Decision Theory and Performance Analysis

EE564: Digital Communication and Coding Systems

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Course Topic (from Syllabus)

- Overview of Comm/Coding
- Signal representation and Random Processes
- Optimal demodulation and decoding
- Uncoded modulations, demod, performance
- Classical FEC
- Modern FEC
- Non-AWGN channels (intersymbol interference)
- Practical consideration (PAPR, synchronization, spectral masks, etc.)

Detection/Demod Topics

- Maximum A Posteriori decision rule for vector-AWGN channel
- Exact performance for binary modulations
- Minimum distance decision rule for M-ary modulation over AWGN
- Performance bounds
- Continuous time model
  - Likelihood functional, sufficient statistics
- Average and generalized likelihood
  - Phase non-coherent demodulation
  - Soft-out demodulation

Decision Problem

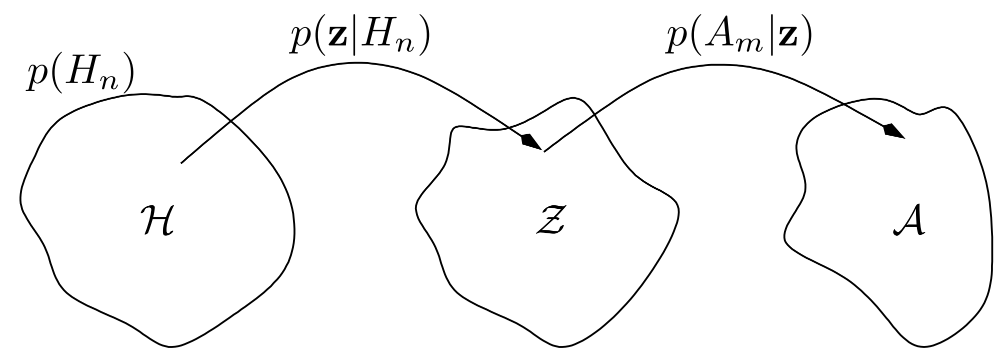


Figure 1.1. The set-up for general decision problems considered.

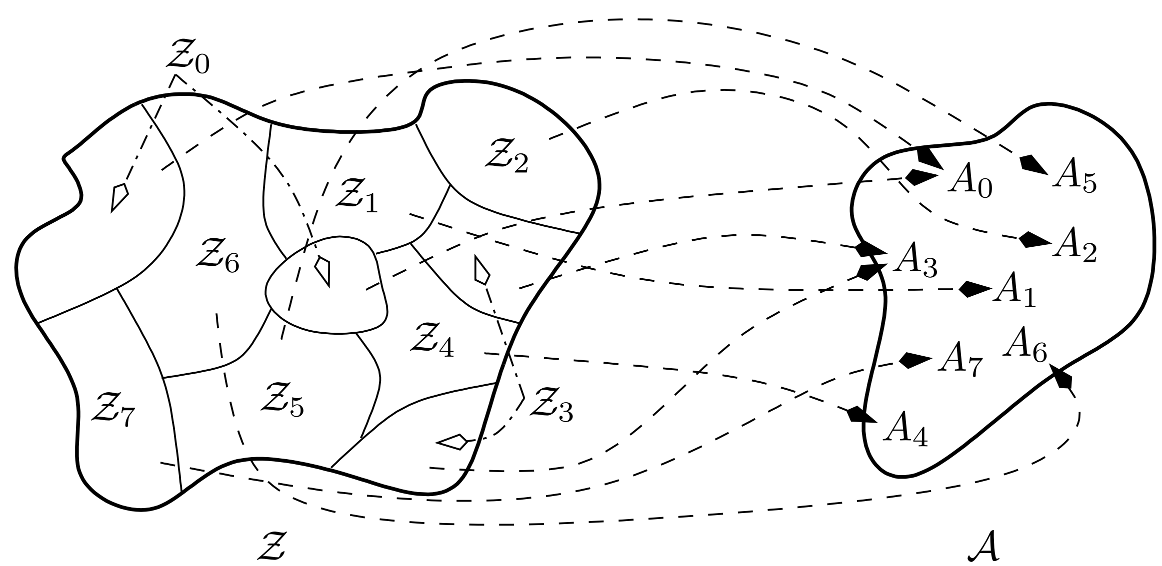


Figure 1.2. Decision rule implemented as a partition of the observation space.

Decision Problem

Bayes risk for decision rule d

$$R(d) = \int_{\mathcal{Z}} p_{\mathbf{z}(\zeta)}(\mathbf{z}) \left[\sum_m d(A_m|\mathbf{z}) C(A_m|\mathbf{z}) \right] d\mathbf{z} \qquad (1.1)$$

Cost for taking action m given observation

$$C(A_m|\mathbf{z}) = \sum_i C(A_m, H_i) p_{H(\zeta)|\mathbf{z}(\zeta)}(H_i|\mathbf{z}) \qquad (1.2)$$

Bayes decision rule

$$\text{Bayes action} = \arg\min_m C(A_m|\mathbf{z}) \qquad (1.3)$$

APP factoring

$$p_{H(\zeta)|\mathbf{z}(\zeta)}(H_m|\mathbf{z}) = \frac{p_{\mathbf{z}(\zeta)|H(\zeta)}(\mathbf{z}|H_m) p_{H(\zeta)}(H_m)}{p_{\mathbf{z}(\zeta)}(\mathbf{z})} \qquad (1.4a)$$

$$\equiv p_{\mathbf{z}(\zeta)|H(\zeta)}(\mathbf{z}|H_m) p_{H(\zeta)}(H_m) \qquad (1.4b)$$

MAP Decision Rule

MAP is special case of Bayesian Decision Rule

The *Maximum A-Posteriori Probability (MAP)* decision rule is the special case of the Bayes rule when A_m corresponds to deciding that H_m is true and $C(A_m, H_i) = 1 - \delta_{m-i}$. This may be seen by substituting these cost coefficients into (1.2) and noting that

$$C(A_m|\mathbf{z}) = \sum_{i \neq m} p_{H(\zeta)|\mathbf{z}(\zeta)}(H_i|\mathbf{z}) = 1 - p_{H(\zeta)|\mathbf{z}(\zeta)}(H_m|\mathbf{z}) \qquad (1.5)$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

$$R(d) = P(\text{decision error})$$

This can be used to reason MAP rule minimizes probability of error

M=2

$$P(\mathcal{E}) = P(\mathcal{E}|\mathcal{H}_0)\pi_0 + P(\mathcal{E}|\mathcal{H}_0)\pi_1$$

$$= \int_{\mathcal{Z}_1} f(\mathbf{z}|\mathcal{H}_0)\pi_0 d\mathbf{z} + \int_{\mathcal{Z}_0} f(\mathbf{z}|\mathcal{H}_1)\pi_1 d\mathbf{z}$$

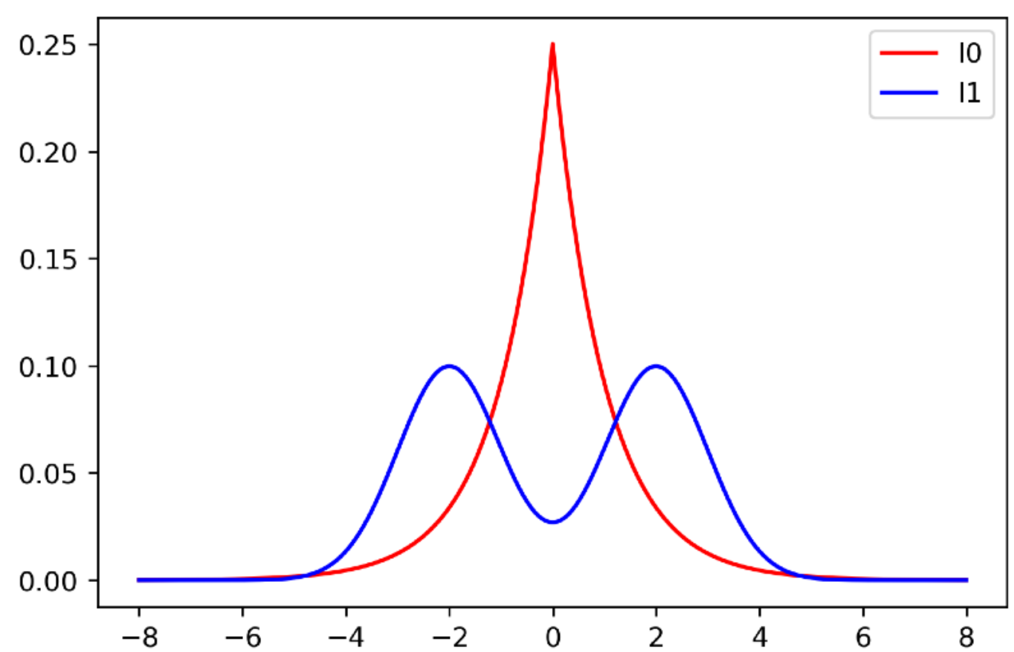
$$f(\mathbf{z}|\mathcal{H}_1)\pi_1 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} f(\mathbf{z}|\mathcal{H}_0)\pi_0$$

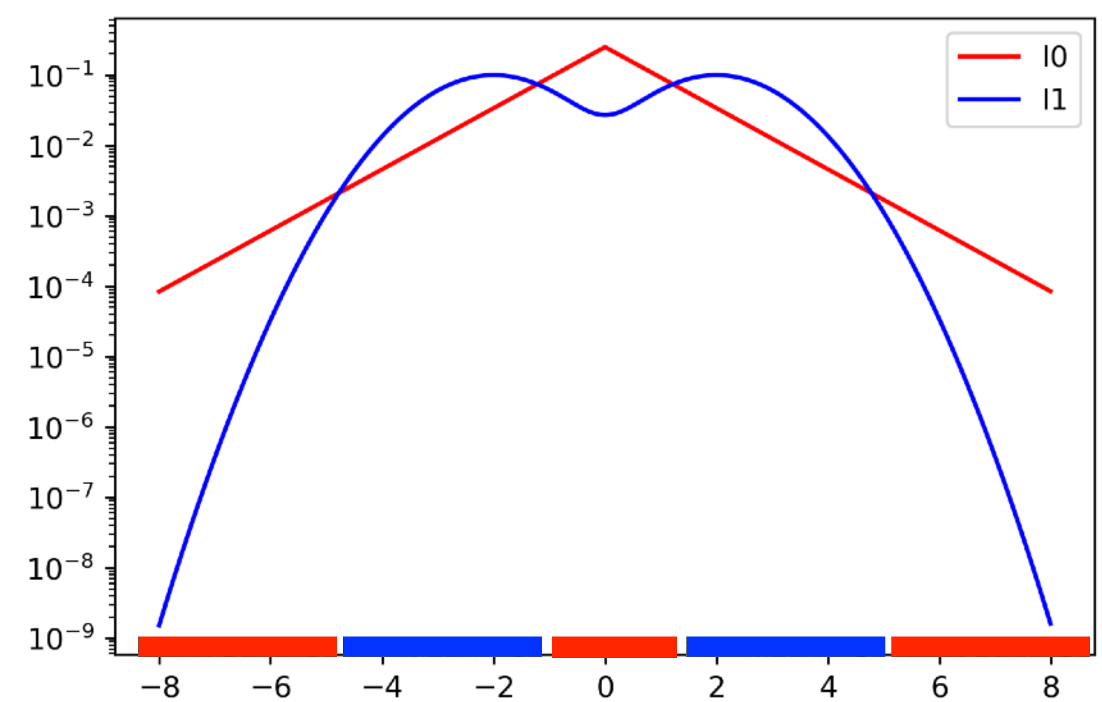
$$\Lambda(\mathbf{z}) = \frac{f(\mathbf{z}|\mathcal{H}_1)}{f(\mathbf{z}|\mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{\pi_0}{\pi_1} = T$$

Likelihood Ratio Test

MAP rule from P_errror Expression

$$\pi_0 = \pi_1 = 0.5 \qquad f_{z(u)}(z|\mathcal{H}_0) = \frac{1}{2}e^{-|z|} \qquad f_{z(u)}(z|\mathcal{H}_1) = \frac{1}{2}\mathcal{N}(z;-2;1) + \frac{1}{2}\mathcal{N}(z;+2;1)$$





$$P(\mathcal{E}) = \int_{\mathcal{Z}_0} \underbrace{\pi_1 f_{z(u)}(z|\mathcal{H}_1)}_{I_1(z)} dz + \int_{\mathcal{Z}_1} \underbrace{\pi_0 f_{z(u)}(z|\mathcal{H}_0)}_{I_0(z)} dz$$

$$I_0(z), I_1(z) \geq 0, \quad \forall\ z$$

$$\mathcal{Z}_0 \cup \mathcal{Z}_1 = \mathcal{R}, \quad \mathcal{Z}_0 \cap \mathcal{Z}_1 = \emptyset$$

Design Z0 and Z1: $\mathcal{Z}_0 = \{z \in \mathcal{R} : I_0(z) > I_1(z)\}$

MAP Rule for Vector-AWGN Channel

$$\mathcal{H}_m : \quad \mathbf{z}(u) = \mathbf{s}_m + \mathbf{w}(u) \qquad (D \times 1)$$

$$P(\mathcal{H}_m|\mathbf{z}) = \frac{f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m)\pi_m}{f_{\mathbf{z}(u)}(\mathbf{z})}$$

$$\equiv f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m)\pi_m$$

$$= \mathcal{N}_D(\mathbf{z}; \mathbf{s}_m; (N_0/2)\mathbf{I})\pi_m$$

$$= \frac{\pi_m}{(\pi N_0)^{D/2}} \exp\left[\frac{-1}{N_0}\|\mathbf{z} - \mathbf{s}_m\|^2\right]$$

$$\equiv \pi_m \exp\left[\frac{-1}{N_0}\|\mathbf{z} - \mathbf{s}_m\|^2\right]$$

$$\max_m P(\mathcal{H}_m|\mathbf{z}) \iff \min_m - \ln\left(P(\mathcal{H}_m|\mathbf{z})\right)$$

$$\iff \min_m \left[- \ln(\pi_m) + \frac{1}{N_0}\|\mathbf{z} - \mathbf{s}_m\|^2\right]$$

$$\iff \min_m \|\mathbf{z} - \mathbf{s}_m\|^2 \qquad \left(\text{when } \pi_m = \frac{1}{M}\right)$$

Other Rules (MAP Special Cases)

Maximum Likelihood (ML): $\max_m f(\mathbf{z}|\mathcal{H}_m)$

Minimum Distance: $\min_m d(\mathbf{z}, \mathbf{s}_m)$

Min. Euclidean (squared) distance: $\min_m \|\mathbf{z} - \mathbf{s}_m\|^2$

M=2

Maximum Likelihood (ML): $f(\mathbf{z}|\mathcal{H}_1) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} f(\mathbf{z}|\mathcal{H}_0)$

Minimum Distance: $d(\mathbf{z}, \mathbf{s}_0) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} d(\mathbf{z}, \mathbf{s}_1)$

Min. Euclidean (squared) distance: $\|\mathbf{z} - \mathbf{s}_0\|^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \|\mathbf{z} - \mathbf{s}_1\|^2$

MAP reduces to ML when a priori probabilities are uniform

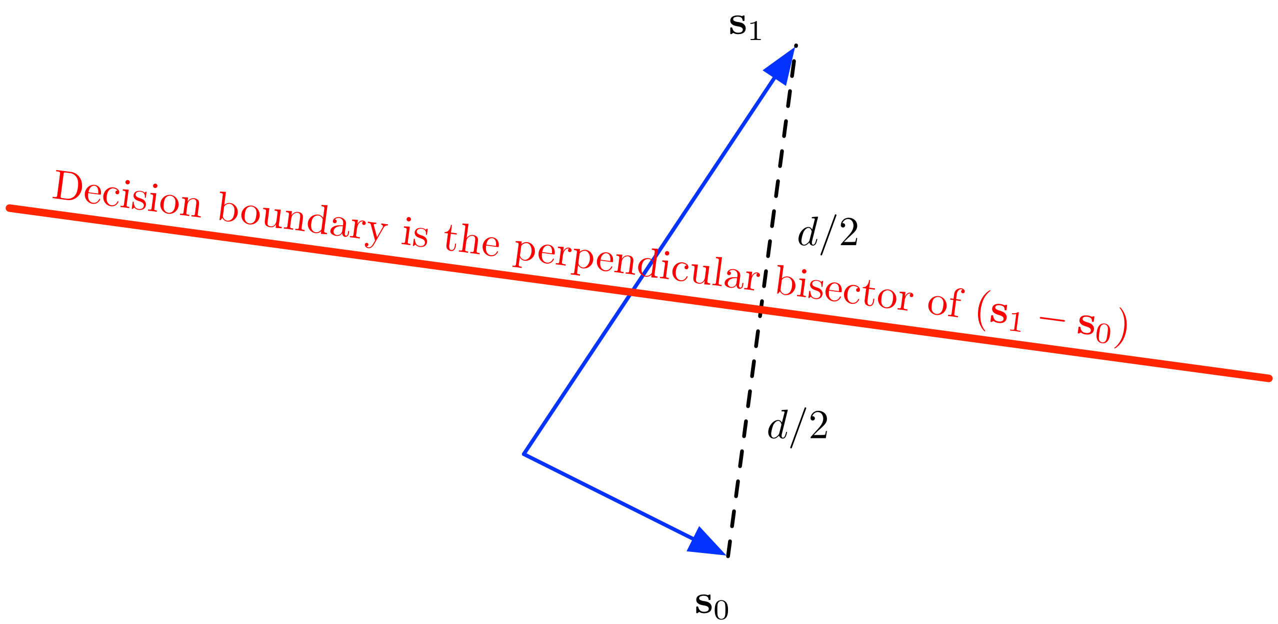
Other Rules (MAP Special Cases)

MAP → ML → Minimum Euclidean Distance

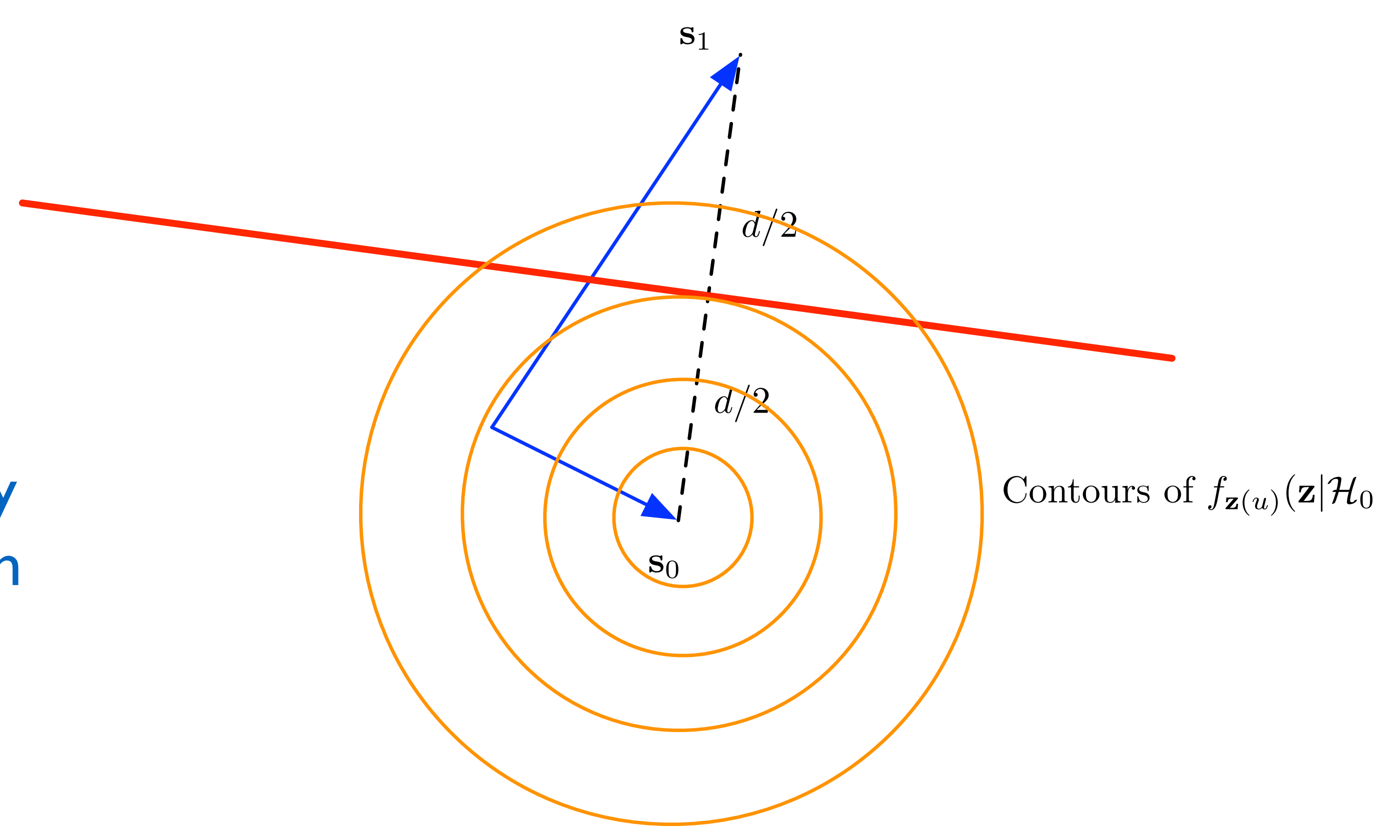
(MAP → ML: uniform priors)

(ML → Minimum Euclidean Distance: AWGN)

Binary MAP Decisions (equal priors)



$$(\mathbf{s}_1 - \mathbf{s}_0)^{\mathrm{t}} \mathbf{z} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2 - N_0 \ln(\pi_1/\pi_0)}{2}$$



Error probability given hypothesis 0 is the probability that noise throws observation over decision boundary

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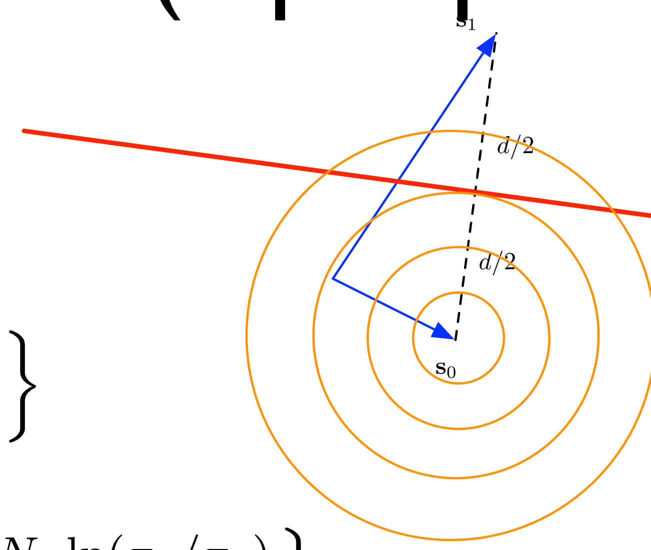
Performance of Binary MAP Decisions (equal priors)



$$P(\mathcal{E}|\mathcal{H}_0) = \mathrm{P}\textsc{r}\left\{(\mathbf{s}_1 - \mathbf{s}_0)^{\mathrm{t}}\mathbf{z}(u) > \tfrac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2 - N_0 \ln(\pi_1/\pi_0)}{2} | \mathcal{H}_0\right\}$$

$$= \mathrm{P}\textsc{r}\left\{(\mathbf{s}_1 - \mathbf{s}_0)^{\mathrm{t}}(\mathbf{s}_0 + \mathbf{w}(u)) > \tfrac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2 - N_0 \ln(\pi_1/\pi_0)}{2}\right\}$$

$$= \mathrm{P}\textsc{r}\left\{(\mathbf{s}_1^{\mathrm{t}}\mathbf{s}_0 - \|\mathbf{s}_0\|^2) + (\mathbf{s}_1 - \mathbf{s}_0)^{\mathrm{t}}\mathbf{w}(u) > \tfrac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2 - N_0 \ln(\pi_1/\pi_0)}{2}\right\}$$

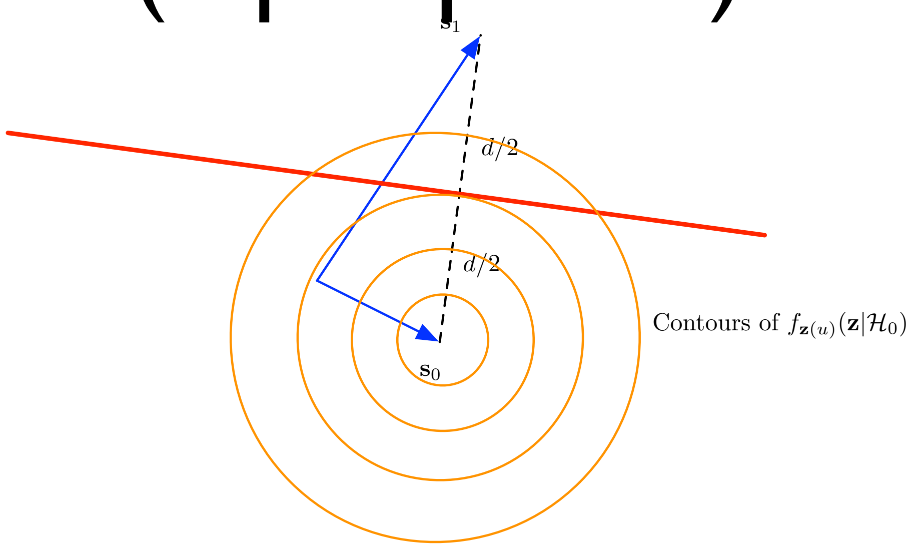
$$= \mathrm{P}\textsc{r}\left\{(\mathbf{s}_1 - \mathbf{s}_0)^{\mathrm{t}}\mathbf{w}(u) > \tfrac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_0\|^2 - 2\mathbf{s}_1^{\mathrm{t}}\mathbf{s}_0 - N_0 \ln(\pi_1/\pi_0)}{2}\right\}$$

$$= \mathrm{P}\textsc{r}\left\{V(u) > \tfrac{1}{2}\left[\|\mathbf{s}_1 - \mathbf{s}_0\|^2 - N_0 \ln(\pi_1/\pi_0)\right]\right\}$$

$$\mathbb{E}\left\{V(u)\right\} = 0$$

$$\sigma_V^2 = \frac{N_0}{2}\|\mathbf{s}_1 - \mathbf{s}_0\|^2$$

Performance of Binary MAP Decisions (equal priors)



$$P(\mathcal{E}|\mathcal{H}_0) = \mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

$$d^2 = \|\mathbf{s}_1 - \mathbf{s}_0\|^2 \qquad (\pi_1 = \pi_0)$$

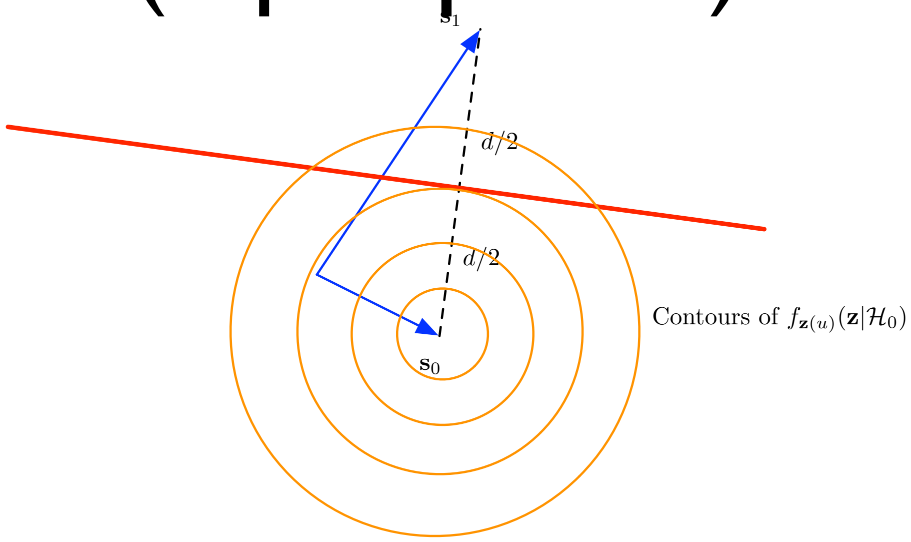
$$P(\mathcal{E}) = P(\mathcal{E}|\mathcal{H}_0)\pi_0 + P(\mathcal{E}|\mathcal{H}_1)\pi_1$$

$$= P(\mathcal{E}|\mathcal{H}_0)(1/2) + P(\mathcal{E}|\mathcal{H}_1)(1/2)$$

$$P(\mathcal{E}) = \mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{\|\mathbf{s}_1 - \mathbf{s}_0\|^2}{2N_0}}\right)$$

