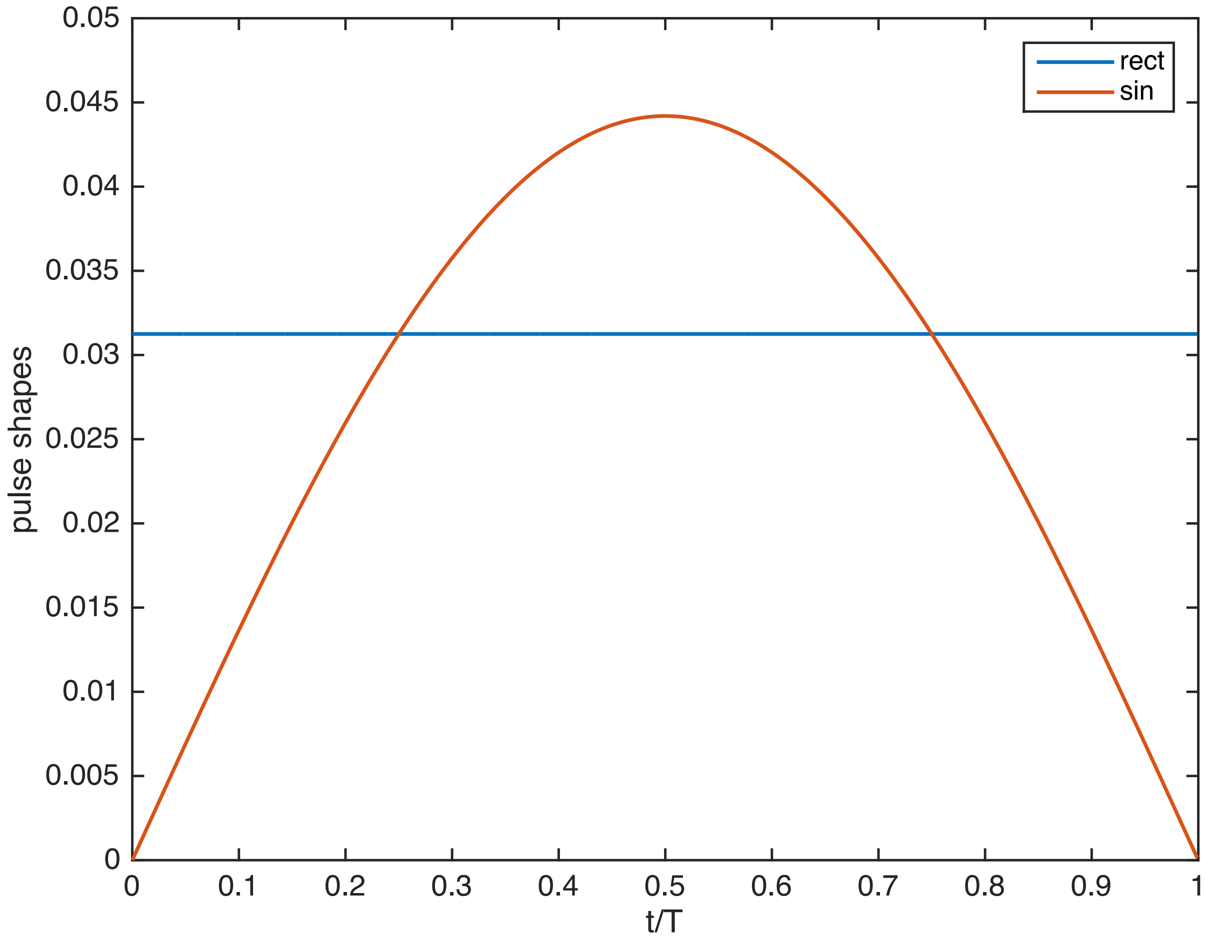
QASK PSD

$$
\begin{aligned}
R_x(\tau) &= \mathbb{E}\left\{x(u,t+\tau)x^*(u,t)\right\} \\
&= \mathbb{E}\left\{\left(\sum_k \overline{X}_k(u)p(t+\tau-kT-\alpha(u))\right)\left(\sum_m \overline{X}_m(u)p(t-mT-\alpha(u))\right)^*\right\} \\
&= \sum_k\sum_m \mathbb{E}\left\{\overline{X}_k(u)\overline{X}_m^*(u)p(t+\tau-kT-\alpha(u))p^*(t-mT-\alpha(u))\right\} \\
&= \sum_k\sum_m \mathbb{E}\left\{\overline{X}_k(u)\overline{X}_m^*(u)\right\}\mathbb{E}\left\{p(t+\tau-kT-\alpha(u))p^*(t-mT-\alpha(u))\right\} \\
&= \sum_k\sum_m \mathbb{E}\left\{|\overline{X}_k(u)|^2\right\}\delta[k-m]\mathbb{E}\left\{p(t+\tau-kT-\alpha(u))p^*(t-mT-\alpha(u))\right\} \\
&= \sigma_{\bar{X}}^2\sum_k \mathbb{E}\left\{p(t+\tau-kT-\alpha(u))p^*(t-kT-\alpha(u))\right\} \\
&= \sigma_{\bar{X}}^2\sum_k \frac{1}{T}\int_0^T p(t+\tau-kT-\alpha)p^*(t-kT-\alpha)d\alpha \\
&= \sigma_{\bar{X}}^2\sum_k \frac{1}{T}\int_{kT-t}^{(k+1)T-t} p(\tau-\lambda)p^*(-\lambda)d\lambda \\
&= \frac{\sigma_{\bar{X}}^2}{T}\int_{-\infty}^{\infty} p(\tau-\lambda)p^*(-\lambda)d\lambda \\
&= \frac{\sigma_{\bar{X}}^2}{T}p(\tau)*p^*(-\tau)
\end{aligned}
$$