Note: not a function of dimension

Performance of Binary MAP Decisions (equal priors)



$$\|\mathbf{s}_1 - \mathbf{s}_0\|^2 = E_1 + E_0 - 2\mathbf{s}_1^t\mathbf{s}_0$$

$$\rho = \frac{\mathbf{s}_1^t\mathbf{s}_0}{\sqrt{E_1 E_0}}$$

$$\|\mathbf{s}_1 - \mathbf{s}_0\|^2 = 2E(1-\rho) \quad \text{(equal energy)}$$

$$\rho = \frac{\mathbf{s}_1^t\mathbf{s}_0}{E} \quad \text{(equal energy)}$$

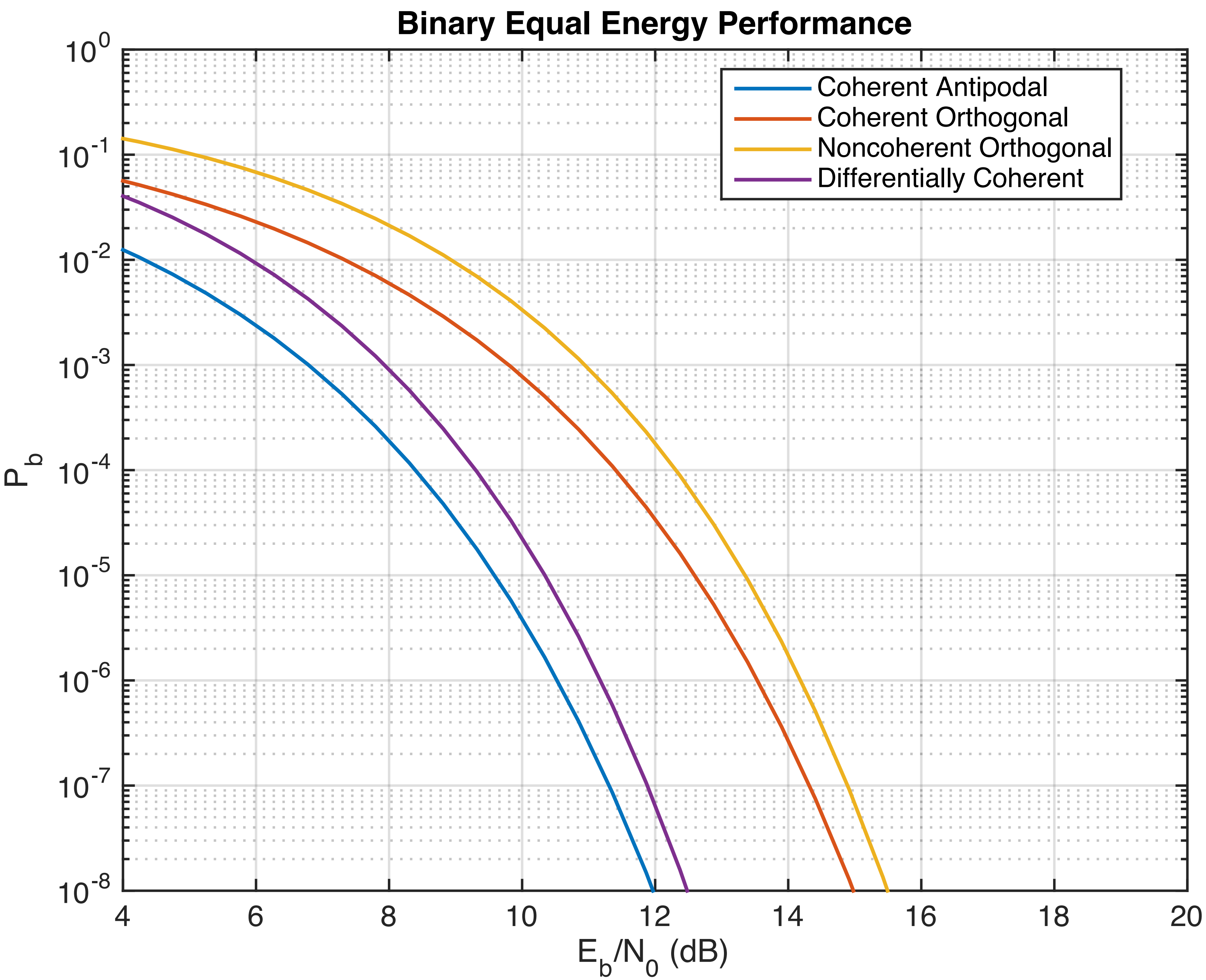
best equal energy binary signals are **antipodal signaling**

$$P(\mathcal{E}) = \mathrm{Q}\left(\sqrt{\frac{2E}{N_0}}\right) \quad \text{(antipodal)}$$

$$P(\mathcal{E}) = \mathrm{Q}\left(\sqrt{\frac{E}{N_0}}\right) \quad \text{(orthogonal, coherent)}$$

Binary orthogonal signaling is 3 dB worse than antipodal

Performance of Binary MAP Decisions (equal priors)



Blue and red curves show results from previous slide

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MAP Decision Regions

$$\text{decide } \mathcal{H}_m \quad \Longleftrightarrow \quad \mathbf{z} \in \mathcal{Z}_m$$

(global) decision region

$$\mathcal{Z}_m = \{\mathbf{z} : f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m)P(\mathcal{H}_m) > f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_j)P(\mathcal{H}_j) \ \forall \ j \neq m\}$$

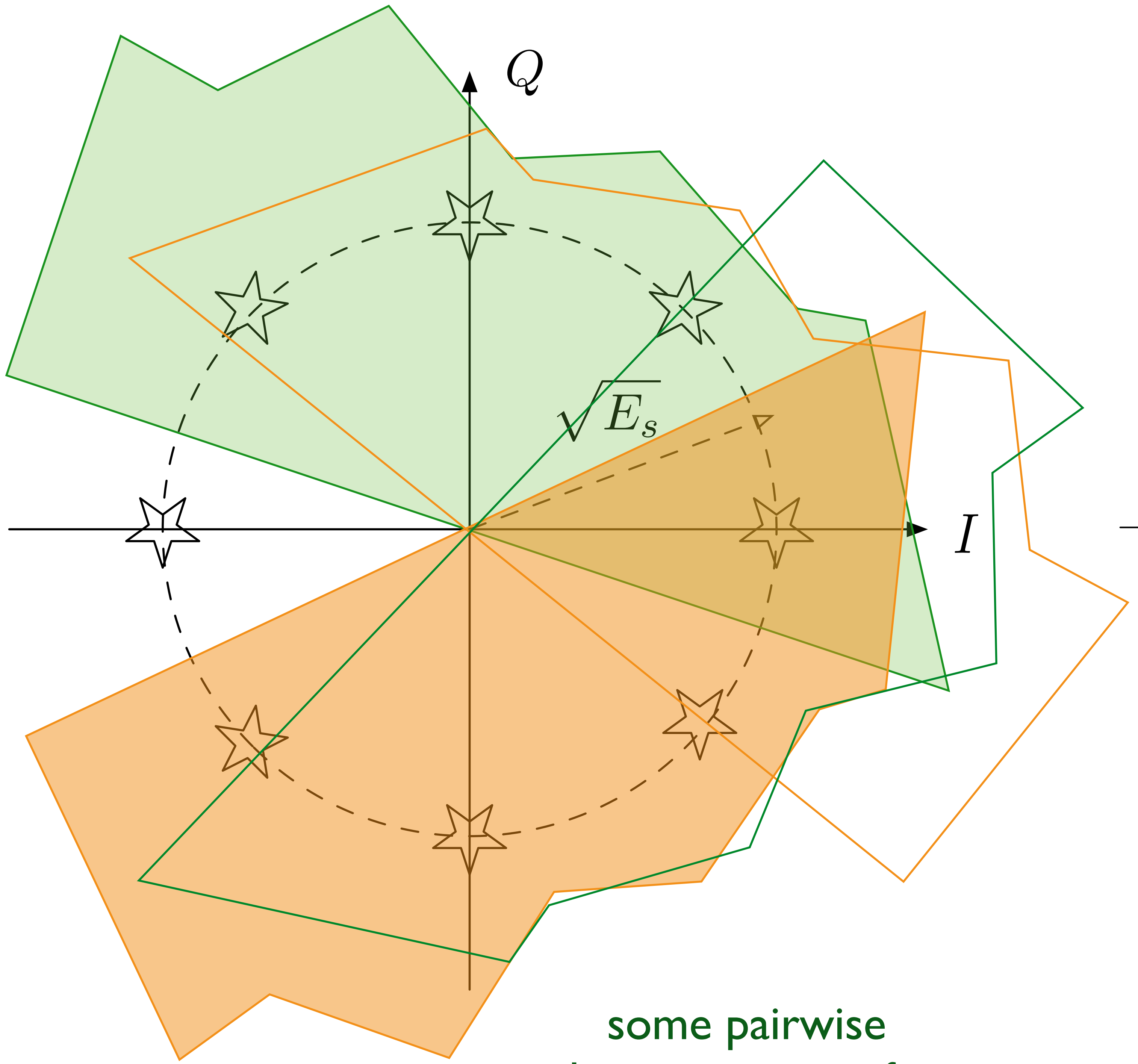
pairwise decision region (m over j)

$$\mathcal{Z}_m^{PW}(j) = \{\mathbf{z} : f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m)P(\mathcal{H}_m) > f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_j)P(\mathcal{H}_j)\}$$

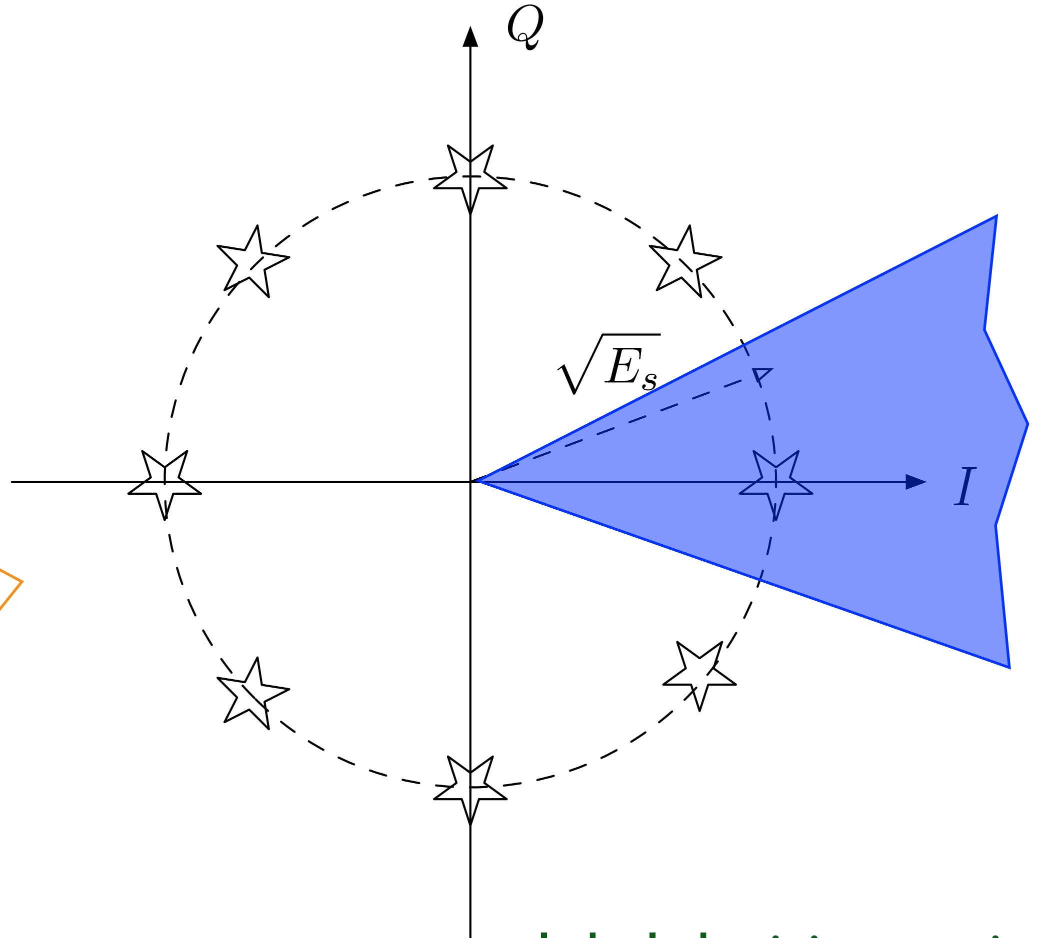
$$\boxed{\mathcal{Z}_m = \bigcap_{j=0, j\neq m}^{M-1} \mathcal{Z}_m^{PW}(j)}$$

In order for 'Hm' to be the decision, it must better than every other hypothesis in a pairwise test

8-PSK Example Min. Distance Rule

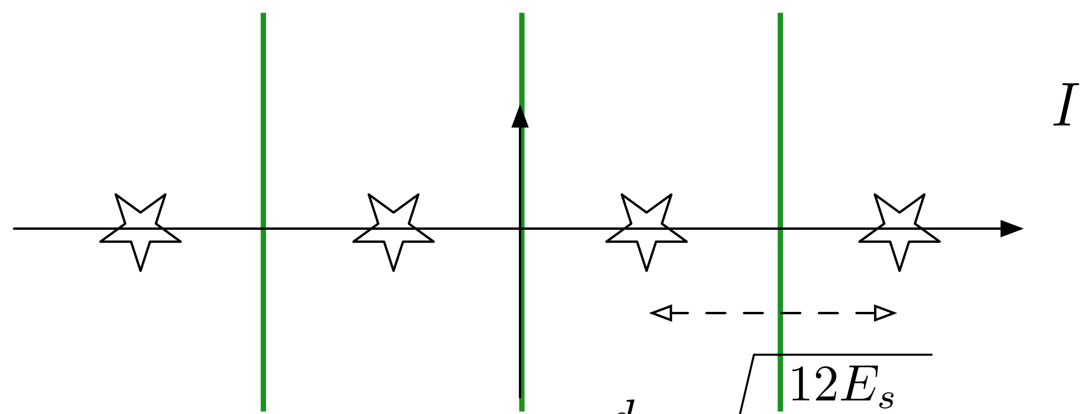


some pairwise decision regions for signal 0

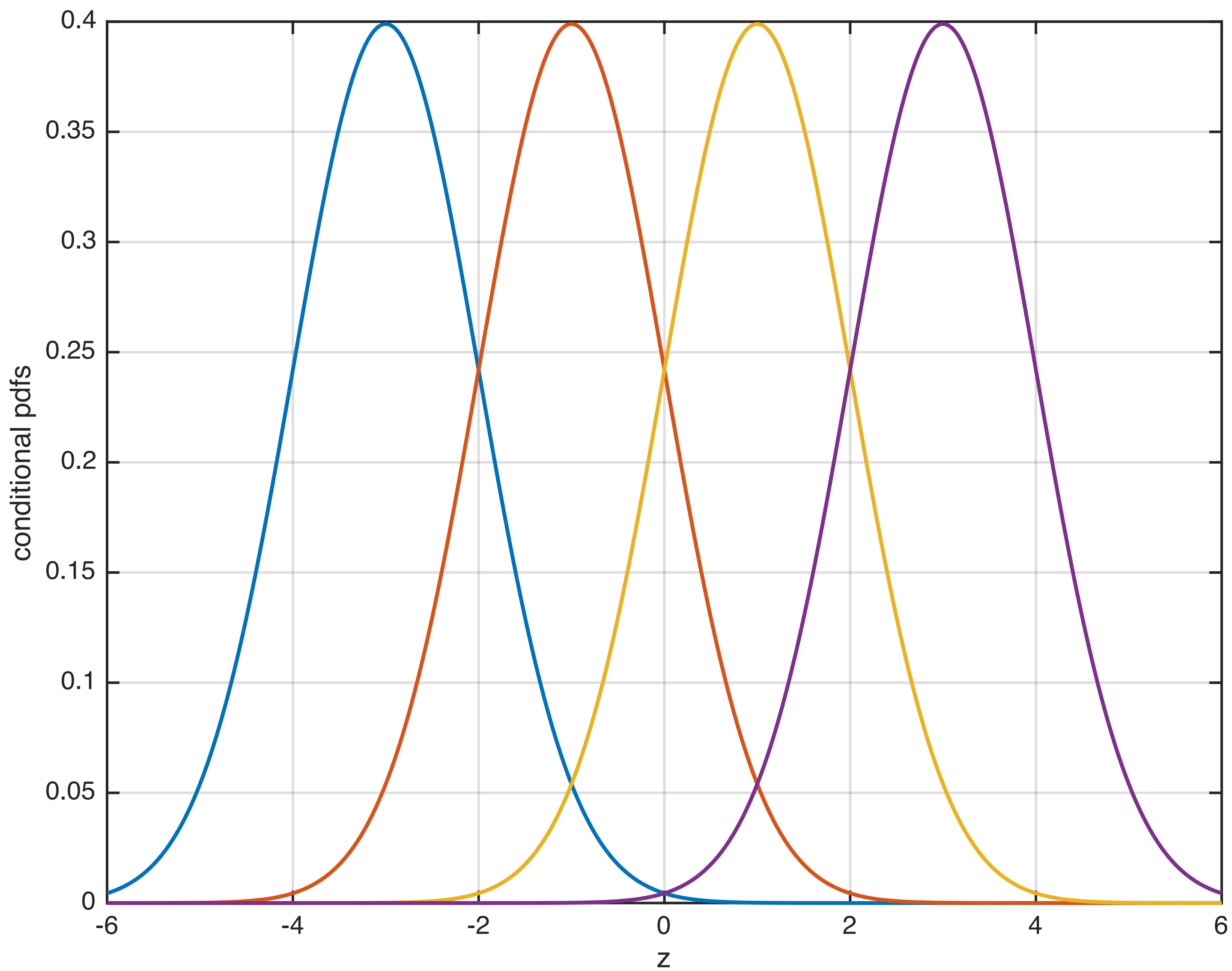


global decision region for signal 0

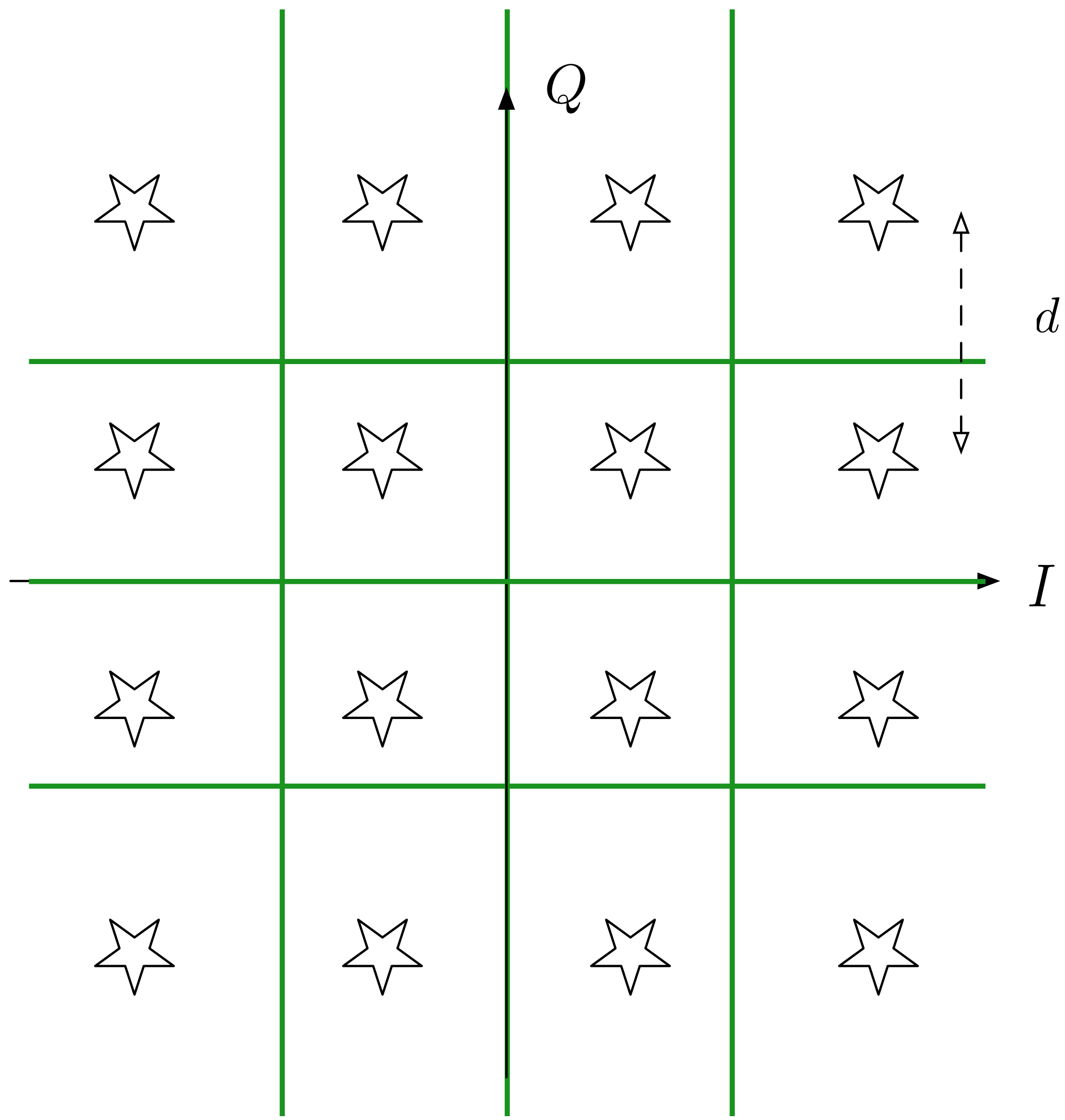
4-PAM Example Min. Distance Rule



$$d = \sqrt{\frac{12E_s}{M^2 - 1}}$$



16-QAM Example Min. Distance Rule



$$d = \sqrt{\frac{6E_s}{(M-1)}}$$

Performance Bounds for M-ary

$$\mathcal{Z}_m = \bigcap_{j=0, j\neq m}^{M-1} \mathcal{Z}_m^{PW}(j)$$

$$\mathcal{Z}_m^c = \bigcup_{j=0, j\neq m}^{M-1} \left[\mathcal{Z}_m^{PW}(j)\right]^c = \bigcup_{j=0, j\neq m}^{M-1} \mathcal{Z}_j^{PW}(m)$$

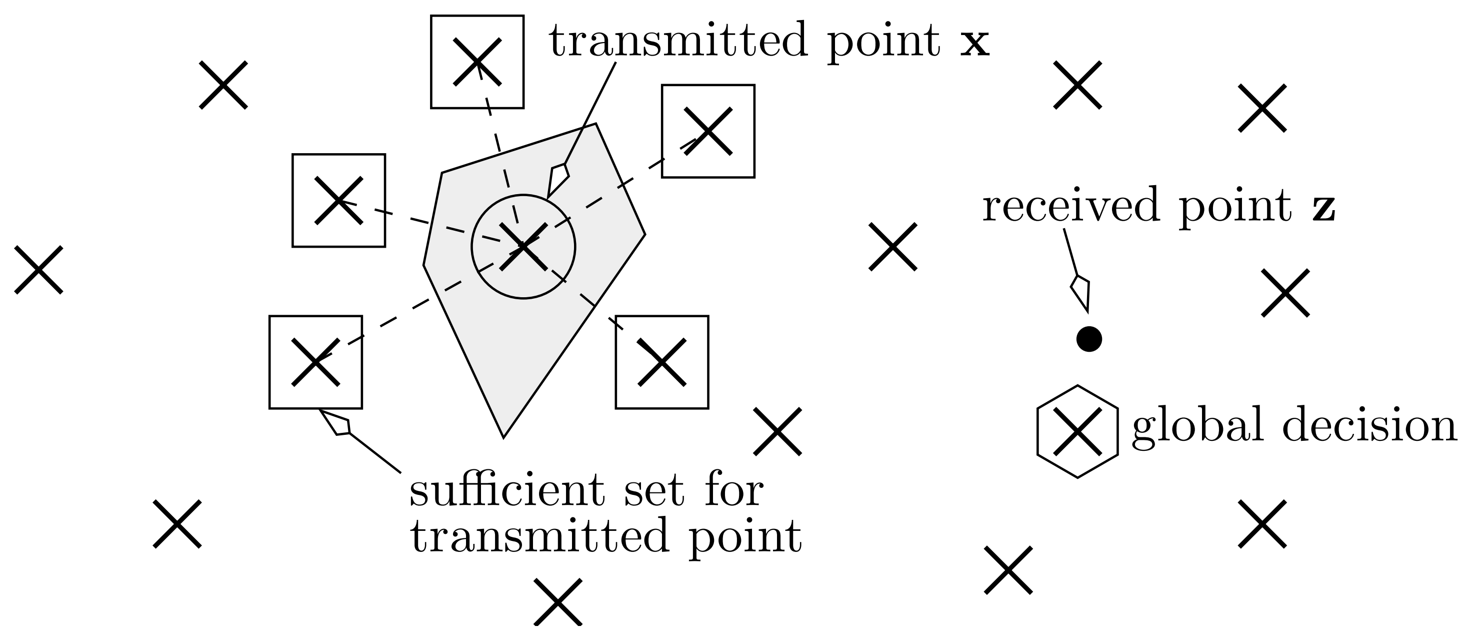
$$P(\mathcal{E}|\mathcal{H}_m) = \Pr\left\{\mathbf{z}(u) \in \mathcal{Z}_m^c | \mathcal{H}_m\right\}$$

$$= \Pr\left\{\mathbf{z}(u) \in \bigcup_{j=0, j\neq m}^{M-1} \mathcal{Z}_j^{PW}(m)\right\}$$

union bound

$$\max_j P_{PW}(j|\mathcal{H}_m) \leq P(\mathcal{E}|\mathcal{H}_m) \leq \sum_{j=0, j\neq m}^{M-1} P_{PW}(j|\mathcal{H}_m)$$

Performance Bounds for M-ary



sufficient set of neighbors

$$\mathcal{Z}_m^c = \bigcup_{j=0, j\neq m}^{M-1} \mathcal{Z}_j^{PW}(m) = \bigcup_{j\in\mathcal{N}_m}^{M-1} \mathcal{Z}_j^{PW}(m)$$

$$\sum_{m=0}^{M-1} P(\mathcal{H}_m)\left[\max_j P_{PW}(j|\mathcal{H}_m)\right] \leq P(\mathcal{E}) \leq \sum_{m=0}^{M-1} P(\mathcal{H}_m) \sum_{j\in\mathcal{N}_m}^{M-1} P_{PW}(j|\mathcal{H}_m)$$

Performance Bounds: M-ary AWGN

equal priors

$$\frac{1}{M}\sum_{m=0}^{M-1} \mathrm{Q}\left(\sqrt{\frac{d^2_{\min}(m)}{2N_0}}\right) \leq P(\mathcal{E}) \leq \frac{1}{M}\sum_{m=0}^{M-1}\sum_{j\in\mathcal{N}_m} \mathrm{Q}\left(\sqrt{\frac{d^2(j,m)}{2N_0}}\right)$$

$$d^2_{\min}(m) = \min_{j\neq m} d^2(j,m) \qquad d^2(j,m) = \|\mathbf{s}_j - \mathbf{s}_m\|^2$$

Book-keeping to combine terms with same distances

$$\sum_i \frac{K_i}{M}\mathrm{Q}\left(\sqrt{\frac{d_i^2}{2N_0}}\right) \leq P(\mathcal{E}) \leq \sum_i \frac{N_i(\{\mathcal{N}_m\})}{M}\mathrm{Q}\left(\sqrt{\frac{d_i^2}{2N_0}}\right)$$

Less tight, but very simple version:

$$\frac{1}{M}\mathrm{Q}\left(\sqrt{\frac{d^2_{\min}}{2N_0}}\right) \leq P(\mathcal{E}) \leq (M-1)\mathrm{Q}\left(\sqrt{\frac{d^2_{\min}}{2N_0}}\right)$$

Performance Bounds: M-ary AWGN

$$\frac{K_1}{M} Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \leq P(\mathcal{E}) \approx \frac{N_1(\{\mathcal{N}_m\})}{M} Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)$$

performance is dominated by the minimum distance

Performance Bounds: Bit Error Probability

$$P(\mathcal{B}_i) = \frac{P(\mathcal{B}_i|\mathcal{E})P(\mathcal{E})}{P(\mathcal{E}|\mathcal{B}_i)} = P(\mathcal{B}_i|\mathcal{E})P(\mathcal{E})$$

$$P_b = \frac{1}{q}\sum_{i=0}^{q-1} P(\mathcal{B}_i) = \frac{1}{q}\sum_{i=0}^{q-1} P(\mathcal{B}_i|\mathcal{E})P(\mathcal{E})$$

$$\frac{1}{q}P(\mathcal{E}) \leq P_b \leq P(\mathcal{E})$$

$$\frac{1}{q}B_L(\mathcal{E}) \leq P_b \leq B_U(\mathcal{E})$$

lower bound on symbol error probability

upper bound on symbol error probability

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Performance Bounds: MPSK

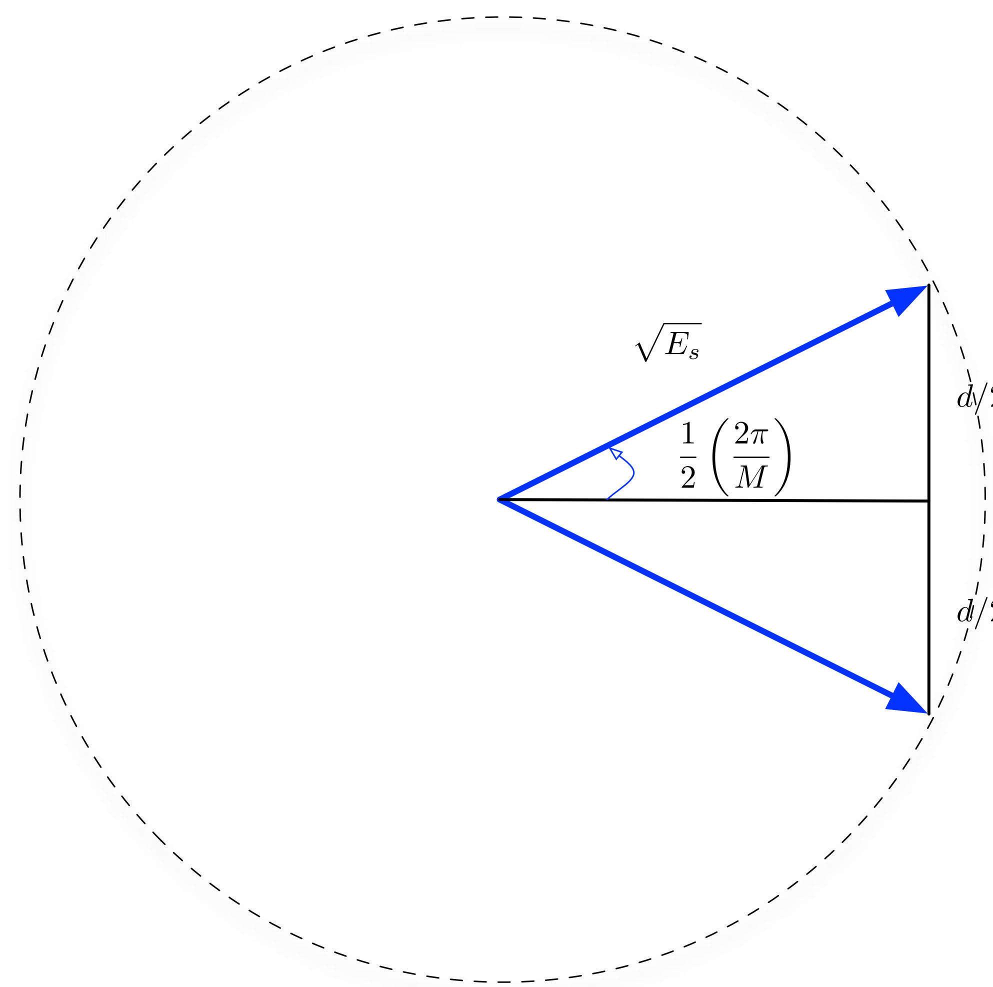
$$d^2(m,n) = \left| \sqrt{E_s} e^{j\frac{2\pi}{M}m} - \sqrt{E_s} e^{j\frac{2\pi}{M}n} \right|^2$$

$$= E_s \left(1 + 1 - 2\Re\left\{ e^{j\frac{2\pi}{M}(m-n)} \right\} \right)$$

$$= 2E_s\left(1 - \cos\left(\frac{2\pi}{M}(m-n)\right)\right)$$

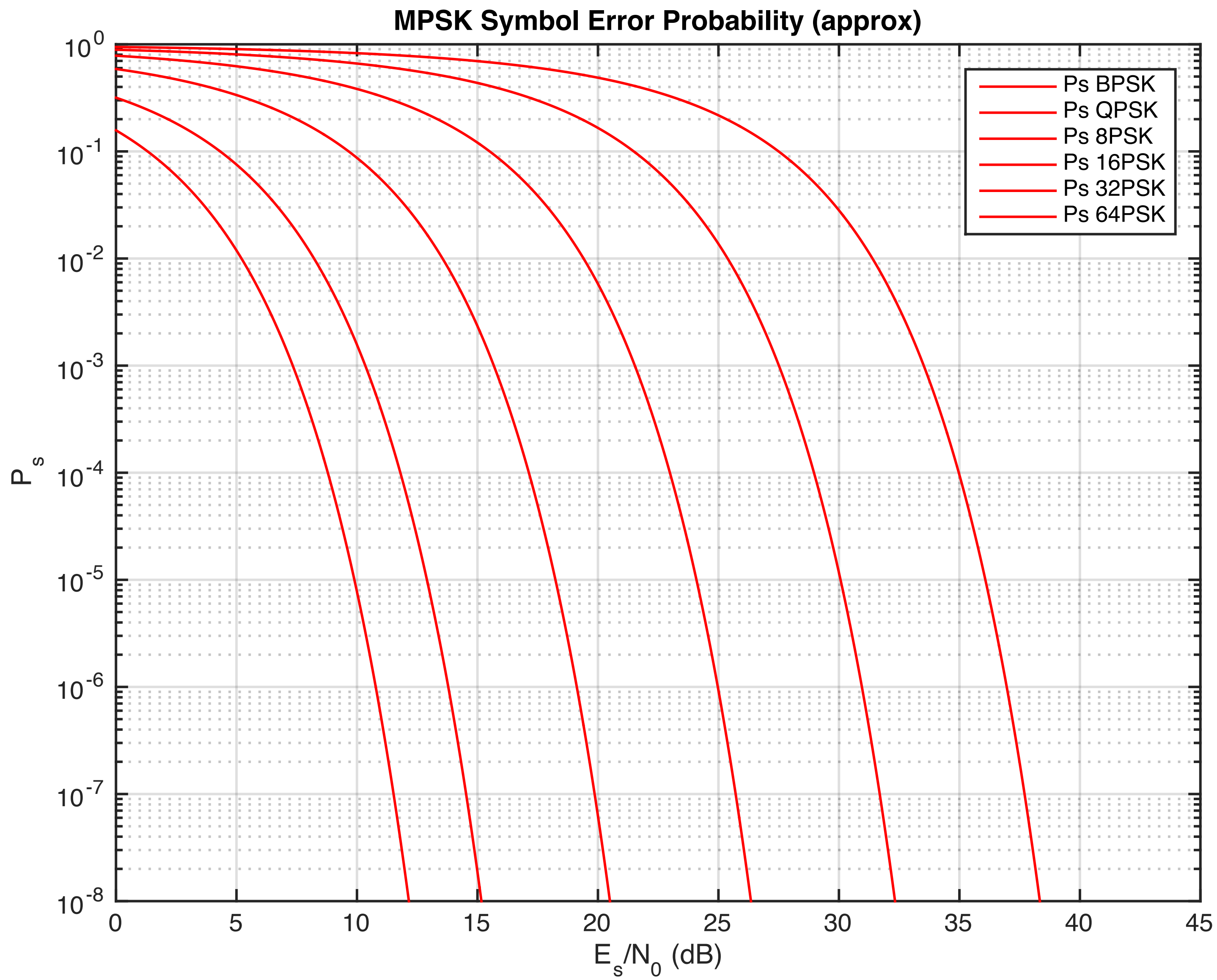
$$= 4E_s \sin^2\left(\frac{\pi}{M}(m-n)\right)$$

$$d^2_{\min} = 4E_s \sin^2\left(\frac{\pi}{M}\right)$$



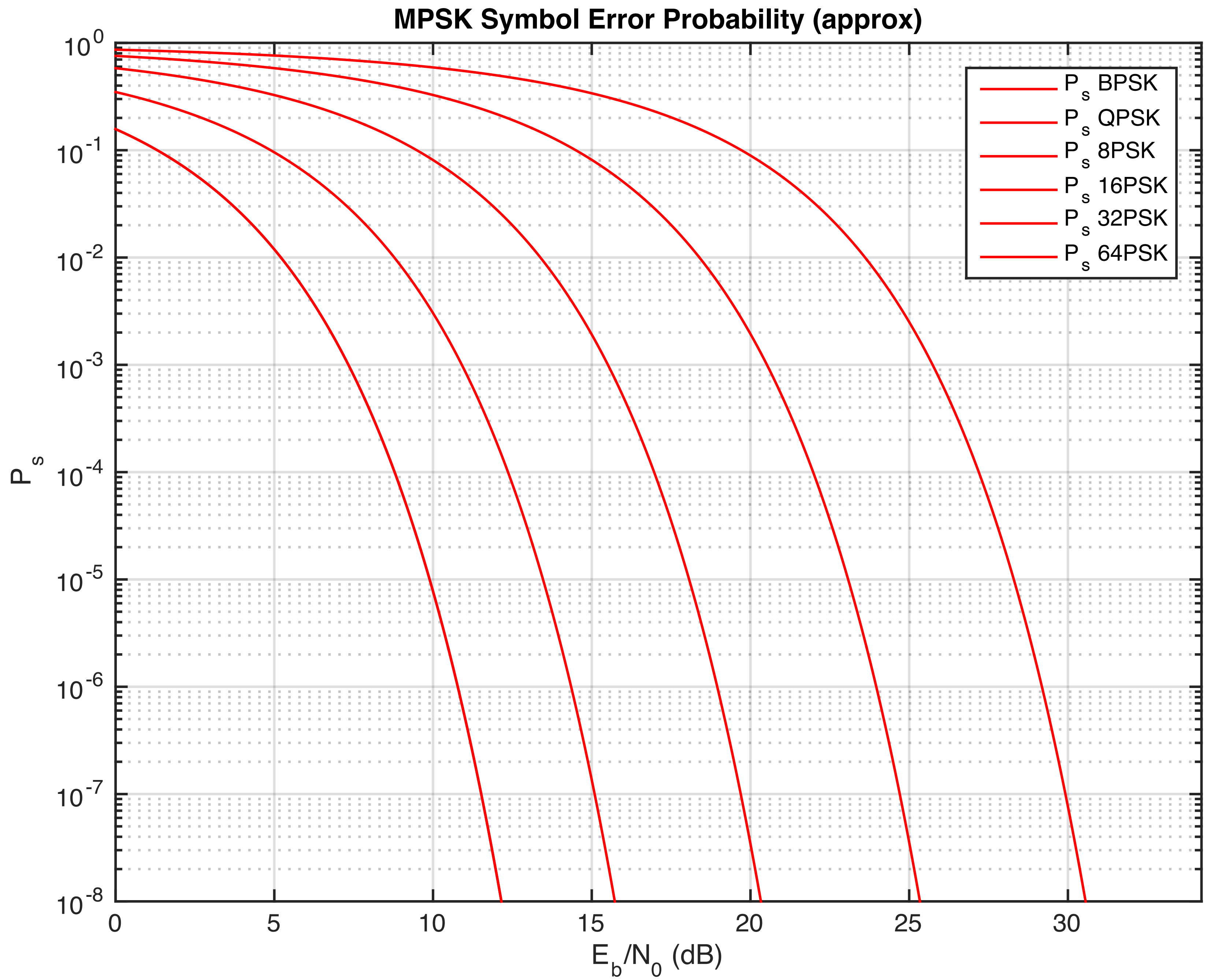
$$\mathrm{Q}\left(\sqrt{\frac{2E_s \sin^2\left(\frac{\pi}{M}\right)}{N_0}}\right) \leq P(\mathcal{E}) \leq 2\mathrm{Q}\left(\sqrt{\frac{2E_s \sin^2\left(\frac{\pi}{M}\right)}{N_0}}\right)$$

Performance Bounds: MPSK



$$Q\left(\sqrt{\frac{2E_s \sin^2\left(\frac{\pi}{M}\right)}{N_0}}\right) \leq P(\mathcal{E}) \leq 2Q\left(\sqrt{\frac{2E_s \sin^2\left(\frac{\pi}{M}\right)}{N_0}}\right)$$

Performance Bounds: MPSK

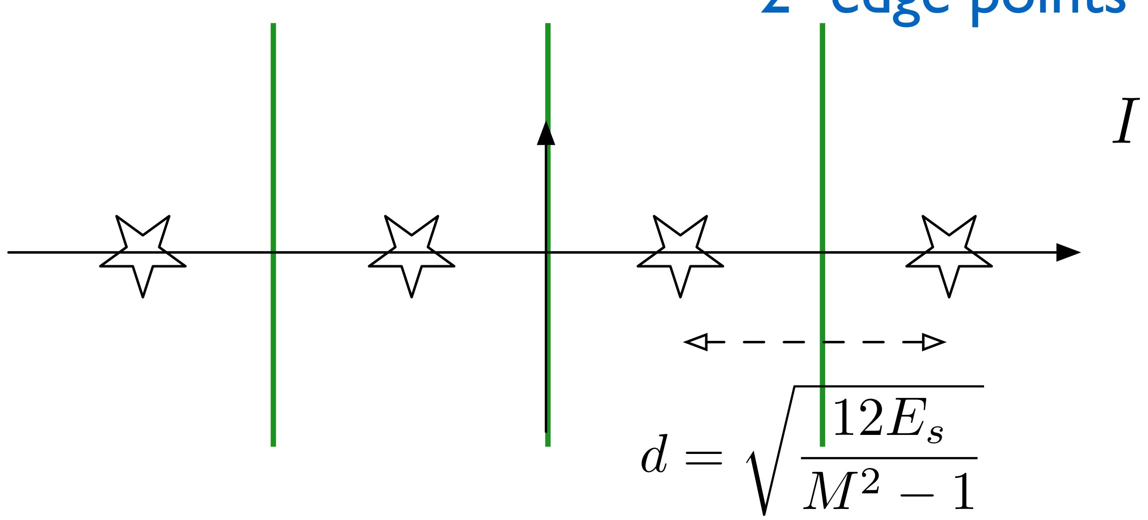


$$E_b = \frac{E_s}{\log_2(M)}$$

Performance (exact): M-PAM

$$P(\mathcal{E}|\mathcal{H}_m) = \mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right) \qquad \text{(edge points)}$$

$$P(\mathcal{E}|\mathcal{H}_m) = 2\mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right) \qquad \text{(interior points)}$$



$$P(\mathcal{E}) = \frac{2}{M}\mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right) + \frac{(M-2)}{M}2\mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

$$= \frac{2(M-1)}{M}\mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

$$P(\mathcal{E}) = \frac{2(M-1)}{M}\mathrm{Q}\left(\sqrt{\left[\frac{3}{M^2-1}\right]\frac{2E_s}{N_0}}\right)$$

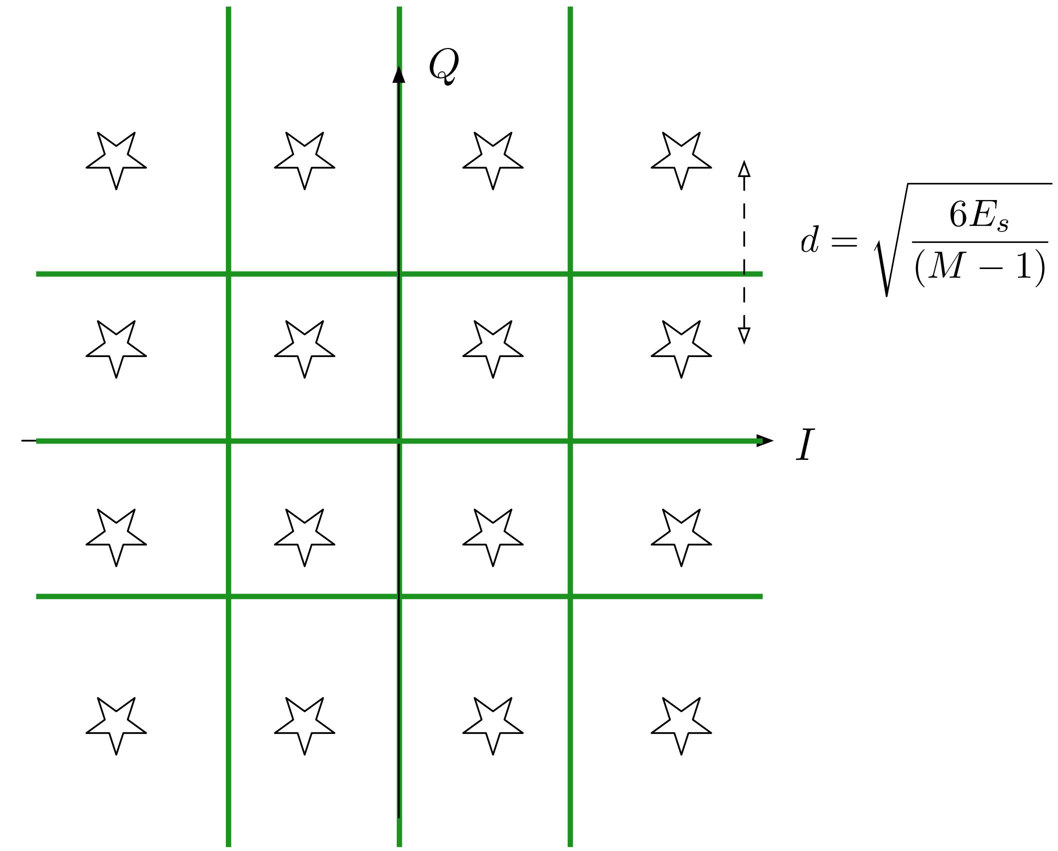
Performance (exact): M-QAM

$$M = M_p^2 = 4^i$$

$$P(\mathcal{E}) = 1 - P(\mathcal{C})$$

$$P(\mathcal{C}) = \left[P_{M_p-\mathrm{PAM}}(\mathcal{C})\right]^2$$

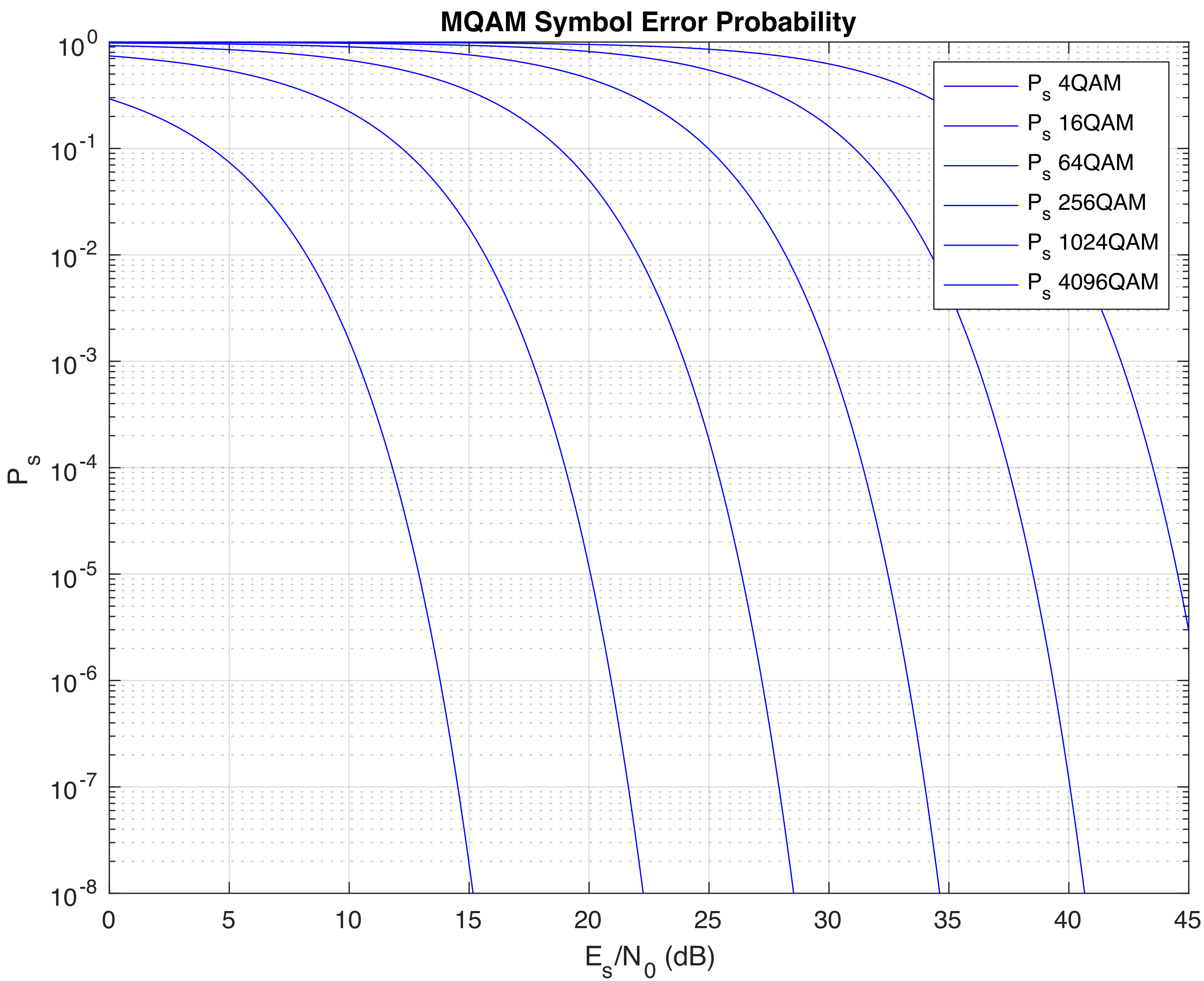
$$= \left[1 - \frac{2(M_p - 1)}{M_p} \mathrm{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right)\right]^2$$