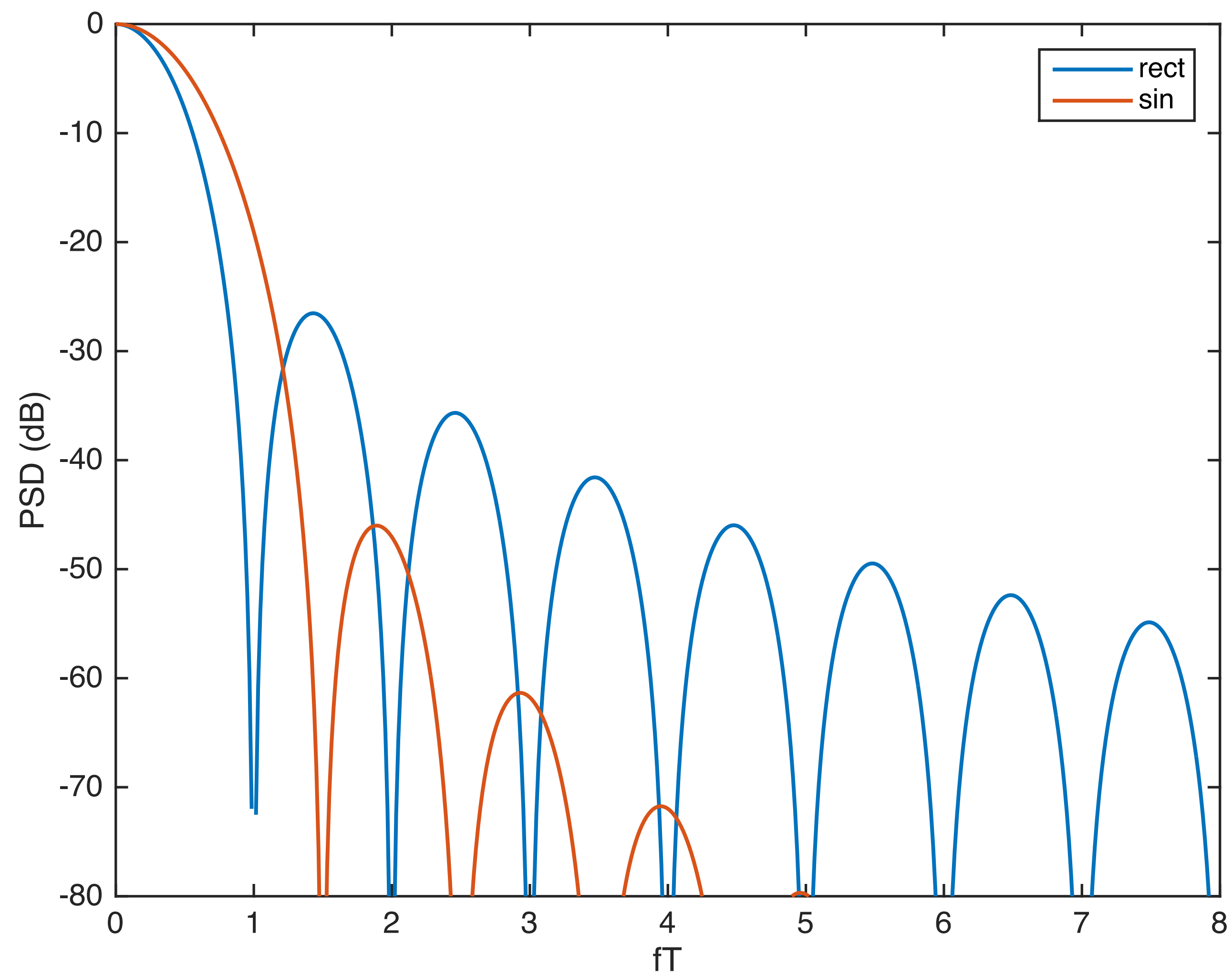
$$\lambda = -(t-kT-\alpha)$$

$$\boxed{S_{\bar{x}}(f) = \frac{\mathbb{E}\left\{|\overline{X}_k(u)|^2\right\}}{T}|P(f)|^2}$$

QASK PSD



rect
sin
pulse shapes
t/T
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



rect
sin
PSD (dB)
fT
0
-10
-20
-30
-40
-50
-60
-70
-80
1
2
3
4
5
6
7
8

Memoryless (Non-linear) Modulations

$$\overline{x}(u,t) = \sum_k \bar{s}_{X_k(u)}(t - kT)$$

$$X_k(u) \in \{0, 1, \ldots M-1\} \quad \text{(uncorrelated)}$$

$$\bar{s}_m(t)(\text{lasts} \leq T \text{ seconds})$$

$$\bar{c}(t) = \frac{1}{M} \sum_{m=0}^{M-1} \bar{s}_m(t)$$

$$\bar{s}_m^c(t) = \bar{s}_m(t) - \bar{c}(t)$$

spectral lines due to periodic, deterministic component due to non-zero centroid

$$S_{\bar{x}}(f) = \frac{1}{MT} \sum_{m=0}^{M-1} |S_m^c(f)|^2 + |C(f)|^2 \sum_k \delta(f - k/T)$$

$$S_m^c(f) = \mathbb{FT}\{\bar{s}_m^c(t)\}$$

$$C(f) = \mathbb{FT}\{c(t)\}$$

PSD of Orthogonal FSK

$$s_m(t) = \sqrt{E_s} p(t) \sqrt{2} \cos\left(2\pi \left[f_c + f_m\right] t\right) \qquad m = 0, 1, \ldots M-1$$

$$\bar{s}_m(t) = \sqrt{E_s} p(t) \exp\left(j 2\pi f_m t\right)$$

$$f_m = \frac{2m + 1 - M}{2} \Delta$$

$$\Delta = \text{tone separation}$$

$$\frac{1}{\sqrt{T}} \bar{S}_m(f) = \sqrt{E} \text{sinc}(T(f - f_m)) \left[e^{-j2\pi(f - f_m)(T/2)} \right]$$

$$\frac{1}{T} |S_m^c(f)|^2 = E \left| \text{sinc}(T(f - f_m)) - \frac{1}{M} \sum_{i=0}^{M-1} \text{sinc}(T(f - f_i)) \right|^2$$

PSD of Orthogonal FSK

$$s_m(t) = \sqrt{E_s} p(t) \sqrt{2} \cos\left(2\pi \left[f_c + \right] t\right) \qquad m = 0, 1, \ldots M-1$$