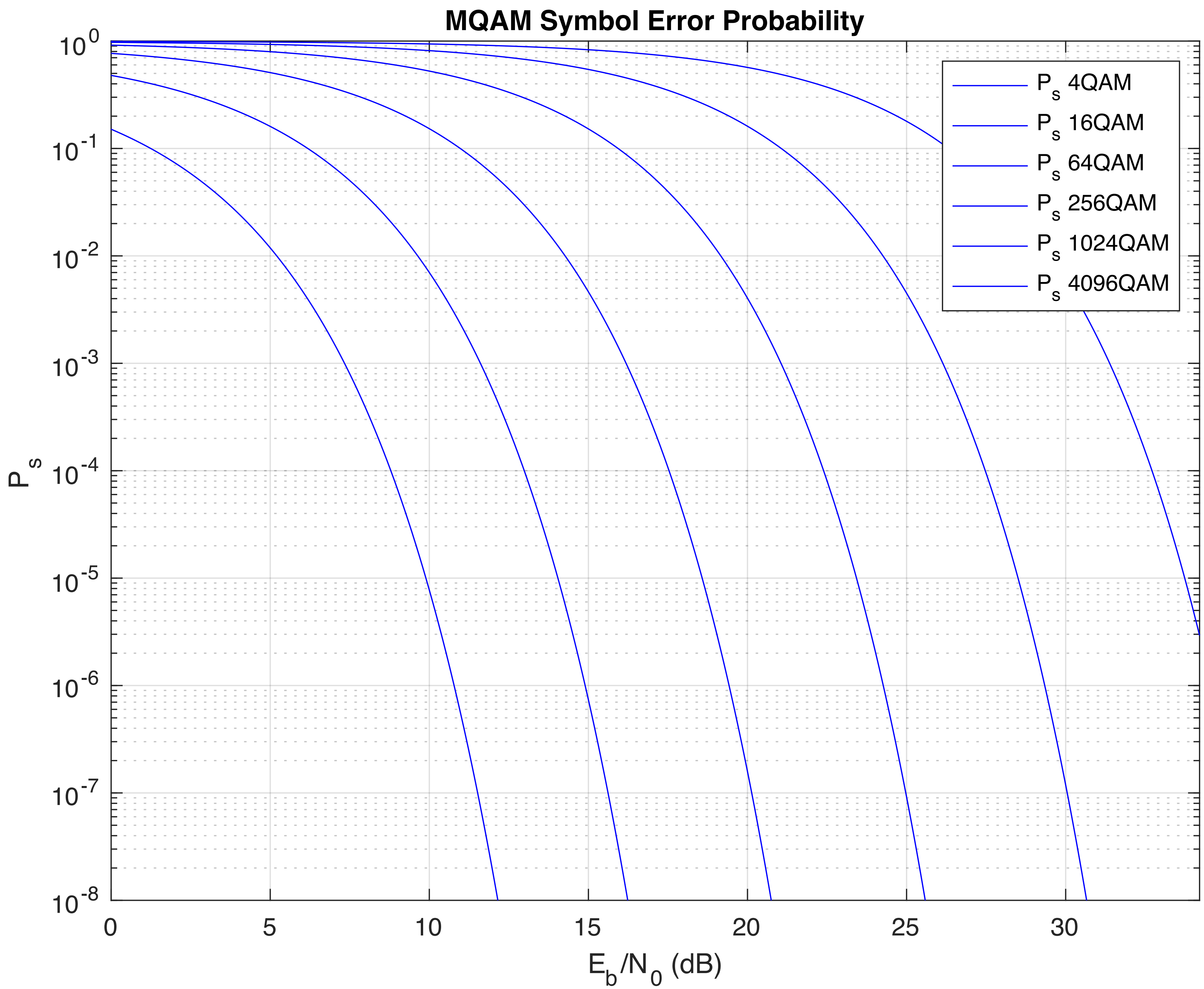
$$P(\mathcal{E}) = 1 - \left[1 - \frac{2(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{3}{2(M-1)}\frac{2E_s}{N_0}}\right)\right]^2$$

$$\approx \frac{4(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{3}{2(M-1)}\frac{2E_s}{N_0}}\right)$$

Performance (exact): M-QAM

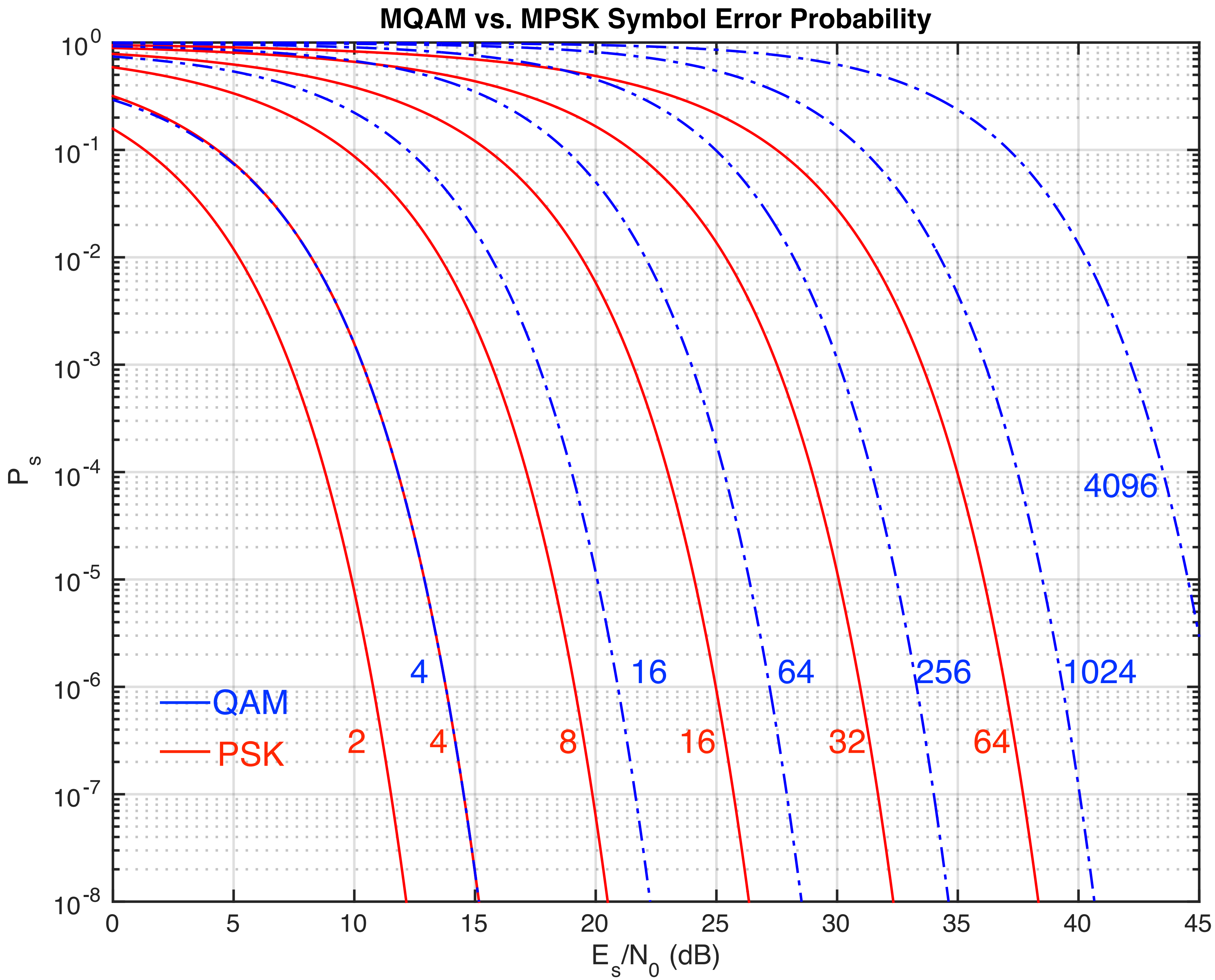


Performance (exact): M-QAM



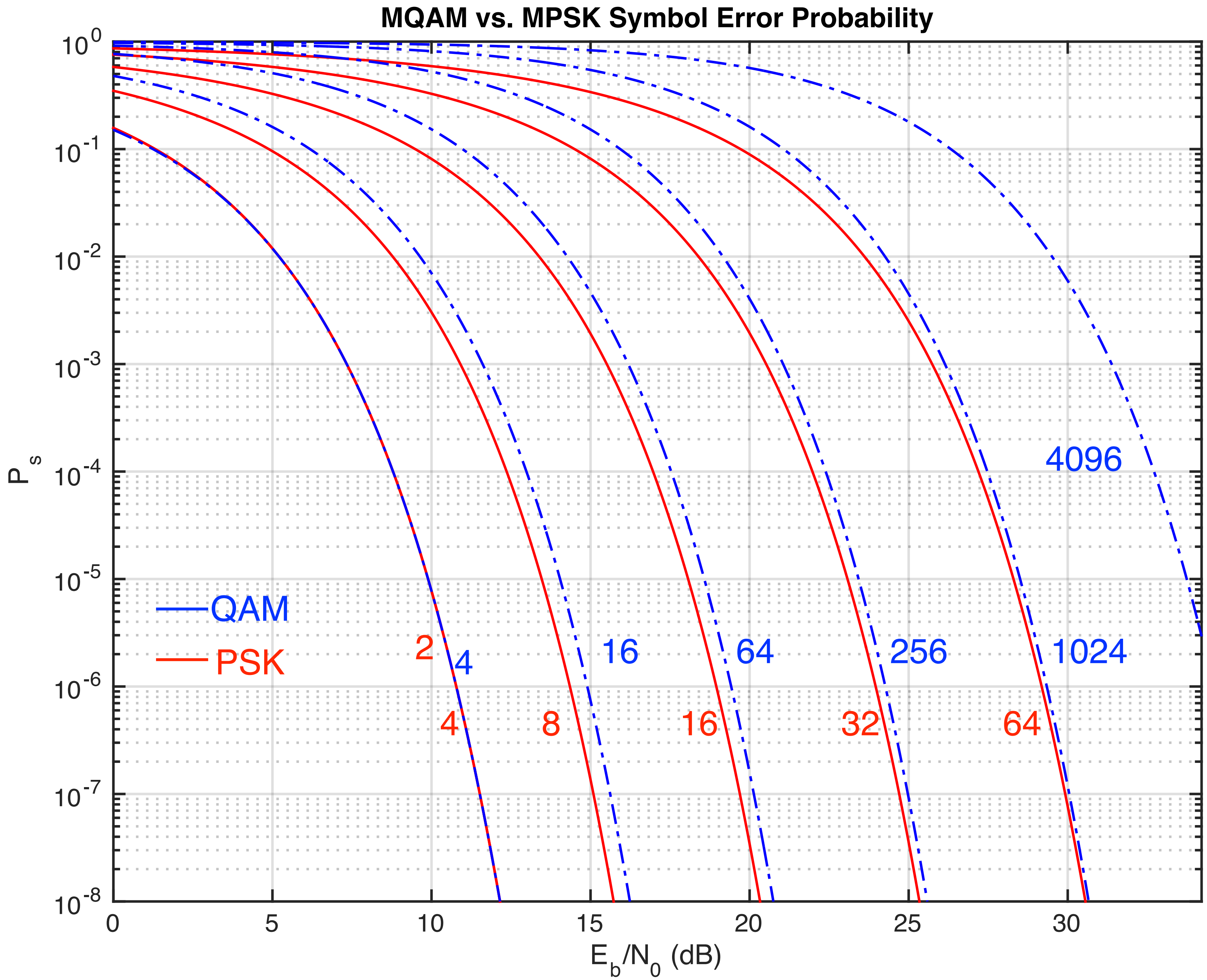
$$E_b = \frac{E_s}{\log_2(M)}$$

QAM/PSK Performance Comparison



	E_s/N_0 loss relative to BPSK (dB)	
M	PSK	QAM
M	$\sin^2(\pi/M)$	$\frac{3}{2(M-1)}$
4	3.0	3.0
8	8.3	
16	14.2	10.0
32	20.2	
64	26.2	16.2
128	32.2	
256	38.2	22.3
1024	50.3	28.3
4096	62.3	34.3

QAM/PSK Performance Comparison



	E_b/N_0 loss relative to BPSK (dB)	
M	PSK	QAM
M	$\sin^2(\pi/M)\log_2(M)$	$\frac{3\log_2(M)}{2(M-1)}$
4	0	0
8	3.6	
16	8.2	4.0
32	13.2	
64	18.4	8.5
128	23.8	
256	29.2	13.3
1024	40.3	18.3
4096	51.5	23.6

QPSK vs. BPSK Comparison

$$P_s = 1 - \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2 \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$ QPSK

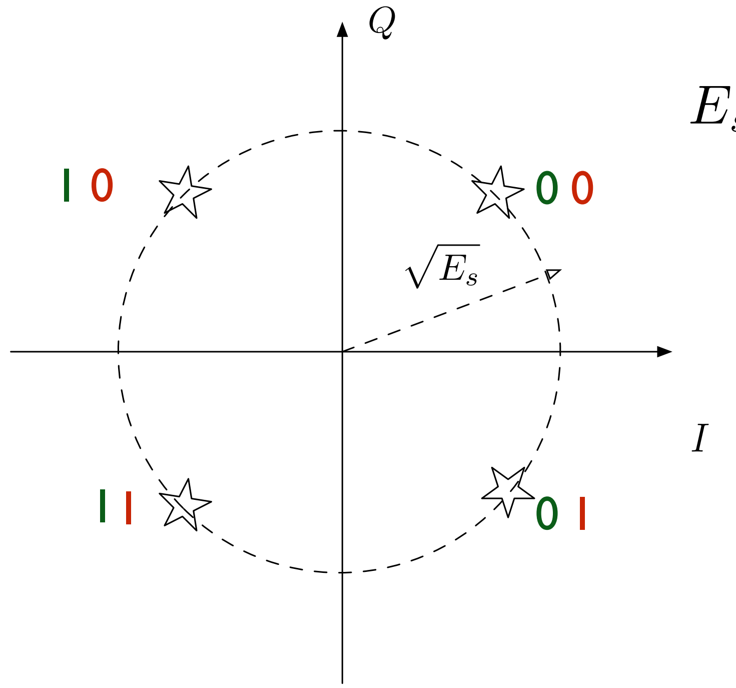
$$P_b \approx Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$ (Gray mapping)

BPSK & Gray-labeled QPSK

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- This is actually exact when using Gray-mapped 4QAM/QPSK

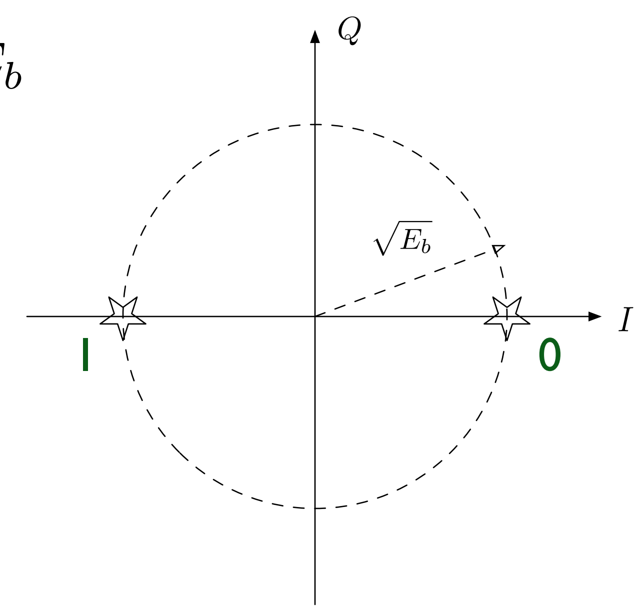
Gray labeled QPSK



$E_s = 2E_b$

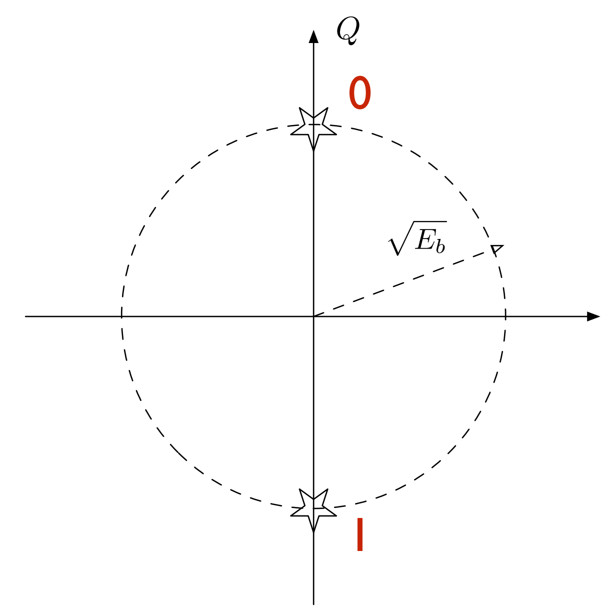
=

BPSK on I



+

BPSK on Q



Noise is independent on the In-phase and Quadrature dimensions

M-ary Orthogonal (coherent demod)

$$\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle = \Re\left\{ \langle \bar{\boldsymbol{s}}_m, \bar{\boldsymbol{s}}_n \rangle \right\} = E_s \delta[m-n]$$

$$\phi_i(t) = \frac{s_i(t)}{\sqrt{E_s}}$$

$$\mathbf{s}_m^{\mathrm{t}} = \sqrt{E_s} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 \dots & 0 \end{pmatrix}$$

$$\|\mathbf{z} - \mathbf{s}_m\|^2 = \|\mathbf{z}\|^2 + \|\mathbf{s}_m\|^2 - 2\mathbf{z}^{\mathrm{t}}\mathbf{s}_m$$

$$\equiv E_s - 2\mathbf{z}^{\mathrm{t}}\mathbf{s}_m$$

$$\equiv -2\sqrt{E_s} z_m$$

ML Rule

$$\max_m z_m$$

$$f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_0) = \mathcal{N}_M\left(\mathbf{z}; \mathbf{s}_0; \frac{N_0}{2}\mathbf{I}\right)$$

$$= \mathcal{N}_1(z_0; \sqrt{E_s}; N_0/2) \prod_{i=1}^{M-1} \mathcal{N}_1(z_i; 0; N_0/2)$$

Given $\mathcal{H}_0$, $\{z_0(u), z_1(u), \dots z_{M-1}(u)\}$ are mutually independent

M-ary Orthogonal (coherent demod)

$$P(\mathcal{C}|\mathcal{H}_0, z_0(u) = z_0) = \text{Pr}\{z_1(u) < z_0, z_2(u) < z_0, \ldots z_{M-1}(u) < z_0 | \mathcal{H}_0, z_0(u) = z_0\}$$

$$= \prod_{i=1}^{M-1} \text{Pr}\{z_i(u) < z_0 | \mathcal{H}_0, z_0(u) = z_0\}$$

$$= \prod_{i=1}^{M-1} \text{Pr}\{z_i(u) < z_0 | \mathcal{H}_0\}$$

$$= \prod_{i=1}^{M-1} \left[1 - \text{Q}\left(\sqrt{\frac{2}{N_0}} z_0\right)\right]$$

$$= \left[1 - \text{Q}\left(\sqrt{\frac{2}{N_0}} z_0\right)\right]^{M-1}$$

M-ary Orthogonal (coherent demod)

$$P(\mathcal{C}|\mathcal{H}_0) = \int_{-\infty}^{\infty} P(\mathcal{C}|\mathcal{H}_0, z_0(u) = z_0) f_{z_0(u)}(z_0|\mathcal{H}_0) dz_0$$

$$= \int_{-\infty}^{\infty} \left[1 - \mathrm{Q}\left(\sqrt{\frac{2}{N_0}} z_0\right)\right]^{M-1} \mathcal{N}_1(z_0; \sqrt{E_s}; N_0/2) dz_0$$

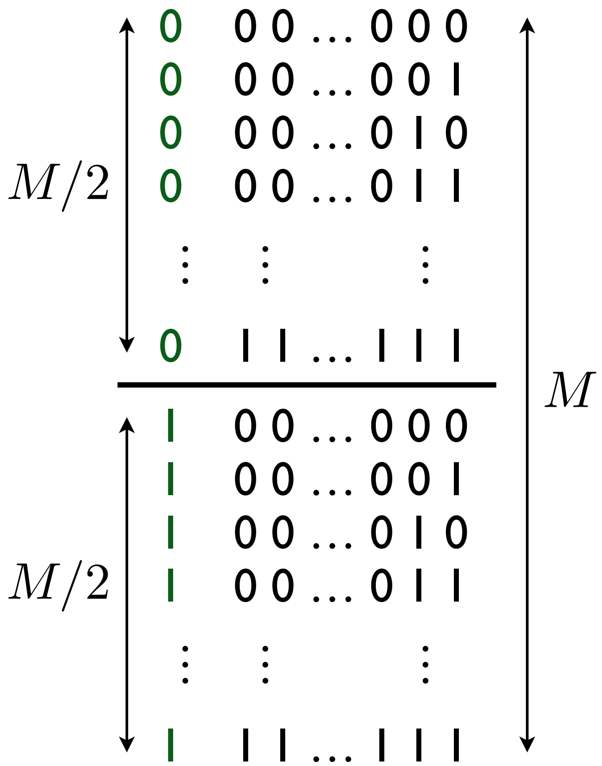
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{x+\sqrt{\frac{2E_b}{N_0}\log_2(M)}} \mathcal{N}_1(v; 0; 1) dv\right]^{M-1} \mathcal{N}_1(x; 0; 1) dx$$

$$= \int_{-\infty}^{\infty} \left[F\left(x + \sqrt{\frac{2E_b}{N_0}\log_2(M)}\right)\right]^{M-1} \mathcal{N}_1(x; 0; 1) dx$$

$F(x) =$ cdf of a standard Gaussian

$$P(\mathcal{E}) = P(E|\mathcal{H}_0) = 1 - \int_{-\infty}^{\infty} \left[F\left(x + \sqrt{\frac{2E_b}{N_0}\log_2(M)}\right)\right]^{M-1} \mathcal{N}_1(x; 0; 1) dx$$

M-ary Orthogonal (coherent demod)

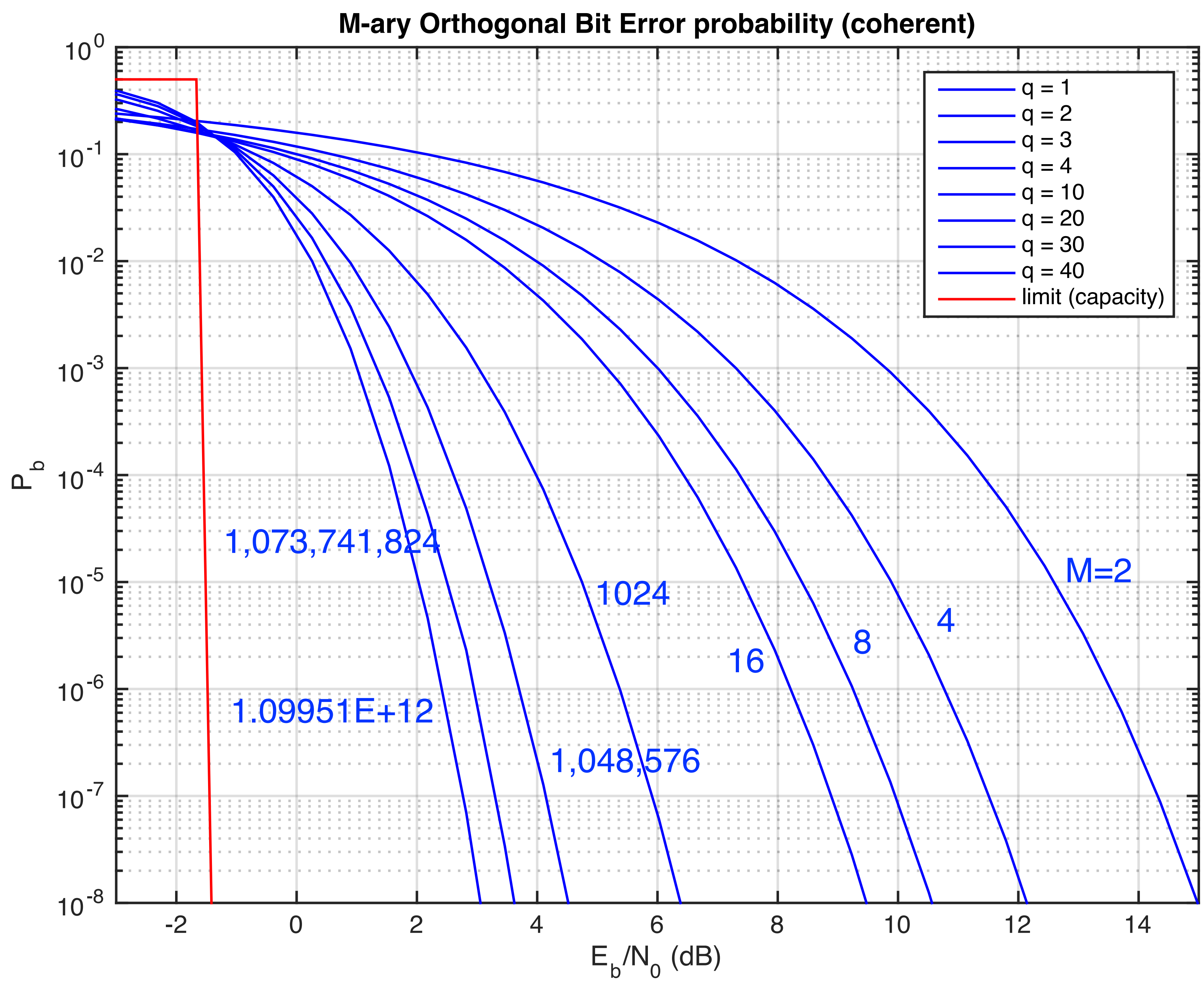


$$P(\mathcal{B}_i|\mathcal{E}) = \frac{(M/2)}{M-1}$$

$$P_b = \frac{M}{2(M-1)} P(\mathcal{E})$$

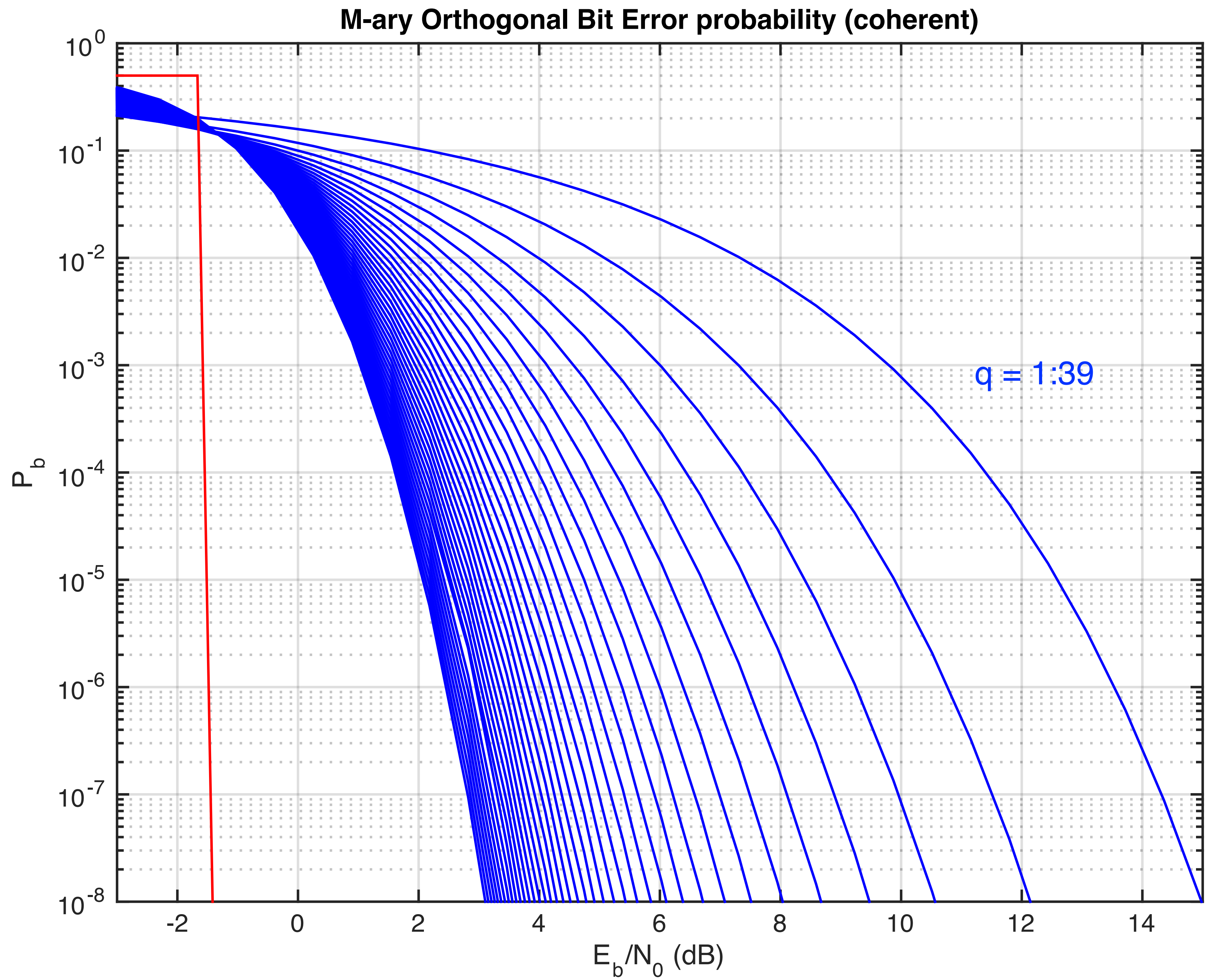
$$P(\mathcal{E}) = P(E|\mathcal{H}_0) = 1 - \int_{-\infty}^{\infty} \left[F\left(x + \sqrt{\frac{2E_b}{N_0}\log_2(M)} \right) \right]^{M-1} \mathcal{N}_1(x;0;1)dx$$

M-ary Orthogonal (coherent demod)



performance improves as M increases — opposite of QASK

M-ary Orthogonal (coherent demod)



performance improves as M increases — opposite of QASK

M-ary Orthogonal (coherent demod)

How far can we take this?

$$\lim_{M\to\infty} P_M(\mathcal{E}) = 1 - \lim_{M\to\infty} P_M(\mathcal{C})$$

$$= 1 - \exp\left[\lim_{M\to\infty} \ln\left(P_M(\mathcal{C})\right)\right]$$

$$\lim_{M\to\infty} \ln\left(P_M(\mathcal{C})\right) = \begin{cases} -\infty & \frac{E_b}{N_0} < \ln(2) \\ 0 & \frac{E_b}{N_0} \geq \ln(2) \end{cases}$$

(use L'Hopital's Rule)

M-ary Orthogonal (coherent demod)

How far can we take this?

$$\lim_{M\to\infty} P_M(\mathcal{E}) = \begin{cases} 1 & \frac{E_b}{N_0} < \ln(2) \\ 0 & \frac{E_b}{N_0} \geq \ln(2) \end{cases}$$

$$\eta_{\text{bits/dim}} = \frac{\log_2(M)}{M}$$

$$\lim_{M\to\infty} P_b(M) = \begin{cases} 1/2 & \frac{E_b}{N_0} < \ln(2) \\ 0 & \frac{E_b}{N_0} \geq \ln(2) \end{cases}$$

$$\lim_{M\to\infty} \eta_{\text{bits/dim}} = 0$$

ln(2) is a threshold on Eb/No for which perfect communication occurs

spectral efficiency (bps/Hz) goes to zero as M increases

We will see that this result shows that orthogonal modulation achieves Shannon Capacity for the AWGN as spectral efficiency goes to 0

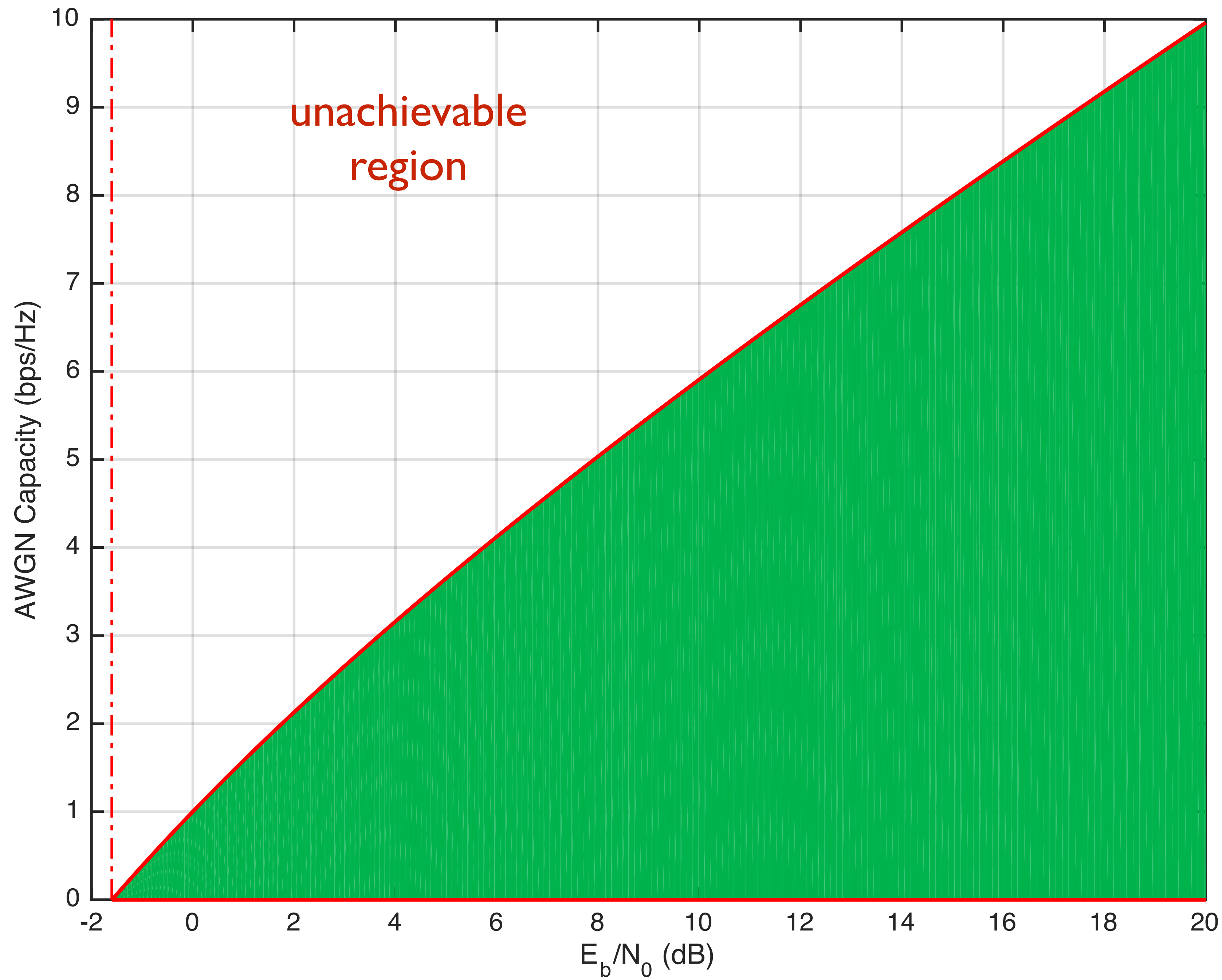
Also, at finite spectral efficiency, the capacity results will show that similar threshold results hold, but for larger values of Eb/No

Eb/No = -1.6 dB is the smallest value of Eb/No for reliable communications on the AWGN channel

M-ary Orthogonal (coherent demod)

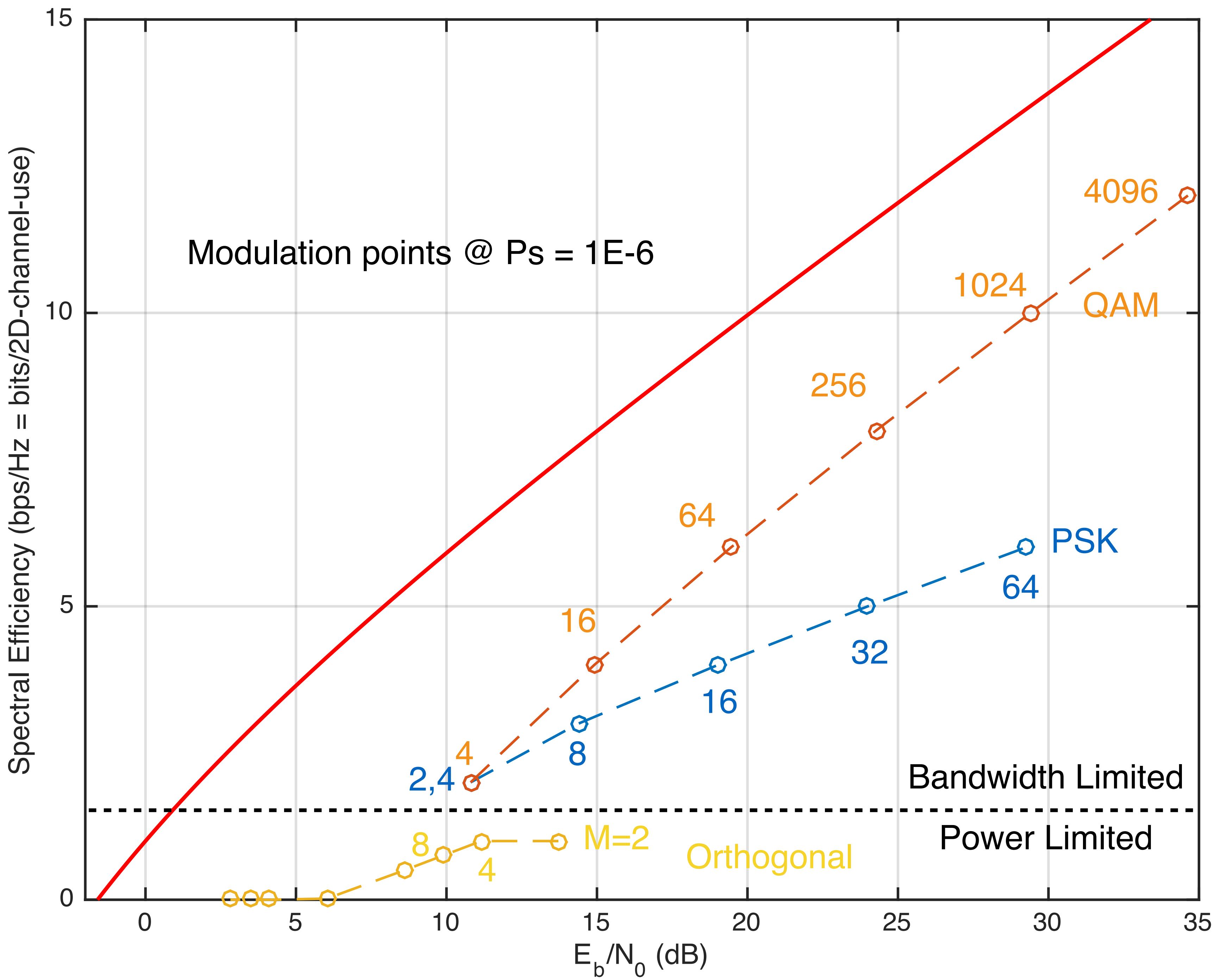
- Other "orthogonal-like" signal sets exhibit similar large M performance trends
  - Bi-orthogonal, simplex
- Result also occurs with phase non-coherent detection

AWGN Capacity (preview)



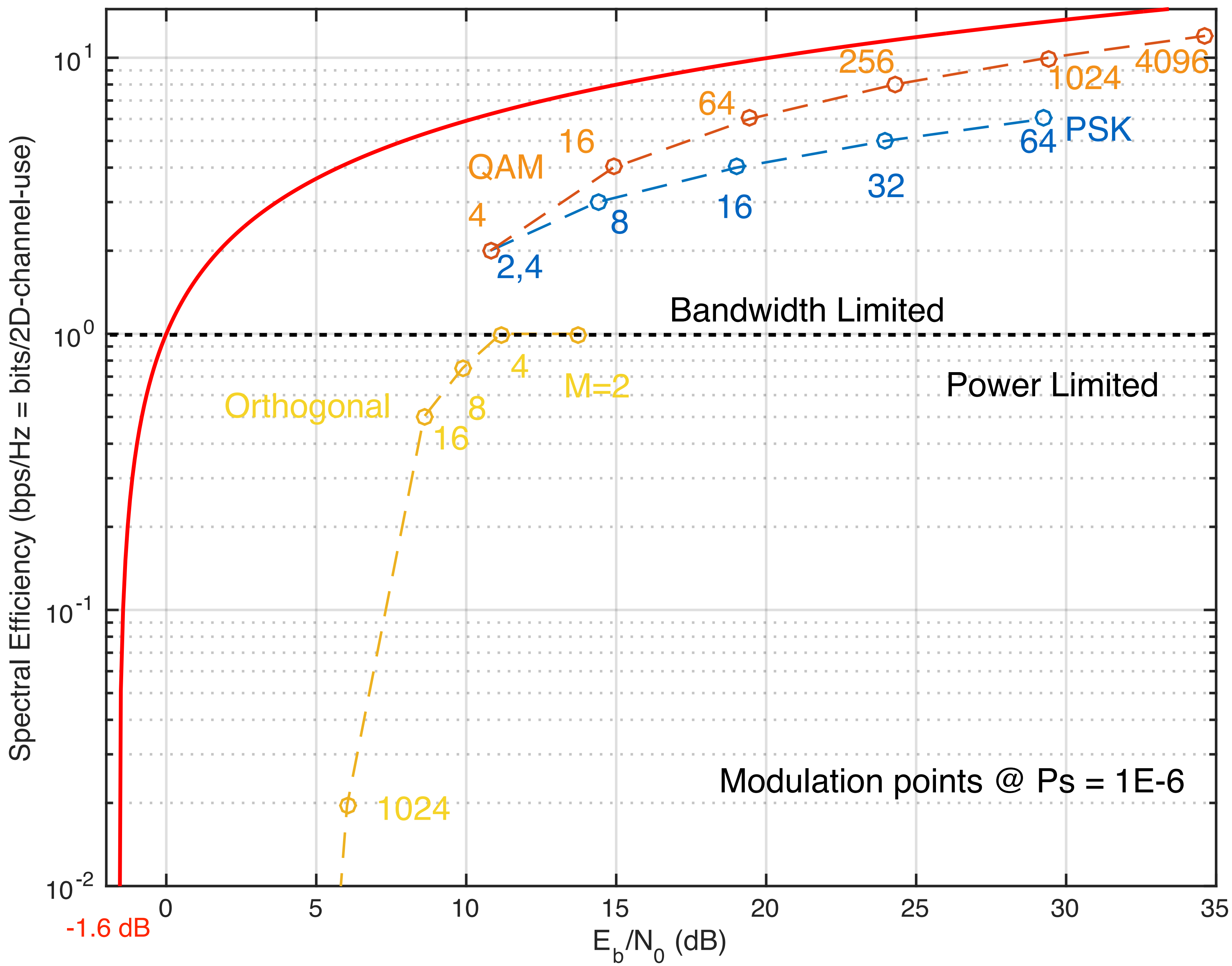
Eb/No = -1.6 dB is the smallest value of Eb/No for reliable communications on the AWGN channel

Modulation Comparison to Capacity



About 10 dB Eb/No gap to capacity for these uncoded QAM

Modulation Comparison to Capacity



About 10 dB Eb/No gap to capacity for orthogonal too

Detection/Demod Topics

- Maximum A Posteriori decision rule for vector-AWGN channel
- Exact performance for binary modulations
- Minimum distance decision rule for M-ary modulation over AWGN
- Performance bounds
  - *Performance of common M-ary modulations*
- Continuous time model
  - Likelihood functional, sufficient statistics
- Average and generalized likelihood
  - Phase non-coherent demodulation
  - Soft-out demodulation

Continuous Time AWGN Likelihood Functional

$$\mathcal{H}_m: \quad \mathbf{z} = \mathbf{s}_m + \mathbf{w} \qquad D \times 1$$

$$f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m) \equiv \exp\left(\frac{-\|\mathbf{z} - \mathbf{s}_m\|^2}{N_0}\right)$$

$$\equiv e^{-E_m/N_0} \exp\left(\frac{2\mathbf{s}_m^t \mathbf{z}}{N_0}\right)$$

vector observation — finite number of random variables

$$\mathcal{H}_m: \quad r(u,t) = s_m(t) + n(u,t) \qquad t \in \mathcal{T}$$

waveform observation — uncountably infinite random variables

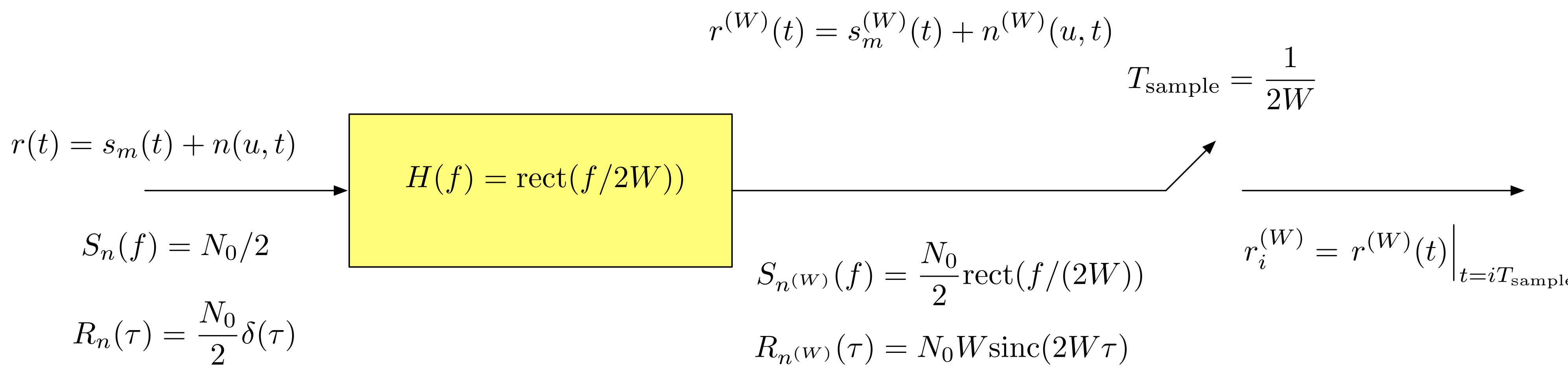
$$L(\boldsymbol{r}|\mathcal{H}_m) \equiv \exp\left(\frac{-1}{N_0}\left[\|\boldsymbol{s}_m\|^2 - 2\langle \boldsymbol{r}, \boldsymbol{s}_m\rangle\right]\right)$$

$$= \exp\left(\frac{-1}{N_0}\left[\int_{\mathcal{T}} s_m^2(t)dt - 2\int_{\mathcal{T}} r(t)s_m(t)dt\right]\right)$$

$$= e^{-E_m/N_0} \exp\left(\frac{2}{N_0}\int_{\mathcal{T}} r(t)s_m(t)dt\right)$$

AWGN
Continuous time likelihood functional
(replaces conditional pdf/pmf)

Continuous Time AWGN Likelihood Functional



$$\mathcal{H}_m : \quad \mathbf{r}^{(W)}(u) = \mathbf{s}_m^{(W)} + \mathbf{n}^{(W)}(u) \qquad (2WT) \times 1$$

$$\mathbf{n}^{(W)}(u) \sim \mathcal{N}_{2WT}\left(\cdot; \mathbf{0}; N_0 W \mathbf{I}\right)$$

$$f_{\mathbf{r}^{(W)}(u)}(\mathbf{r}^{(W)}|\mathcal{H}_m) \equiv \exp\left(\frac{-\|\mathbf{r}^{(W)} - \mathbf{s}_m^{(W)}\|^2}{2WN_0}\right)$$

$$\equiv e^{-\|\mathbf{s}_m^{(W)}\|^2/(2WN_0)} \exp\left(\frac{2\left[\mathbf{s}_m^{(W)}\right]^{\mathrm{t}}\mathbf{r}^{(W)}}{2WN_0}\right)$$

$$\frac{1}{2WN_0}\|\mathbf{s}_m^{(W)}\|^2 = \frac{1}{N_0}\sum_i \left(s_m^{(W)}\left(\frac{i}{2W}\right)\right)^2 \frac{1}{2W}$$

$$\frac{1}{2WN_0}\left[\mathbf{s}_m^{(W)}\right]^{\mathrm{t}}\mathbf{r}^{(W)} = \frac{1}{N_0}\sum_i r^{(W)}\left(\frac{i}{2W}\right) s_m^{(W)}\left(\frac{i}{2W}\right)\frac{1}{2W}$$

As W goes to infinity, this converges to the integral

Continuous Time AWGN Likelihood Functional

More rigorous development from a generalized Fourier Series expansion of the observed signal - Karhunen-Loeve Expansion

$$\mathcal{H}_m : \quad r(u,t) = s_m(t) + n(u,t) \qquad t \in \mathcal{T}$$

$$\mathcal{H}_m : \quad R(u,i) = S_m(i) + N(u,i) \qquad i = 1, 2, 3, \ldots$$

$$R(u,i) = \int_{\mathcal{T}} r(u,t)\phi_i(t)dt$$

$$S_m(i) = \int_{\mathcal{T}} s_m(t)\phi_i(t)dt$$

$$N(u,i) = \int_{\mathcal{T}} n(u,t)\phi_i(t)dt$$

Continuous Time AWGN Likelihood Functional

More rigorous development from a generalized Fourier Series expansion of the observed signal - Karhunen-Loeve Expansion

$$N(u,i) = \int_{\mathcal{T}} n(u,t)\phi_i(t)dt$$

$$\mathbb{E}\left\{N(u,i)\right\} = \int_{\mathcal{T}} \mathbb{E}\left\{n(u,t)\right\}\phi_i(t)dt = 0$$

$$\mathbb{E}\left\{N(u,i)N(u,k)\right\} = \int_{\mathcal{T}}\int_{\mathcal{T}} \phi_i(t_1)\mathbb{E}\left\{n(u,t_1)n(u,t_2)\right\}\phi_k(t_2)dt_1dt_2$$

$$= \int_{\mathcal{T}}\int_{\mathcal{T}} \phi_i(t_1)K_n(t_1,t_2)\phi_k(t_2)dt_1dt_2$$

Continuous Time AWGN Likelihood Functional

Karhunen-Loeve Expansion comes from solving

$$\int_{\mathcal{T}} K_n(t_1, t_2)\phi_k(t_2)dt_2 = \lambda_k \phi_k(t_1)$$

This implies that the noise coefficients are uncorrelated

$$\mathbb{E}\left\{N(u,i)N(u,k)\right\} = \lambda_k \delta[i-k]$$

For the generalized FS we have

$$\sum_i X(i)Y(i) = \int_{\mathcal{T}} x(t)y(t)dt$$

In the limiting case of AWGN

Any CONS works

$$\int_{\mathcal{T}} K_n(t_1, t_2)\phi_k(t_2)dt_2 = \int_{\mathcal{T}} \frac{N_0}{2}\delta(t_1 - t_2)\phi_k(t_2)dt_2 = \frac{N_0}{2}\phi_k(t_1)$$

Continuous Time AWGN Likelihood Functional

Summary of Karhunen-Loeve Expansion for AWGN Limiting Sase

$$N(u,i) = \int_{\mathcal{T}} n(u,t)\phi_i(t)dt$$

Any CONS

$$\mathbb{E}\left\{N(u,i)\right\} = 0$$

$$\mathbb{E}\left\{N(u,i)N(u,k)\right\} = \frac{N_0}{2}\delta[i-k]$$

Jointly Gaussian (iid) coefficients

Related Facts About Correlating AWGN

$$N_a(u) = \int_{\mathcal{T}} a(t) n(u,t) dt$$

$$N_b(u) = \int_{\mathcal{T}} b(t) n(u,t) dt$$

$$\mathbb{E}\left\{N_a(u)\right\} = \mathbb{E}\left\{N_b(u)\right\} = 0$$

$$\mathbb{E}\left\{N_a^2(u)\right\} = \frac{N_0}{2} \int_{\mathcal{T}} a^2(t) dt$$

$$\mathbb{E}\left\{N_b^2(u)\right\} = \frac{N_0}{2} \int_{\mathcal{T}} b^2(t) dt$$

$$\mathbb{E}\left\{N_a(u) N_b(u)\right\} = \frac{N_0}{2} \int_{\mathcal{T}} a(t) b(t) dt$$

Continuous Time AWGN Likelihood Functional

KL expansion for AWGN leads to Likelihood Functional

$$f(\{R(i)\}_{i=1}^{N}|\mathcal{H}_m) \equiv \exp\left(\frac{-1}{N_0}\sum_{i=1}^{N} S_m^2(i) + \frac{2}{N_0}\sum_{i=1}^{N} R(i)S_m(i)\right)$$

$$\lim_{N\to\infty} f(\{R(i)\}_{i=1}^{N}|\mathcal{H}_m) \equiv \exp\left(\frac{-1}{N_0}\int_{\mathcal{T}} s_m^2(t)dt + \frac{2}{N_0}\int_{\mathcal{T}} r(t)s_m(t)dt\right)$$

$$\text{CONS} = \{\phi_1(t), \phi_2(t), \ldots \phi_D(t)\} \bigcup \{\phi_{D+1}(t), \phi_{D+2}(t), \phi_{D+3}(t), \ldots\}$$

Orthonormal basis for signal space

orthonormal completion of first D functions

Continuous Time AWGN Likelihood Functional

KL expansion for AWGN leads to Likelihood Functional

$$f(\{R(i)\}_{i=1}^{D+k}|\mathcal{H}_m) = \mathcal{N}_{D+k}\left(\begin{bmatrix} R(1) \\ R(2) \\ \vdots \\ R(D) \\ R(D+1) \\ \vdots \\ R(D+k) \end{bmatrix}; \begin{bmatrix} S_m(1) \\ S_m(2) \\ \vdots \\ S_m(D) \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \begin{bmatrix} \frac{N_0}{2}\mathbf{I}_D & \mathbf{O} \\ \mathbf{O} & \frac{N_0}{2}\mathbf{I}_k \end{bmatrix}\right)$$

$$= \mathcal{N}_D\left(\begin{bmatrix} R(1) \\ R(2) \\ \vdots \\ R(D) \end{bmatrix}; \begin{bmatrix} S_m(1) \\ S_m(2) \\ \vdots \\ S_m(D) \end{bmatrix}; \frac{N_0}{2}\mathbf{I}_D\right) \mathcal{N}_k\left(\begin{bmatrix} R(D+1) \\ R(D+2) \\ \vdots \\ R(D+k) \end{bmatrix}; \mathbf{0}; \frac{N_0}{2}\mathbf{I}_k\right)$$

Continuous Time AWGN Likelihood Functional

KL expansion for AWGN leads to Likelihood Functional

$$f(\{R(i)\}_{i=1}^{D+k}|\mathcal{H}_m) \equiv \mathcal{N}_D\left(\begin{bmatrix} R(1) \\ R(2) \\ \vdots \\ R(D) \end{bmatrix}; \begin{bmatrix} S_m(1) \\ S_m(2) \\ \vdots \\ S_m(D) \end{bmatrix}; \frac{N_0}{2}\mathbf{I}_D\right)$$

$$\equiv \exp\left(\frac{-\|\mathbf{z}-\mathbf{s}_m\|^2}{N_0}\right)$$

$$\mathbf{z} = \begin{bmatrix} R(1) & R(2) & \cdots & R(D) \end{bmatrix}^{\mathrm{t}}$$

$$\mathbf{s}_m = \begin{bmatrix} S_m(1) & S_m(2) & \cdots & S_m(D) \end{bmatrix}^{\mathrm{t}}$$

Vector-AWGN Model is Equivalent to Processing Continuous AWGN observation optimally!