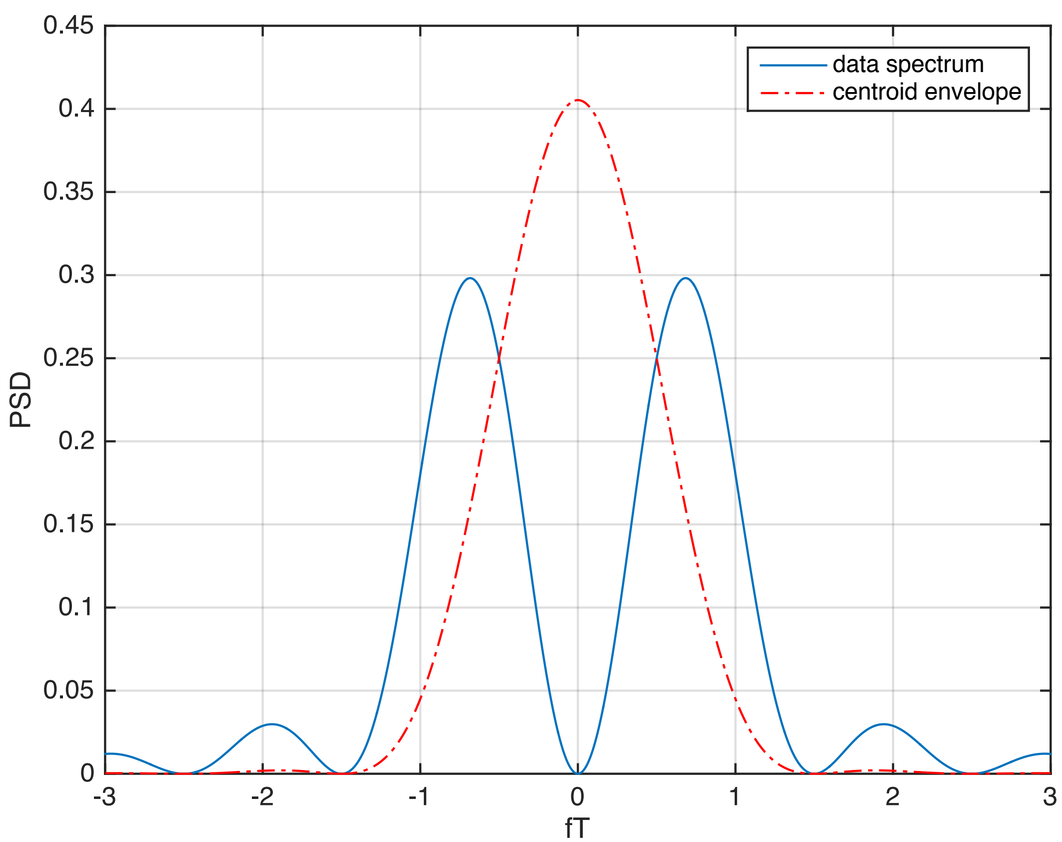
$$\bar{s}_m(t) = \sqrt{E_s} p(t) \exp\left(j 2\pi f_m t\right)$$