Continuous Time AWGN Likelihood Functional

$$\mathbf{z}^{\mathrm{t}}\mathbf{s}_m = \int_{\mathcal{T}} r(t)s_m(t)dt = \sum_{i=1}^{D} R(i)S_m(i)$$

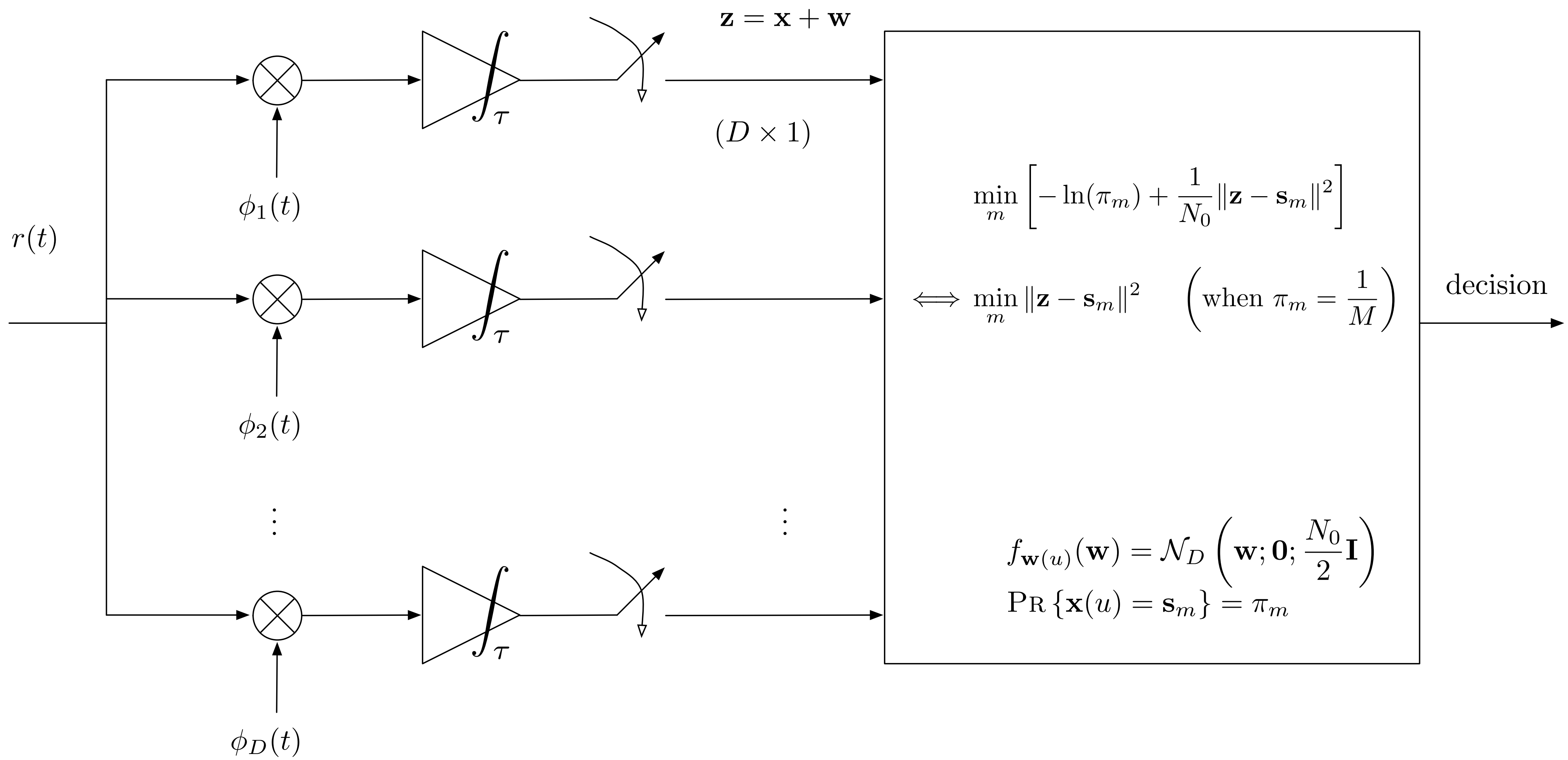
$$E_m = \int_{\mathcal{T}} s_m^2(t)dt = \sum_{i=1}^{D} S_m^2(i)$$

$$\begin{aligned}
L(\boldsymbol{r}|\mathcal{H}_m) &\equiv \exp\left(\frac{-1}{N_0}\left[\|\boldsymbol{s}_m\|^2 - 2\langle \boldsymbol{r}, \boldsymbol{s}_m\rangle\right]\right) \\
&= \exp\left(\frac{-1}{N_0}\left[\int_{\mathcal{T}} s_m^2(t)dt - 2\int_{\mathcal{T}} r(t)s_m(t)dt\right]\right) \\
&= e^{-E_m/N_0}\exp\left(\frac{2}{N_0}\int_{\mathcal{T}} r(t)s_m(t)dt\right) \\
&= e^{-E_m/N_0}\exp\left(\frac{2}{N_0}\mathbf{z}^{\mathrm{t}}\mathbf{s}_m\right)
\end{aligned}$$

Vector-AWGN Model is Equivalent to Processing Continuous AWGN observation optimally!

One-shot MAP Receiver in AWGN

(correlation to orthonormal basis)



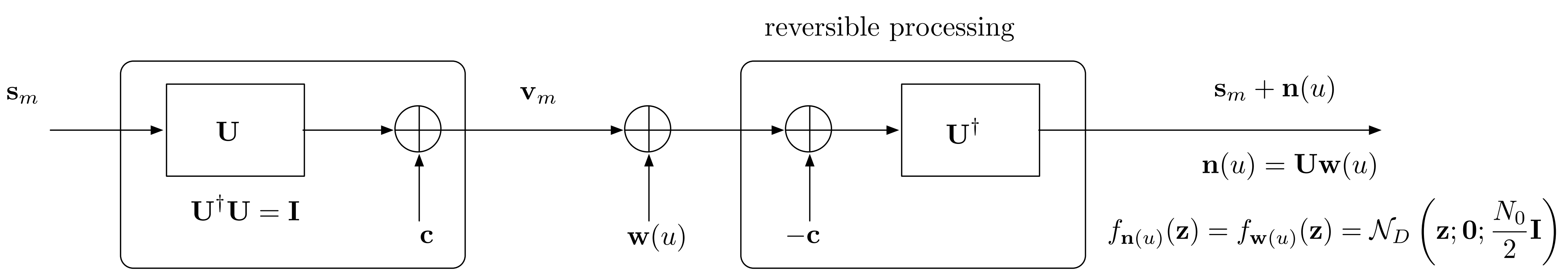
Vector-AWGN model is a model for the output of the bank of correlates to the orthonormal basis for the signal space

Sufficient Stats and Related Topics

- Note that the processing from r(t) to **z** is is not reversible
  - Cannot recover r(t) from **z**
- In general, how do we know that we are not throwing out useful information?
  - Notion of a ***set of sufficient statistics***
  - Engineering lingo (Wozencraft & Jacobs)
    - *Theorem of Reversibility*
    - *Theorem of Irrelevance*

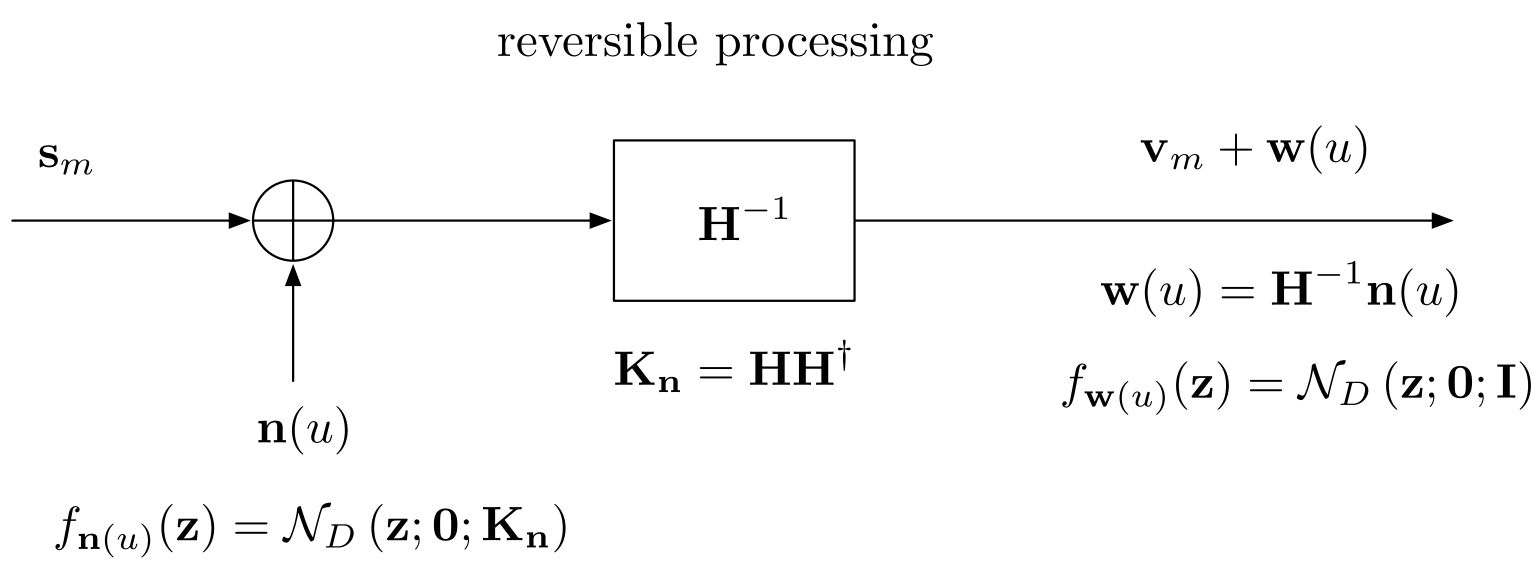
Theorem of Reversibility

- Any reversible (invertible) signal processing operation can be performed on the observation without losing information relevant to the decision problem



1. A nonzero centroid does not help performance and wastes energy
2. A unitary transformation of the signals (e.g., rotation, reflection) does not affect performance in AWGN

Theorem of Reversibility



Colored Noise MAP receiver realized using a whitening filter

Theorem of Irrelevance

Suppose we have two observations

$$f(\mathbf{z}_1, \mathbf{z}_2 | \mathcal{H}_m) = f(\mathbf{z}_1 | \mathbf{z}_2, \mathcal{H}_m) f(\mathbf{z}_2 | \mathcal{H}_m)$$

If the following holds

$$f(\mathbf{z}_1 | \mathbf{z}_2, \mathcal{H}_m) = f(\mathbf{z}_1 | \mathbf{z}_2) \qquad m = 0, 1, \ldots M-1$$

Then we say that z1 is irrelevant given z2 for the purposes of making a decision on the hypotheses

We have used this when dropping multiplicative terms and in discarding the AWGN outside the signal space

Set of Sufficient Statistics

A set of sufficient statistics for a hypothesis testing problem is a function of the observation that makes the observation irrelevant

$$
\begin{aligned}
f(\mathbf{z}, \mathbf{g}(\mathbf{z})|\mathcal{H}_m) &= f(\mathbf{z}|\mathbf{g}(\mathbf{z}), \mathcal{H}_m) f(\mathbf{g}(\mathbf{z})|\mathcal{H}_m) \\
&= f(\mathbf{z}|\mathbf{g}(\mathbf{z})) f(\mathbf{g}(\mathbf{z})|\mathcal{H}_m) \qquad m = 0, 1, \ldots M-1 \\
&\equiv f(\mathbf{g}(\mathbf{z})|\mathcal{H}_m)
\end{aligned}
$$

Examples:

$\{\mathbf{z}^{\mathrm{t}}\mathbf{s}_m\}_{m=0}^{M-1}$ is a set of sufficient stats for the vector AWGN channel

$\left\{\int_{\mathcal{T}} r(t) s_m(t) dt\right\}_{m=0}^{M-1}$ is a set of sufficient stats for the AWGN channel

$\left\{\int_{\mathcal{T}} r(t) \phi_i(t) dt\right\}_{i=0}^{D}$ is a set of sufficient stats for the AWGN channel

Set of Sufficient Statistics

If you start with the likelihood (functional) and you simplify to the only hypothesis-dependent terms that are a function of the observation, then these are a set of sufficient stats

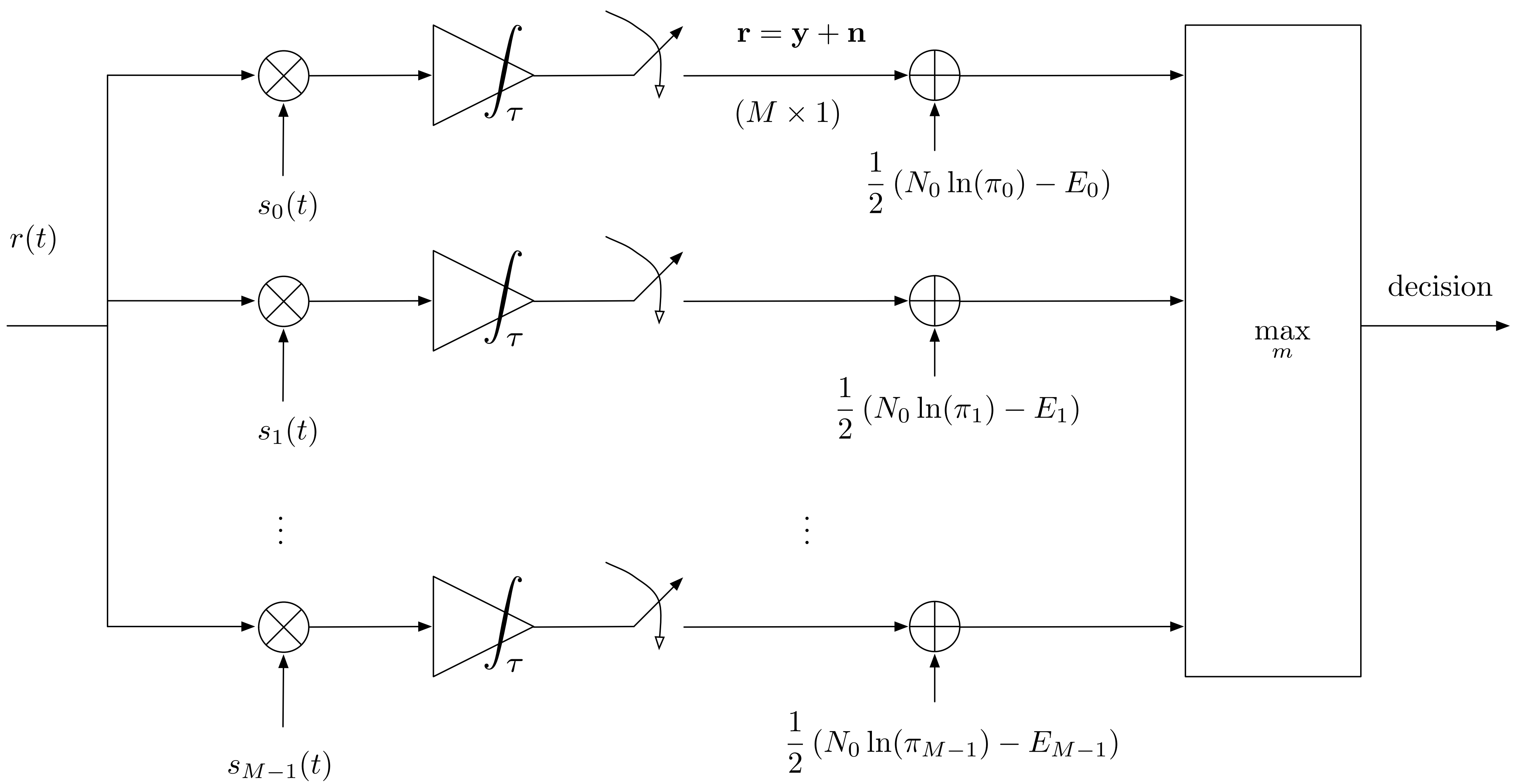
Whenever you have a set of sufficient statistics, they can be treated as an equivalent observation and the hypothesis testing problem can be reformulated using the equivalent observation

Example: We reformulated the waveform AWGN channel problem in terms of the equivalent vector model which we now see is a set of sufficient stats

Why not use the other set of sufficient stats from previous slide?

One-shot MAP Receiver in AWGN

(correlation to signals)



Requires M >= D correlators

One-shot MAP Receiver in AWGN

Post-correlator model for processing of previous slide

$$\mathcal{H}_m : \quad \mathbf{r}(u) = \mathbf{v}_m + \mathbf{n}(u)$$

$$\mathbf{v}_m = \begin{bmatrix} \langle \boldsymbol{s}_m, \boldsymbol{s}_0 \rangle & \langle \boldsymbol{s}_m, \boldsymbol{s}_1 \rangle & \langle \boldsymbol{s}_m, \boldsymbol{s}_2 \rangle & \cdots & \langle \boldsymbol{s}_m, \boldsymbol{s}_{M-1} \rangle \end{bmatrix}^{\mathrm{t}}$$

$$\mathbf{m}_{\mathbf{n}} = \mathbf{0}$$

$$\mathbf{K}_{\mathbf{n}} = \frac{N_0}{2} \begin{bmatrix} \langle \boldsymbol{s}_0, \boldsymbol{s}_0 \rangle & \langle \boldsymbol{s}_0, \boldsymbol{s}_1 \rangle & \langle \boldsymbol{s}_0, \boldsymbol{s}_2 \rangle & \dots & \langle \boldsymbol{s}_0, \boldsymbol{s}_{M-1} \rangle \\ \langle \boldsymbol{s}_1, \boldsymbol{s}_0 \rangle & \langle \boldsymbol{s}_1, \boldsymbol{s}_1 \rangle & \langle \boldsymbol{s}_1, \boldsymbol{s}_2 \rangle & \dots & \langle \boldsymbol{s}_1, \boldsymbol{s}_{M-1} \rangle \\ \langle \boldsymbol{s}_1, \boldsymbol{s}_0 \rangle & \langle \boldsymbol{s}_1, \boldsymbol{s}_1 \rangle & \langle \boldsymbol{s}_1, \boldsymbol{s}_2 \rangle & \dots & \langle \boldsymbol{s}_1, \boldsymbol{s}_{M-1} \rangle \\ \vdots & & & \ddots & \vdots \\ \langle \boldsymbol{s}_{M-1}, \boldsymbol{s}_0 \rangle & \langle \boldsymbol{s}_{M-1}, \boldsymbol{s}_1 \rangle & \langle \boldsymbol{s}_{M-1}, \boldsymbol{s}_2 \rangle & \dots & \langle \boldsymbol{s}_{M-1}, \boldsymbol{s}_{M-1} \rangle \end{bmatrix}$$

This matrix of inner products is called the Gramian of the signal set

One-shot MAP Receiver in AWGN

(correlation to signals)

- This processing is not preferred because
  - More correlates than needs (high complexity)
  - Noise vector covariance matrix will have rank D which means it is singular unless D=M
- For orthogonal signaling, the two approaches are the same!

This post-correlator model illustrates that the performance in AWGN is completely determined by the Gramian of the signal set — i.e., the inner products between signals

Complex BB CT Likelihood Functional

Recall:

$$\mathcal{H}_m: \quad r(u,t) = s_m(t) + n(u,t) \quad t \in \mathcal{T} \qquad \text{(narrowband)}$$

$$\mathcal{H}_m: \quad \bar{r}(u,t) = \bar{s}_m(t) + \bar{n}(u,t) \quad t \in \mathcal{T} \qquad \text{(complex BB)}$$

I and Q components of complex BB equivalent AWGN are each AWGN processes that are independent

$$\langle \boldsymbol{r}, \boldsymbol{s}_m \rangle = \int_{\mathcal{T}} r(t) s_m(t) dt = \Re\left\{ \langle \bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}_m \rangle \right\} = \Re\left\{ \int_{\mathcal{T}} r(t) s_m^*(t) dt \right\}$$

Complex BB CT Likelihood Functional

$$
\begin{aligned}
L(\boldsymbol{r}|\mathcal{H}_m) &\equiv \exp\left(\frac{-1}{N_0}\left[\|\boldsymbol{s}_m\|^2 - 2\langle \boldsymbol{r}, \boldsymbol{s}_m\rangle\right]\right) \\
&= \exp\left(\frac{-1}{N_0}\left[\int_{\mathcal{T}} s_m^2(t)dt - 2\int_{\mathcal{T}} r(t)s_m(t)dt\right]\right) \\
&= e^{-E_m/N_0}\exp\left(\frac{2}{N_0}\int_{\mathcal{T}} r(t)s_m(t)dt\right) \\
&= \exp\left(\frac{-1}{N_0}\left[\int_{\mathcal{T}} |\bar{s}_m(t)|^2 dt - 2\Re\left\{\textstyle\int_{\mathcal{T}} r(t)s_m^*(t)dt\right\}\right]\right) \\
&= L(\bar{\boldsymbol{r}}|\mathcal{H}_m)
\end{aligned}
$$

(narrowband)

(complex BB)

Detection of a Digital Sequence

- For the PSD, we considered a sequence of digital symbols sent through the channel

memoryless (nonlinear) modulation

$$\overline{x}(u,t) = \sum_k \bar{s}_{X_k(u)}(t - kT)$$

$$X_k(u) \in \{0, 1, \ldots M-1\} \quad \text{(independent)}$$

$$\bar{s}_m(t) (\text{lasts} \leq T \text{ seconds})$$

e.g., FSK

linear (QASK) modulation

$$\overline{x}(u,t) = \sum_k \overline{X}_k(u) p(t - kT)$$

$$\overline{X}_k(u) \sim \text{independent, distributed over QASK constellation}$$

- So far we have considered only the "one shot" detection problem
  - Let's use the continuous time likelihood functional to solve this sequence detection problem

Detection of a Digital Sequence

memoryless (nonlinear) modulation

$$\bar{x}(u,t) = \sum_k \bar{s}_{X_k(u)}(t-kT)$$

$$X_k(u) \in \{0, 1, \ldots M-1\} \quad \text{(independent)}$$

$$\bar{s}_m(t) (\text{lasts} \leq T \text{ seconds})$$

$$L(\bar{\boldsymbol{r}} | \{X_k(u) = a_k\}_k) = \exp\left(\frac{-1}{N_0} \left[\int_{\mathcal{T}} |\bar{x}(u,t;\mathbf{a})|^2 dt - 2\Re\left\{ \int_{\mathcal{T}} \bar{r}(t) \bar{x}^*(u,t;\mathbf{a}) dt \right\} \right] \right)$$

$$= \exp\left(\frac{-1}{N_0} \left[\sum_k \int_{kT}^{(k+1)T} |\bar{s}_{a_k}(t)|^2 dt - 2\Re\left\{ \sum_k \int_{kT}^{(k+1)T} \bar{r}(t) \bar{s}^*_{a_k}(t) dt \right\} \right] \right)$$

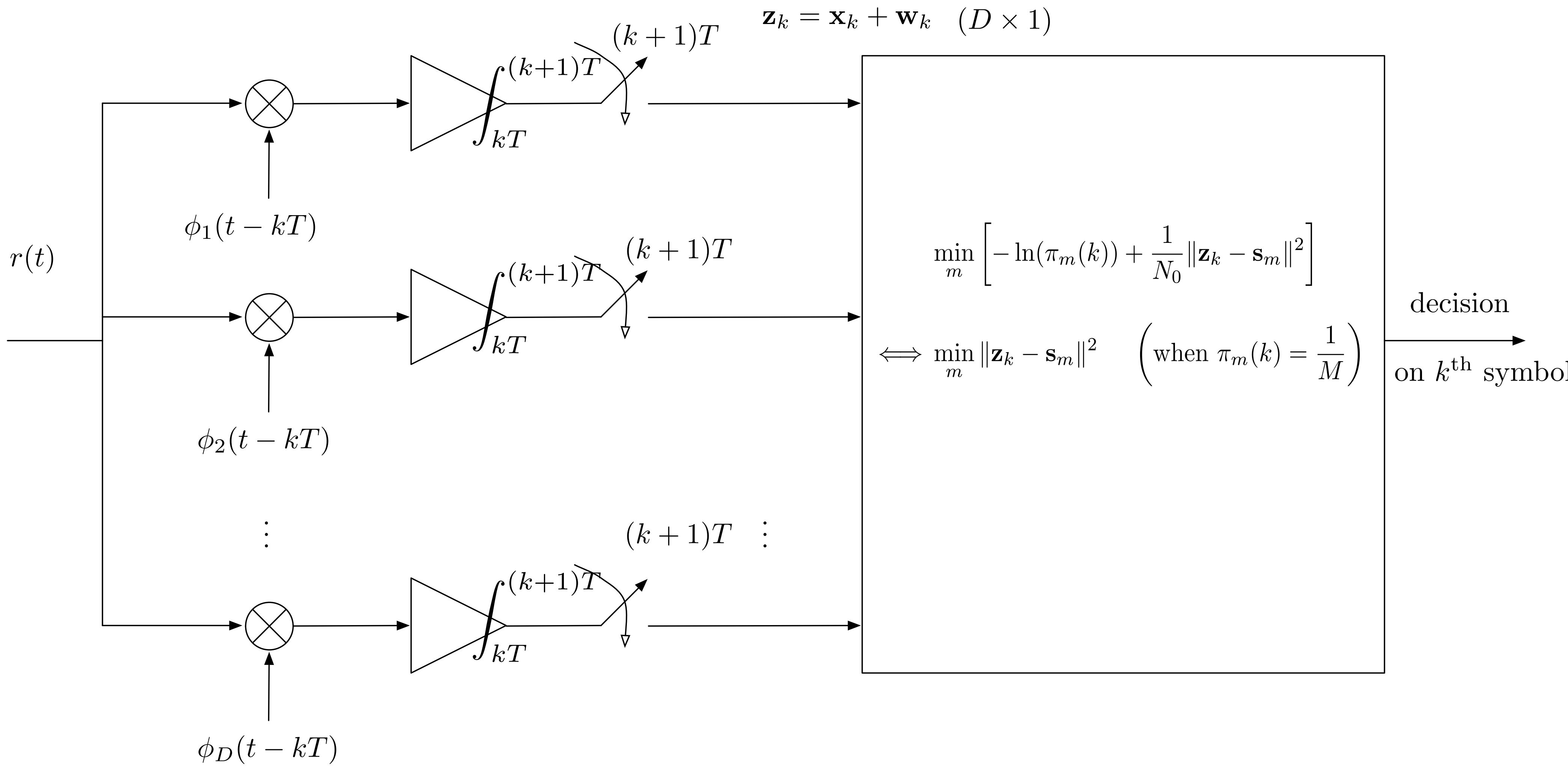
$$= \prod_k \exp\left(\frac{-1}{N_0} \left[\int_{kT}^{(k+1)T} |\bar{s}_{a_k}(t)|^2 dt - 2\Re\left\{ \int_{kT}^{(k+1)T} \bar{r}(t) \bar{s}^*_{a_k}(t) dt \right\} \right] \right)$$

$$= \prod_k L(\bar{\boldsymbol{r}}_k | X_k(u) = a_k)$$

For independent modulation symbols, the likelihood functional factors and the optimal processing is to repeat the one-shot MAP detector each symbol time