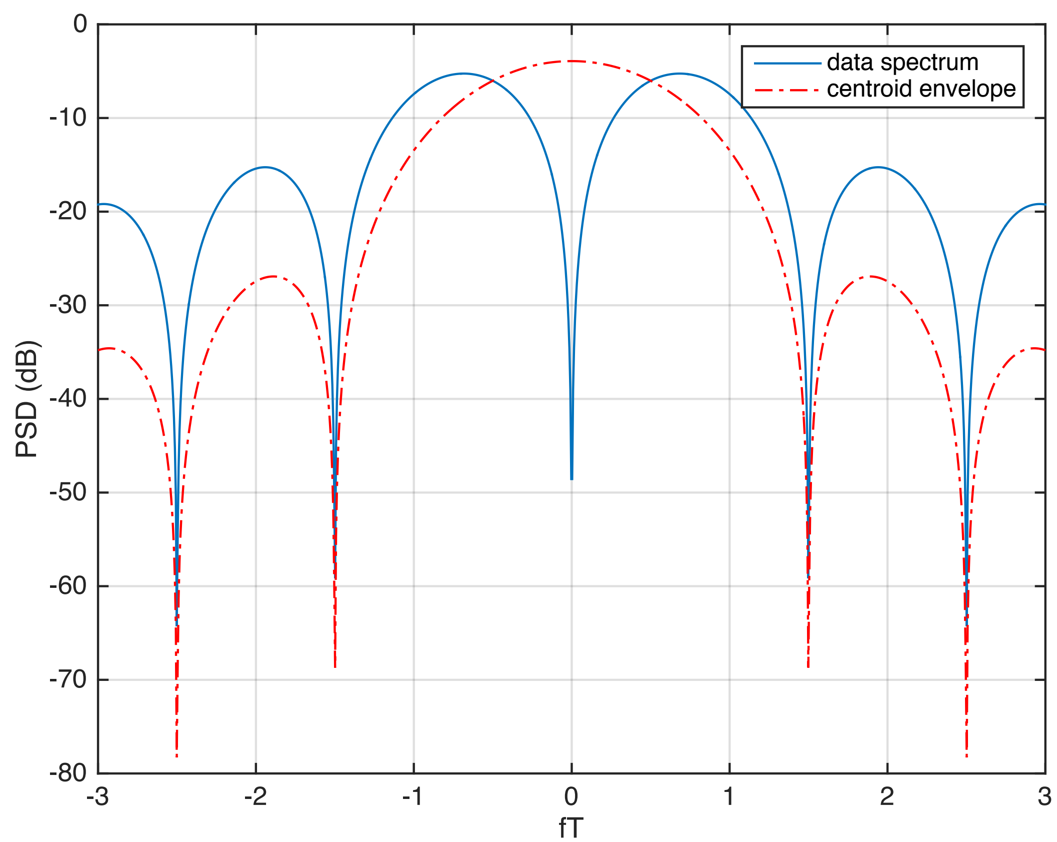
$$f_m = \frac{2m + 1 - M}{2} \Delta$$

$$\Delta = \text{tone separation}$$

M = 2, Delta = 1/T

(compare w/ Benedetto Fig. 5.19)





There are Dirac deltas at integer fT with area falling the red curve

PSD of Orthogonal FSK

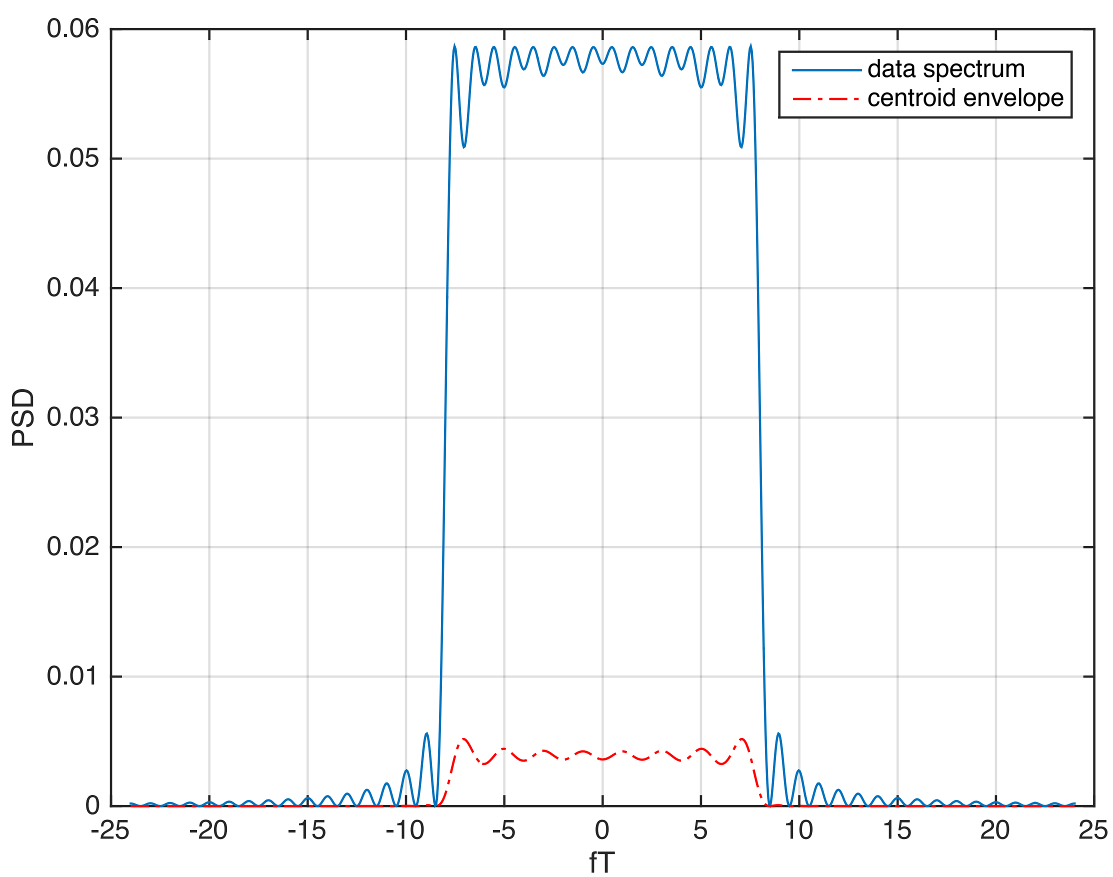
$$s_m(t) = \sqrt{E_s} p(t) \sqrt{2} \cos\left(2\pi \left[f_c + \right] t\right) \qquad m = 0, 1, \ldots M-1$$