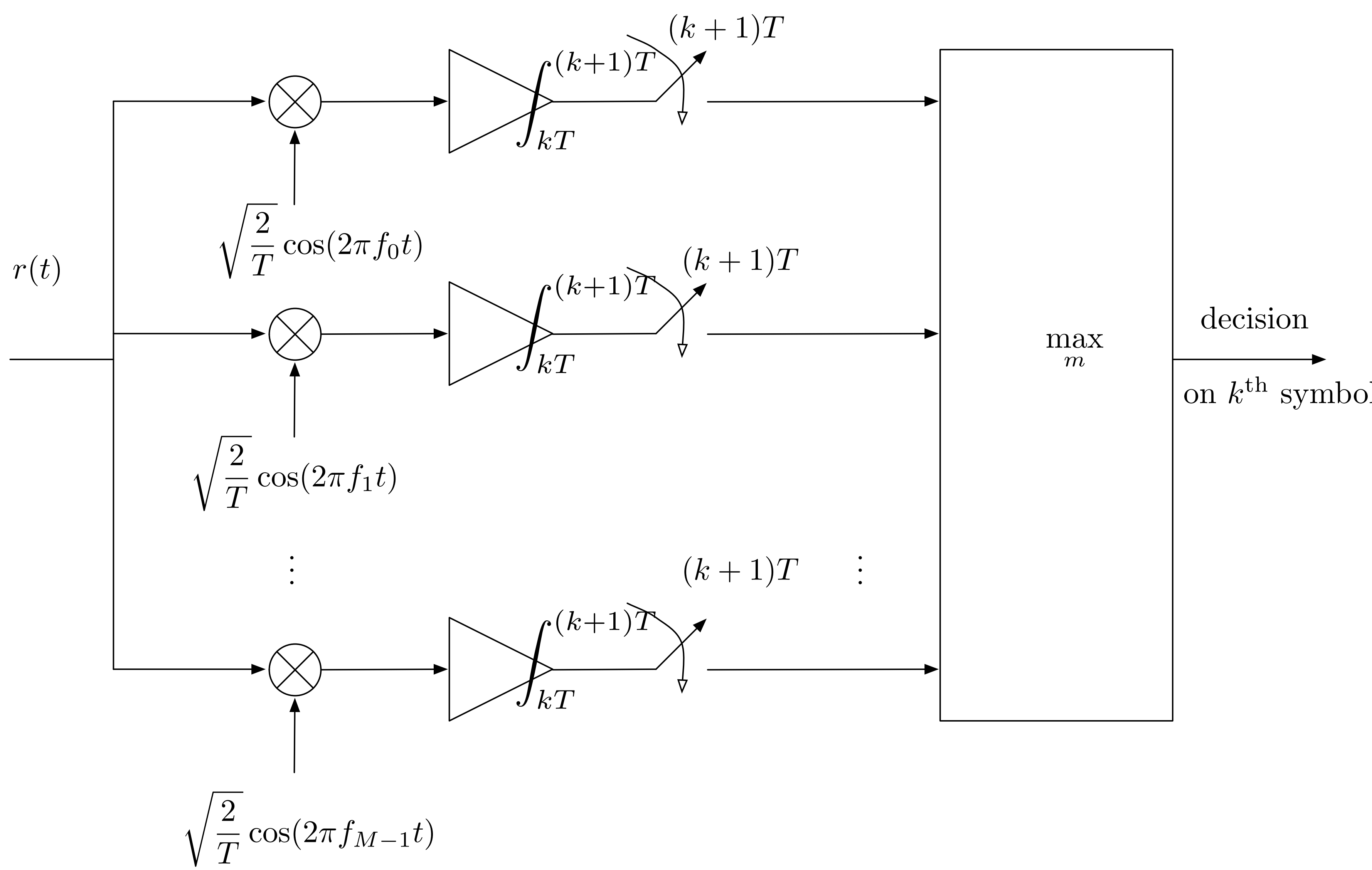
MAP Receiver in AWGN

(correlation to orthonormal basis)



Can think of this as just resetting the one-shot detector and repeating each symbol time

Example: MFSK Orthogonal

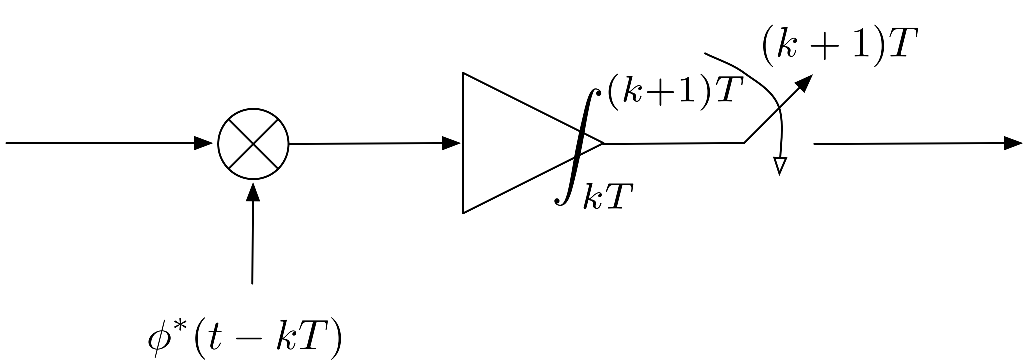


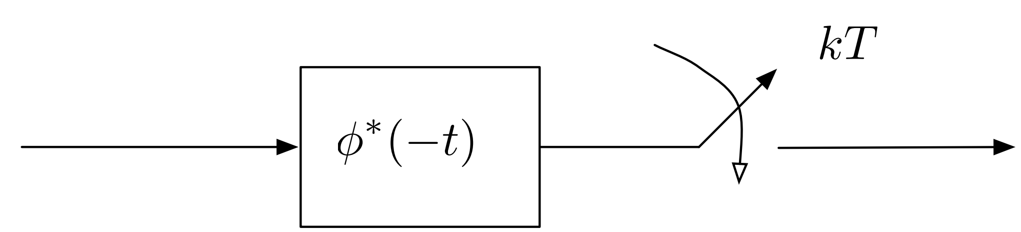
Correlation vs Matched Filter

$$x(t) * v(t) = \int x(\tau) v(t-\tau) d\tau$$

$$x(t) * v^*(-t) = \int x(\tau) v^*(\tau - t) d\tau$$

$$x(t) * v^*(-t)\big|_{t=kT} = \int x(\tau) v^*(\tau - kT) d\tau$$

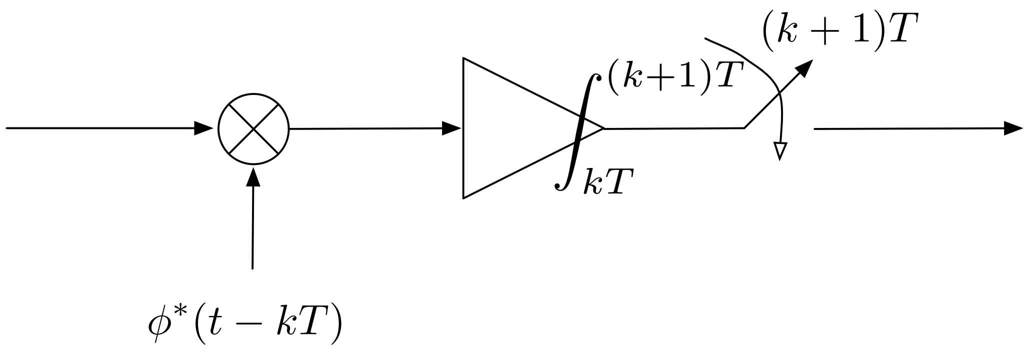




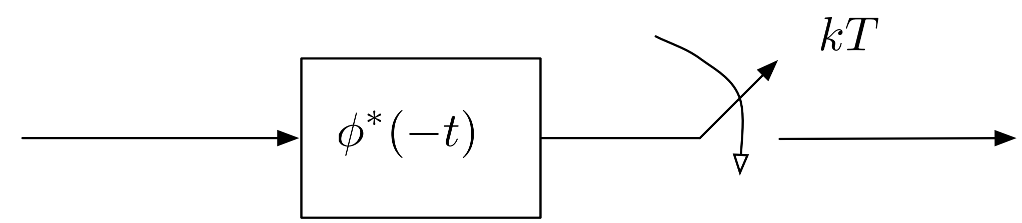
correlator ⟷ matched filter

Correlation vs Matched Filter

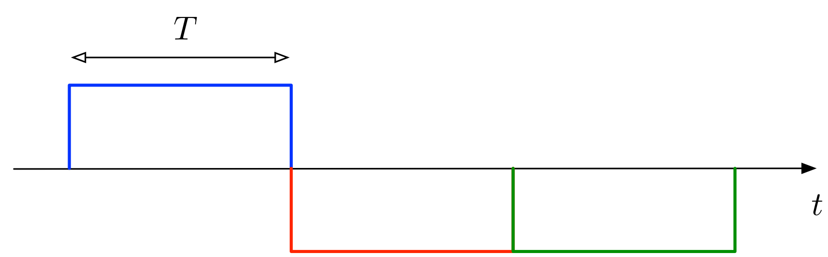
correlator



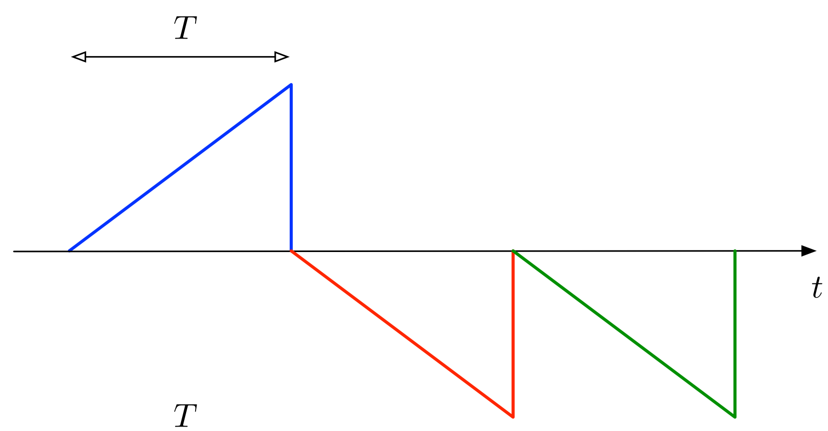
matched filter



example input with rect pulse

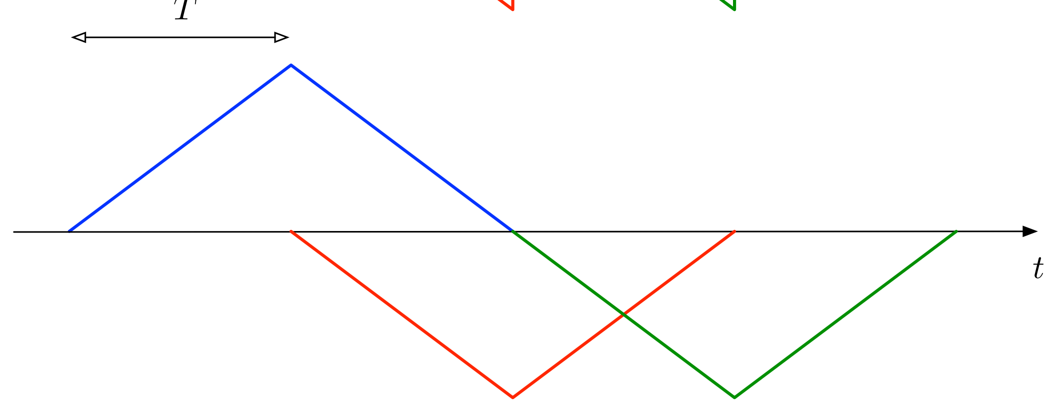


example correlator output



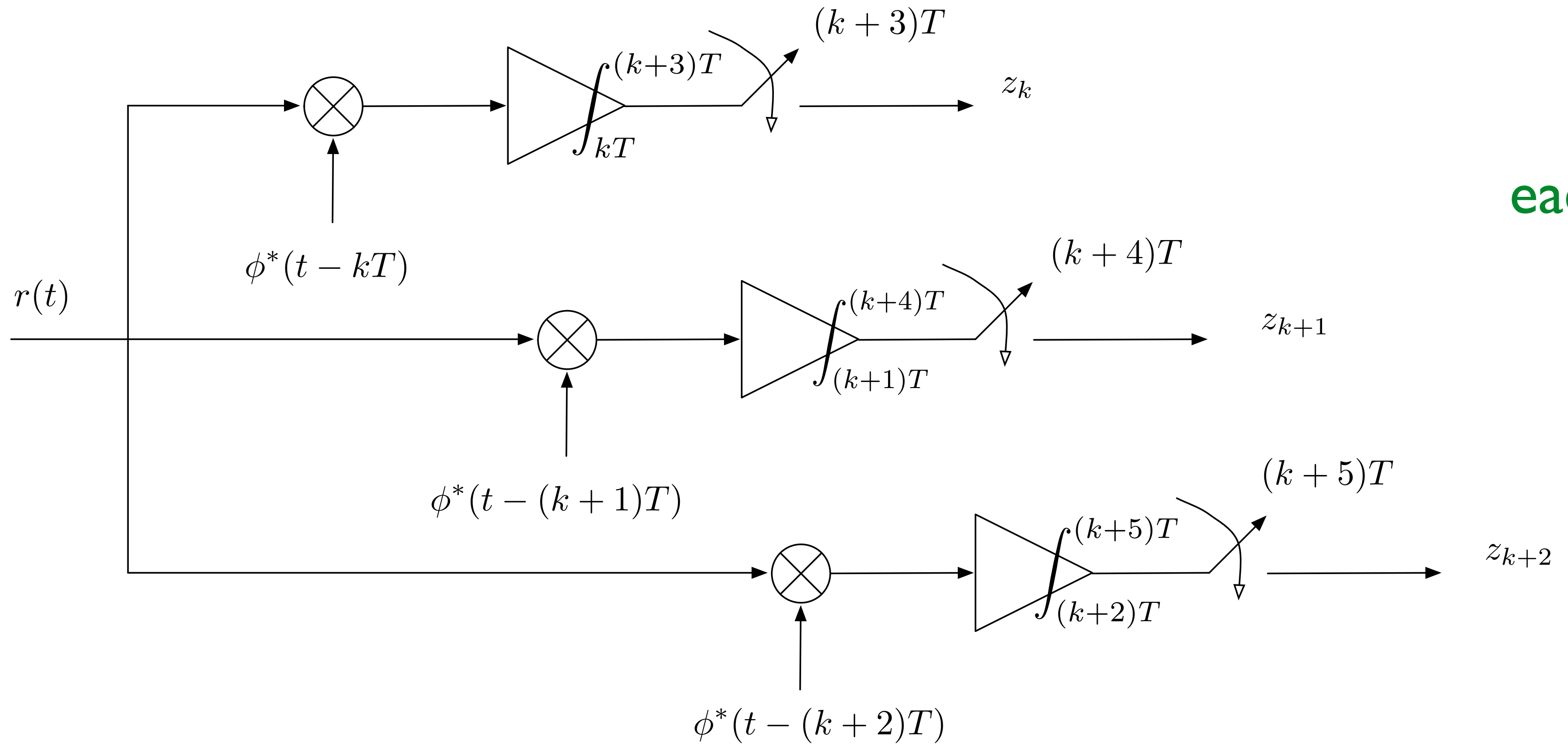
no noise is shown

example matched-filter output



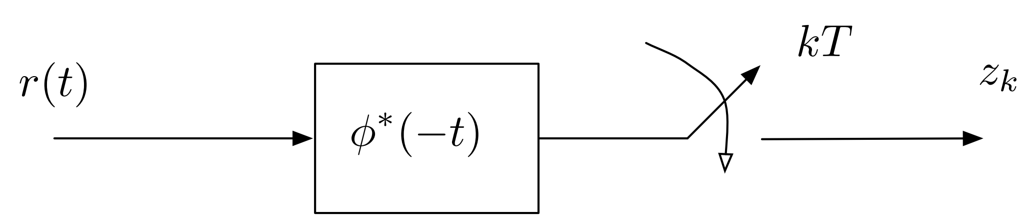
Correlation vs Matched Filter

bank of correlates for signal that last 3T



each correlation takes 3T to complete, can be reused after that

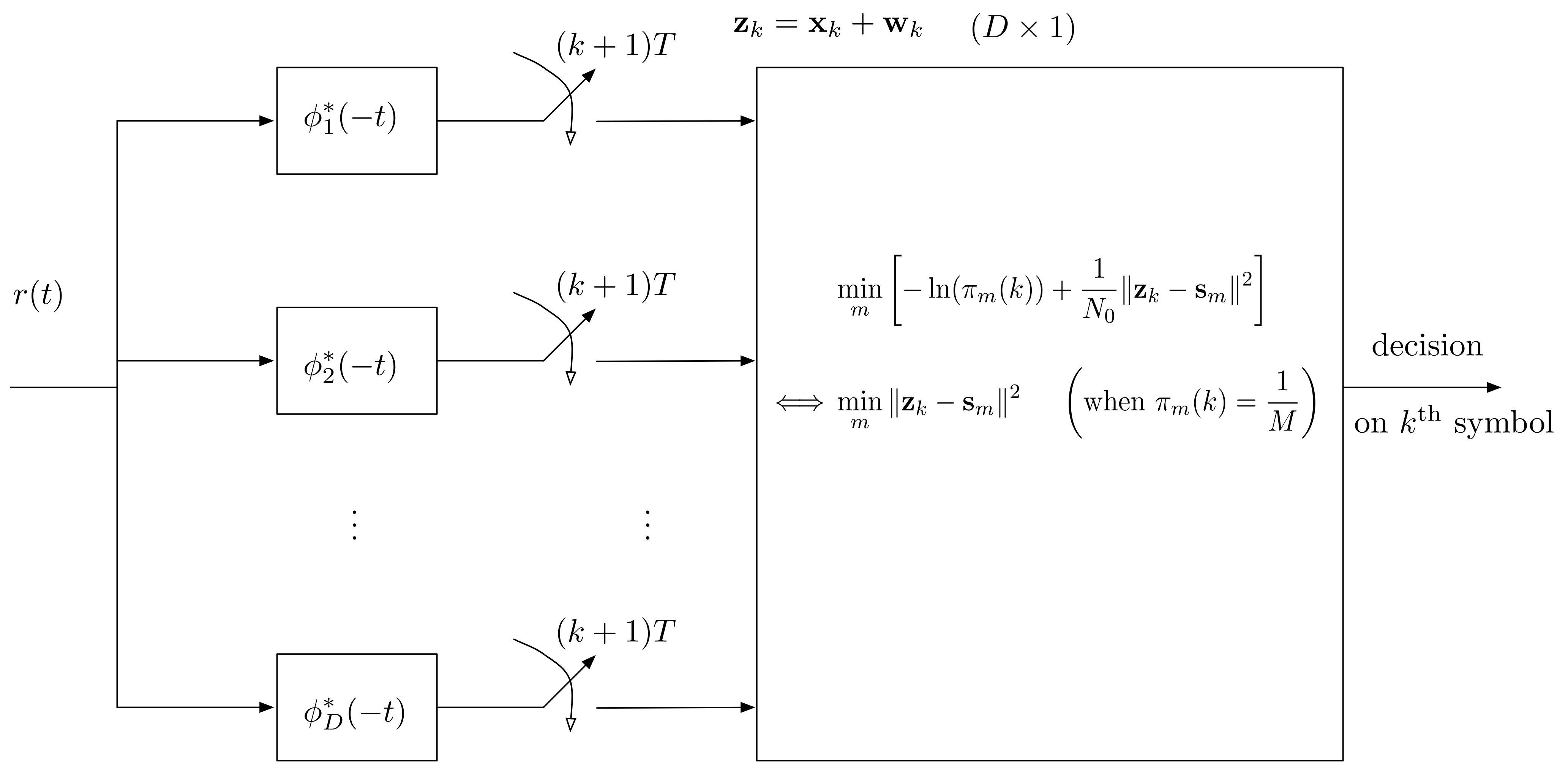
matched filter



single matched-filter required even if signal phi lasts multiple symbol times

MAP Receiver in AWGN

(matched-filters to orthonormal basis)



correlator form is common with rect-pulses and called an "integrate and dump"

Detection of a Digital Sequence - QASK

linear modulation

(consider arbitrary pulse shape)

$$\overline{x}(u,t) = \sum_k \overline{X}_k(u) p(t-kT)$$

$\overline{X}_k(u) \sim$ independent, distributed over QASK constellation

$$L(\bar{\boldsymbol{r}}|\{\bar{X}_k(u)=\bar{a}_k\}_k) = \exp\left(\frac{-1}{N_0}\left[\int_{\mathcal{T}} |\bar{x}(u,t;\bar{\mathbf{a}})|^2 dt - 2\Re\left\{\int_{\mathcal{T}} \bar{r}(t)\bar{x}^*(u,t;\bar{\mathbf{a}})dt\right\}\right]\right)$$

$$= \exp\left(\frac{-1}{N_0}\left[\int_{\mathcal{T}} |\bar{x}(u,t;\bar{\mathbf{a}})|^2 dt\right]\right)\exp\left(\frac{-1}{N_0}\left[2\Re\left\{\int_{\mathcal{T}} \bar{r}(t)\sum_k \bar{a}_k^* p^*(t-kT)dt\right\}\right]\right)$$

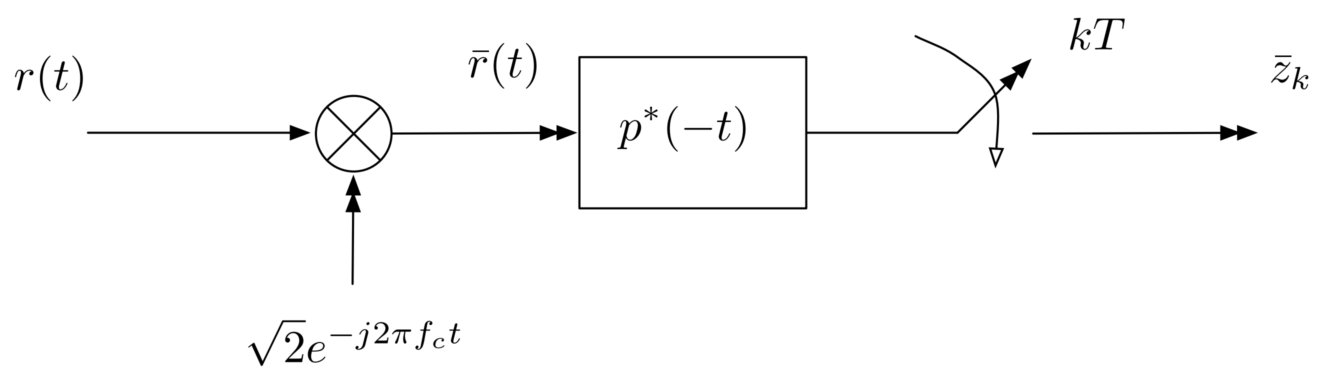
$$= \exp\left(\frac{-1}{N_0}\left[\int_{\mathcal{T}} |\bar{x}(u,t;\bar{\mathbf{a}})|^2 dt\right]\right)\exp\left(\frac{-1}{N_0}\left[2\Re\left\{\sum_k \bar{a}_k^* \int_{\mathcal{T}} \bar{r}(t)p^*(t-kT)dt\right\}\right]\right)$$

$$\bar{z}_k = \int \bar{r}(t)p^*(t-kT)dt$$

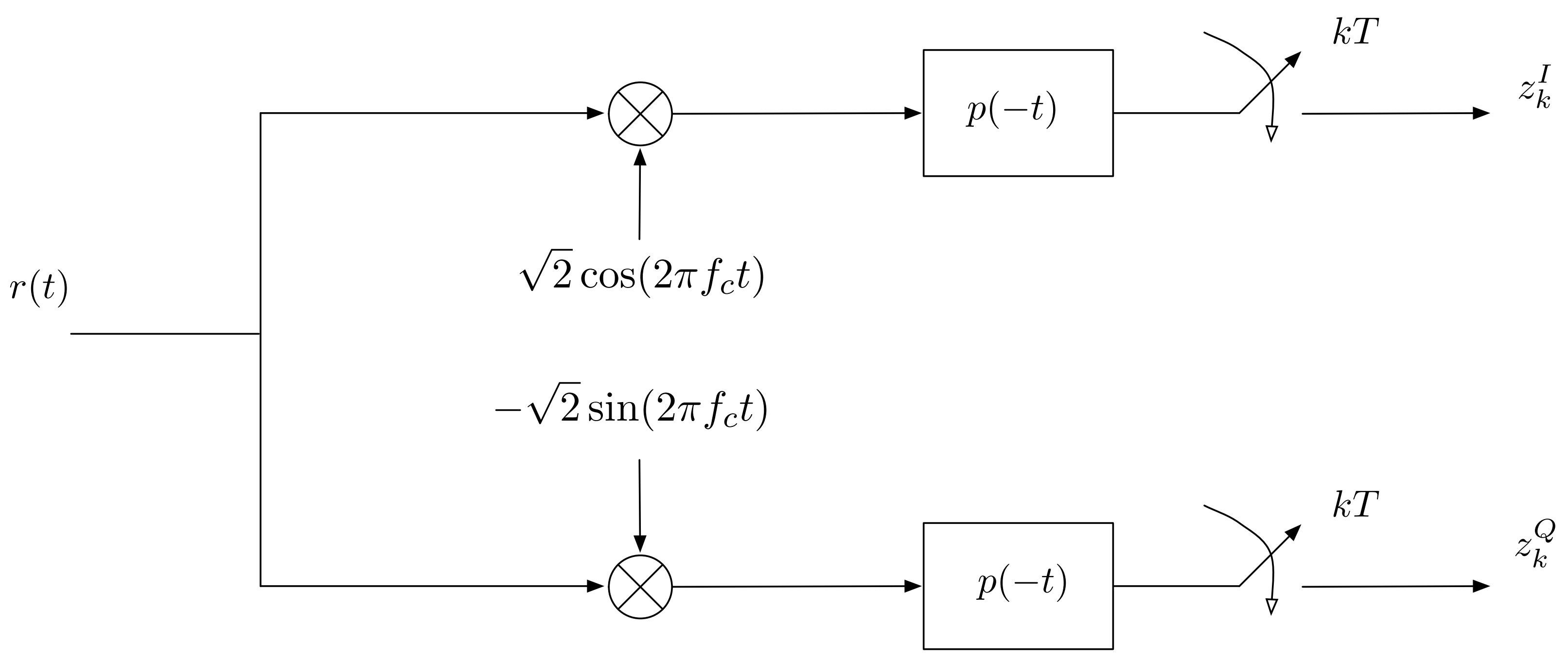
$\{\bar{z}_k\}_k$ is a set of sufficient statistics