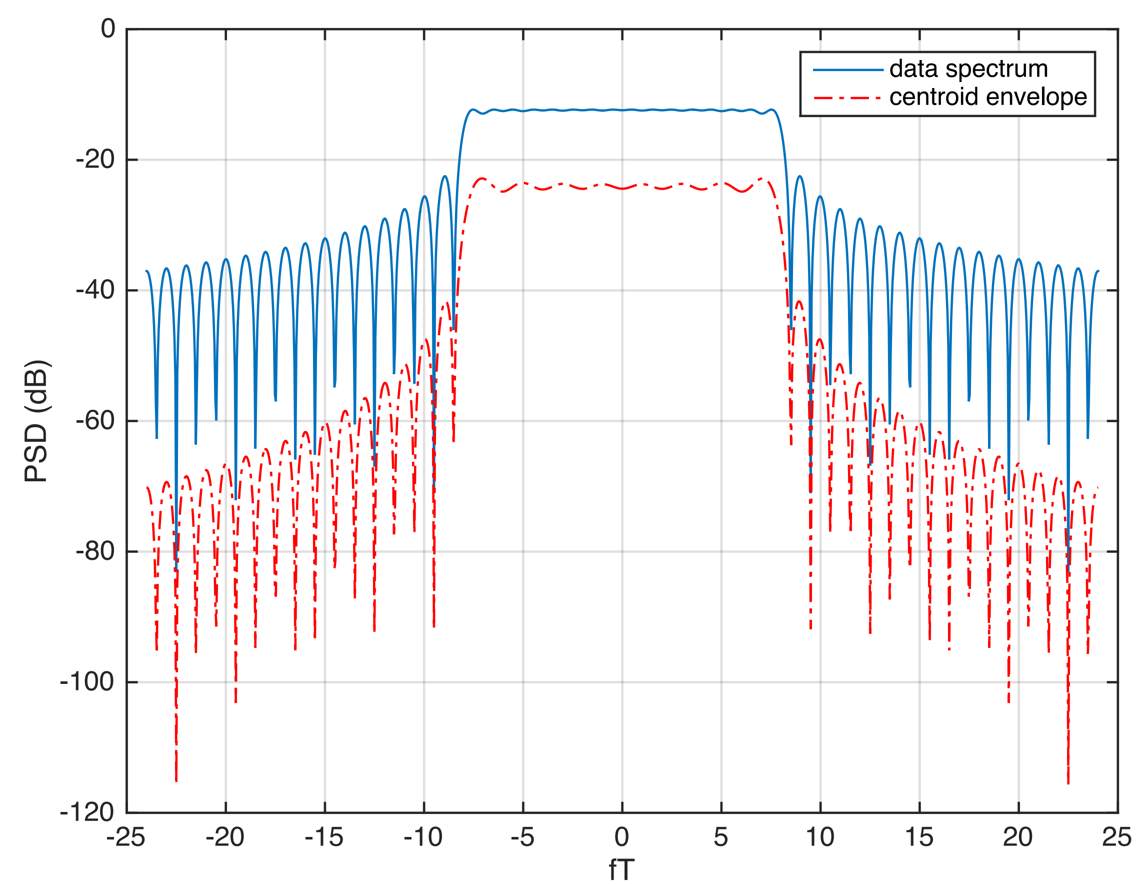
$$\bar{s}_m(t) = \sqrt{E_s} p(t) \exp\left(j 2\pi f_m t\right)$$

$$f_m = \frac{2m + 1 - M}{2} \Delta$$

$$\Delta = \text{tone separation}$$

M = 16, Delta = 1/T





There are Dirac deltas at integer fT with area following the red curve