complex baseband matched filter outputs

Detection of a Digital Sequence - QASK

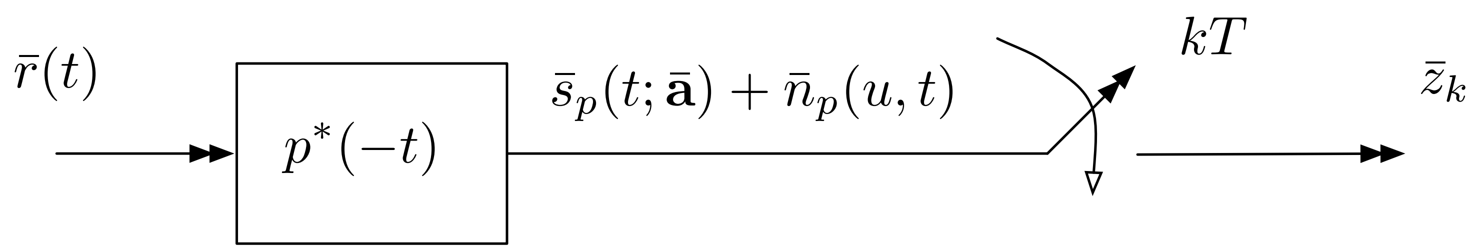


complex baseband



narrowband signal
processing for real pulse

Detection of a Digital Sequence - QASK



$$\bar{s}_p(t;\bar{\mathbf{a}}) = \left[\sum_i \bar{a}_i p(t-iT)\right] * p^*(-t)$$

$$= \left[\sum_i \bar{a}_i \delta(t-iT)\right] * p(t) * p^*(-t)$$

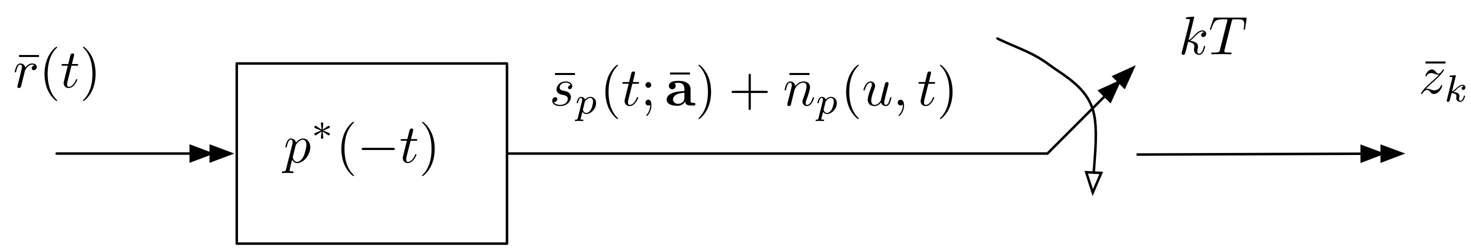
$$= \left[\sum_i \bar{a}_i \delta(t-iT)\right] * R_p(t)$$

$$= \sum_i \bar{a}_i R_p(t-iT)$$

$$\bar{n}_p(u,t) = \bar{n}(u,t) * p^*(-t)$$

$$R_{\bar{n}_p}(\tau) = N_0 R_p(\tau)$$

Detection of a Digital Sequence - QASK

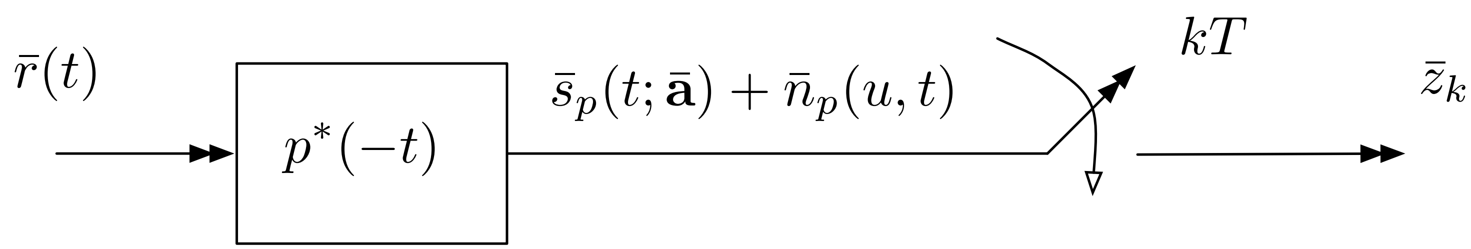


$$\bar{z}_k(u) = [\bar{s}_p(t;\bar{\mathbf{a}}) + \bar{n}_p(u,t)]|_{t=kT}$$

$$= \sum_i \bar{a}_i R_p((k-i)T) + \bar{n}_k(u)$$

$$\mathbb{E}\left\{\bar{n}_{k+m}(u)\bar{n}_k^*(u)\right\} = N_0 R_p(mT)$$

Detection of a Digital Sequence - QASK



Nyquist Condition on pulse shape

$$R_p(mT) = p(t) * p^*(-t)|_{t=mT} = C\delta[m]$$

When p(t) satisfies the Nyquist condition

- There is no inter symbol interference
- The noise at the output of the MF is CC-AWGN

$$\bar{z}_k(u) = \bar{a}_k + \bar{w}_k(u)$$

$$\mathbb{E}\left\{\bar{w}_{k+m}(u)\bar{w}_k^*(u)\right\} = N_0\delta[m]$$

The Nyquist condition is satisfied for any pulse that is zero outside of [0,T]

Can a pulse that lasts longer than T satisfy this?

Nyquist Condition for No ISI

time domain

$$R_p(mT) = p(t) * p^*(-t)|_{t=mT} = C\delta[m]$$

$$\mathbb{FT}\left\{R_p(t)\right\} = |P(f)|^2$$

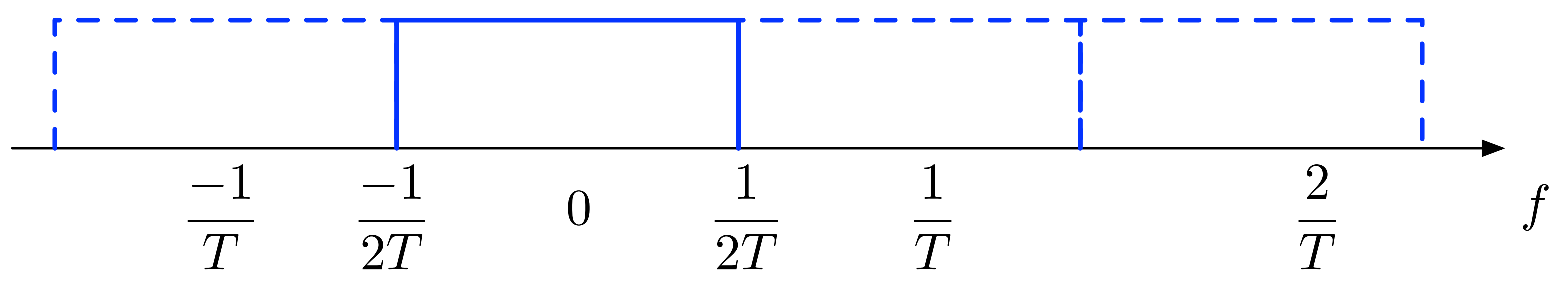
frequency domain

$$\frac{1}{T}\sum_k |P(f - k/T)|^2 = C$$

folded-spectrum should be flat

Nyquist Condition on pulse shape (freq domain)

Nyquist Pulse Shape: sinc()



$$|P(f)|^2 = T\text{rect}(fT)$$

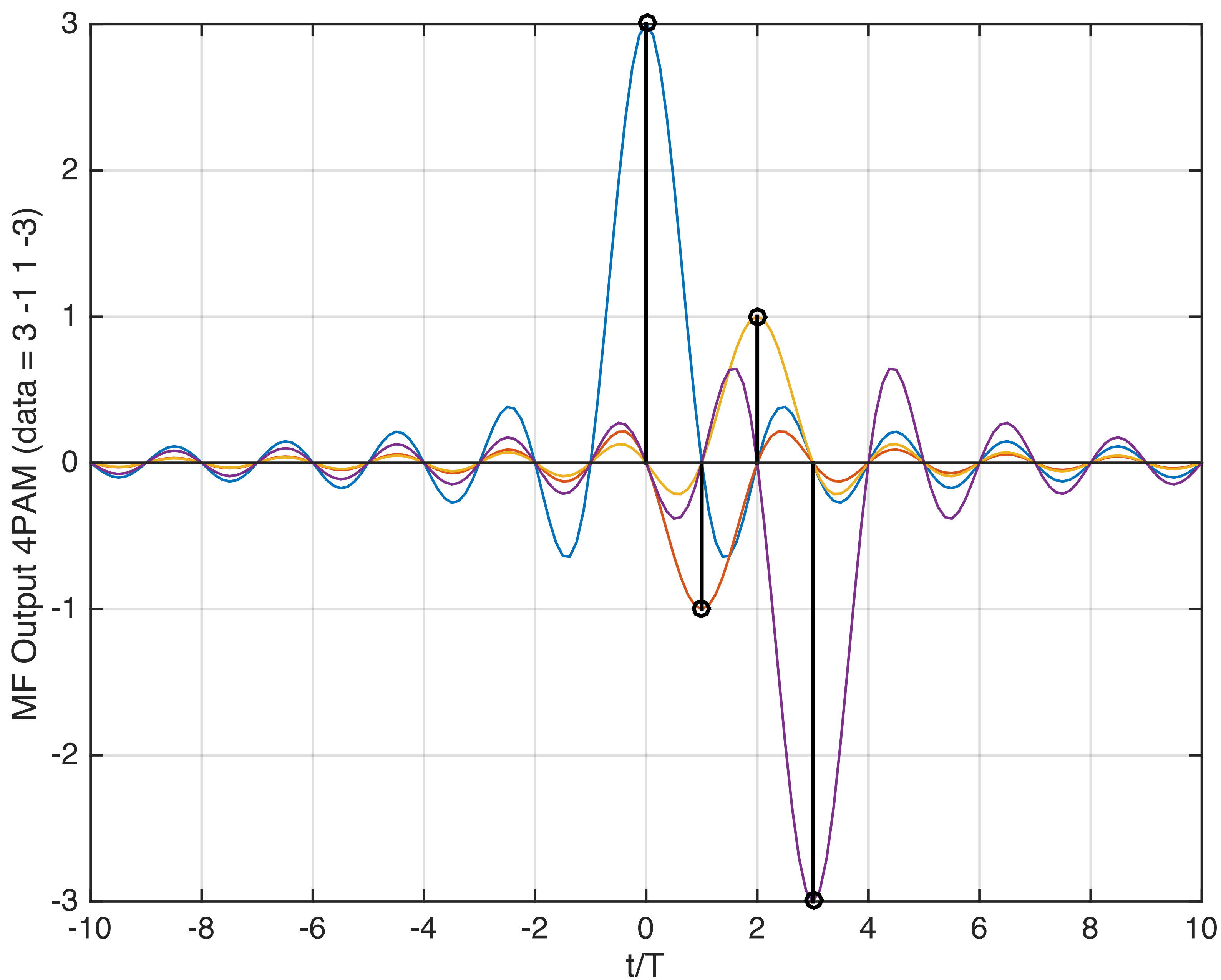
$$R_p(t) = \text{sinc}(t/T)$$

$$P(f) = \sqrt{T}\text{rect}(fT)$$

$$p(t) = \frac{1}{\sqrt{T}}\text{sinc}(t/T)$$

Nyquist Condition on pulse shape (freq domain)

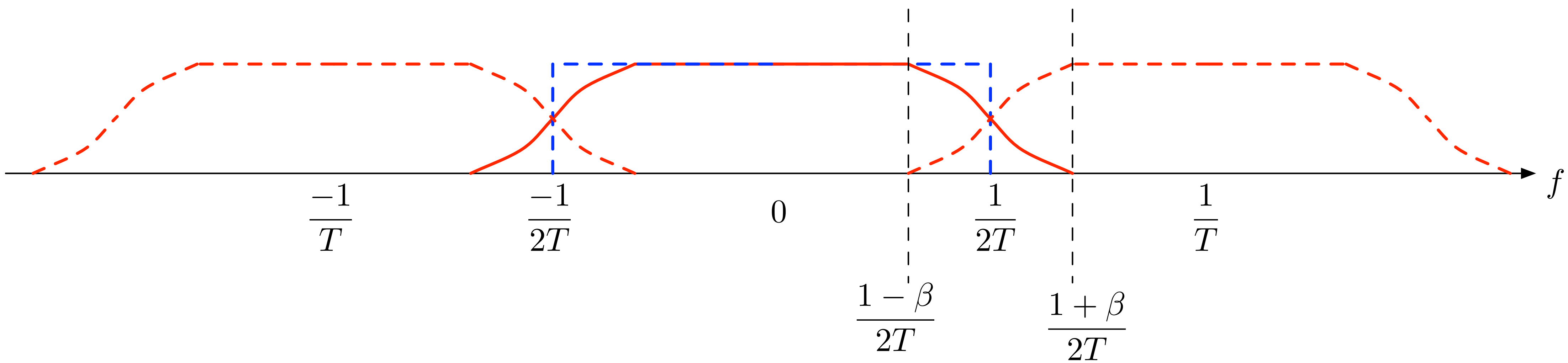
Nyquist Pulse Shape: sinc()



- pulse correlation falls off like 1/t
  - sensitive to sample timing error

sample waveform for 4PAM with sinc() pulse shape (matched filter output)

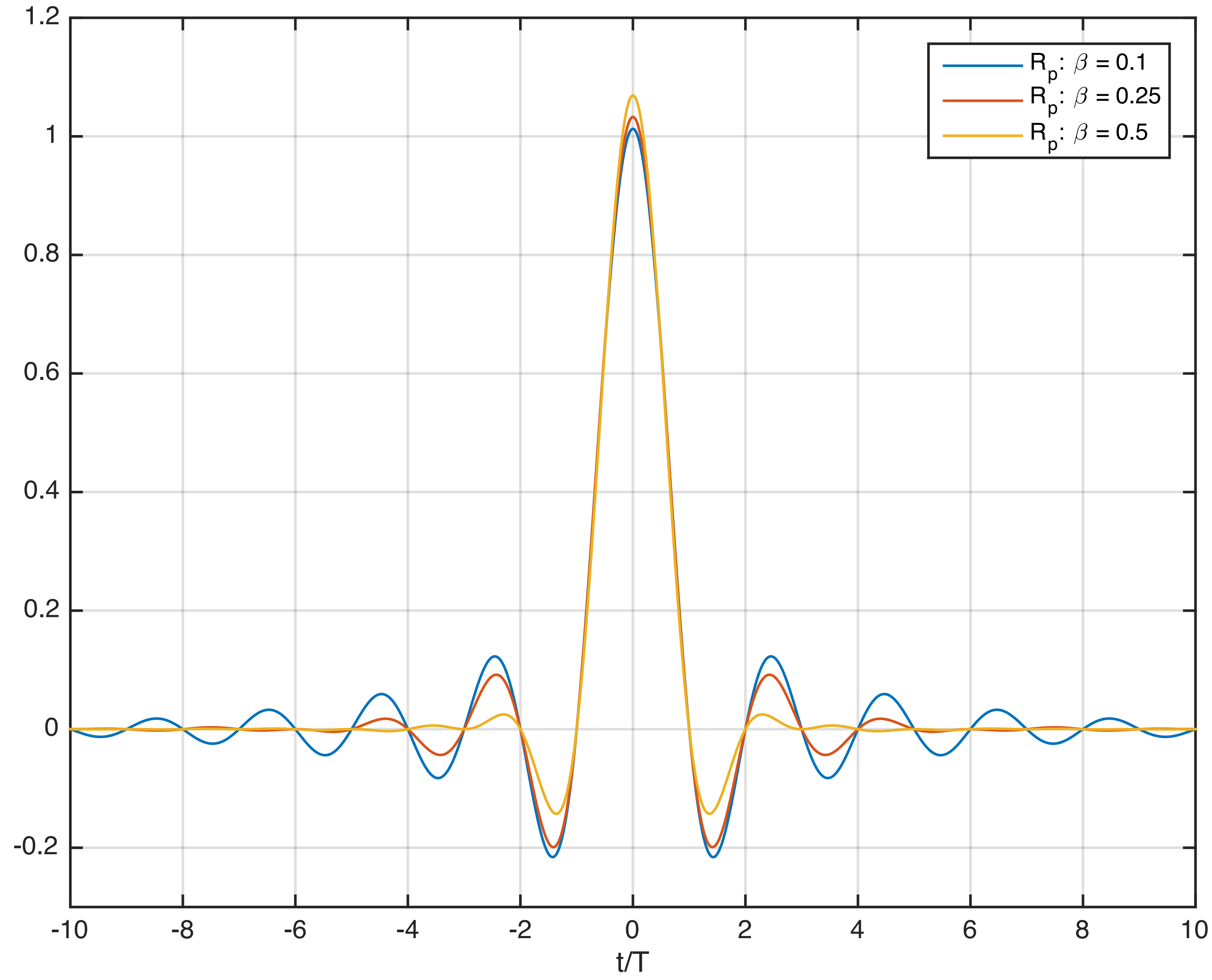
Nyquist Pulse Shape: Raised Cosine Spectrum



$$|P(f)|^2 = \begin{cases} T & |f| < \frac{1-\beta}{2T} \\ \frac{T}{2}\left[1 - \sin\left(\frac{\pi T}{\beta}\left(f - \frac{1}{2T}\right)\right)\right] & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases}$$

$\beta \in [0, 1)$ fractional excess bandwidth

Nyquist Pulse Shape: Raised Cosine Pulse Rp



note that pulse correlation passes through zero at integer multiples of T

Nyquist Pulse Shape: Raised Cosine Spectrum

"raised cosine pulse" $$R_p(t) = \text{sinc}(t/T)\frac{\cos(\beta\pi t/T)}{1-4\beta^2(t/T)^2}$$

$$P(f) = |P(f)|$$

"root raised cosine pulse" $$p(t) = 4\beta\frac{\cos((1+\beta)\pi t/T) + \sin((1-\beta)\pi(t/T))\,[4\beta(t/T)]^{-1}}{\pi\sqrt{T}\,[1-(4\beta t/T)^2]}$$

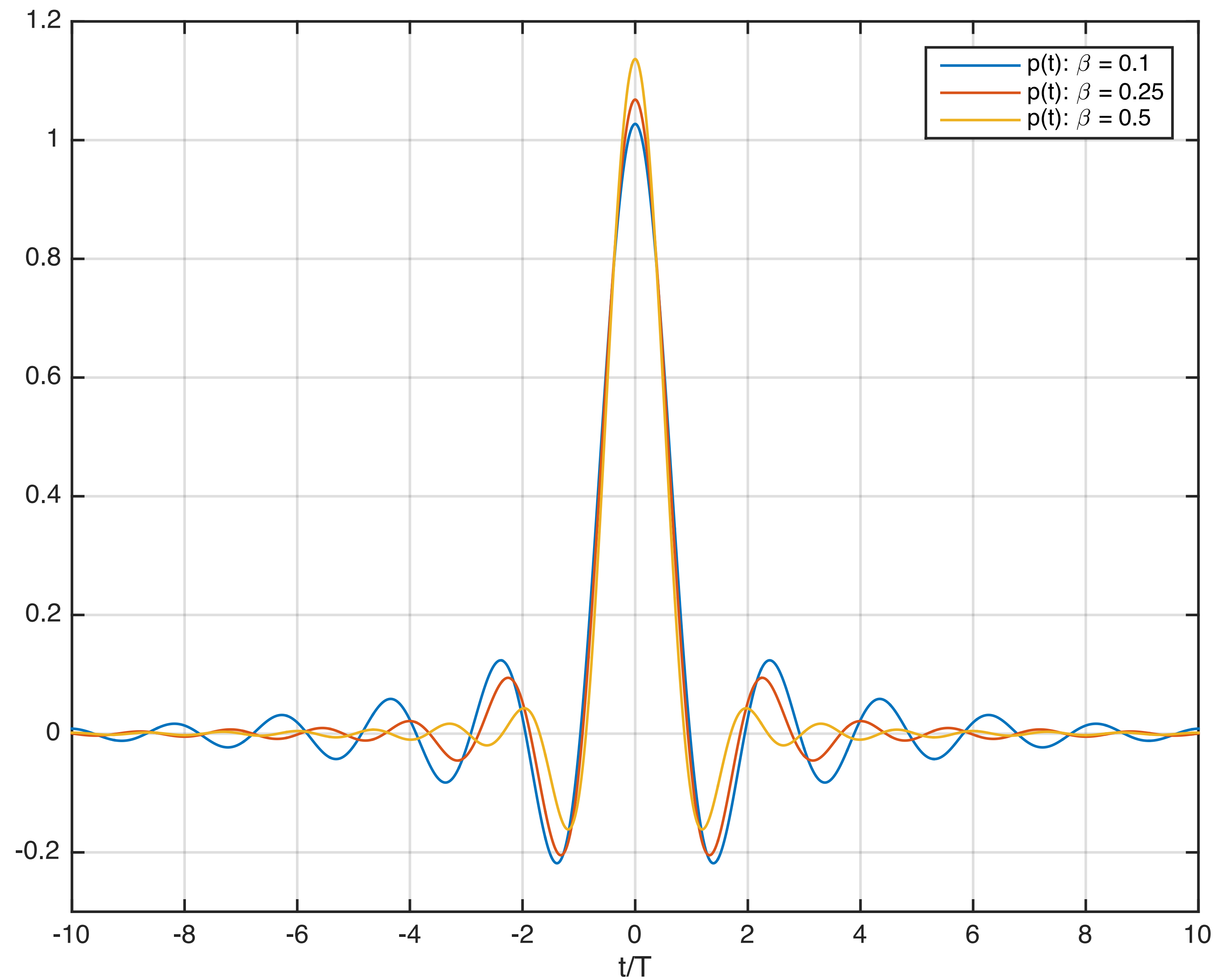
These are built into Matlab!

Raised cosine `rcosdesign(0.35,40,N_sps,'norm');`

Roots-Raised cosine `rcosdesign(0.35,40,N_sps,'sqrt');`

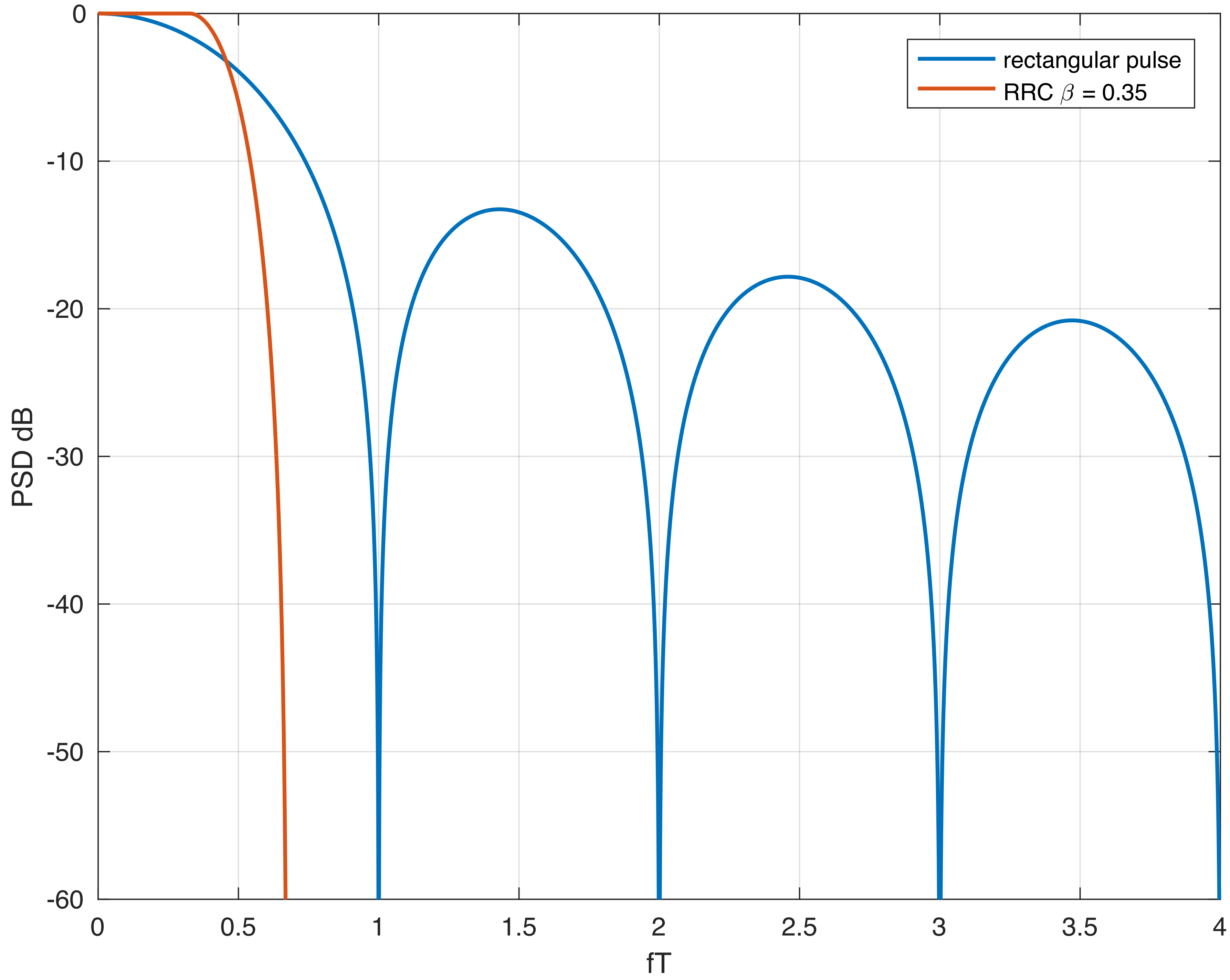
beta = 0.35, truncated to 40 symbols length, number of samples per symbol

Nyquist Pulse Shape: Root Raised Cosine Pulse

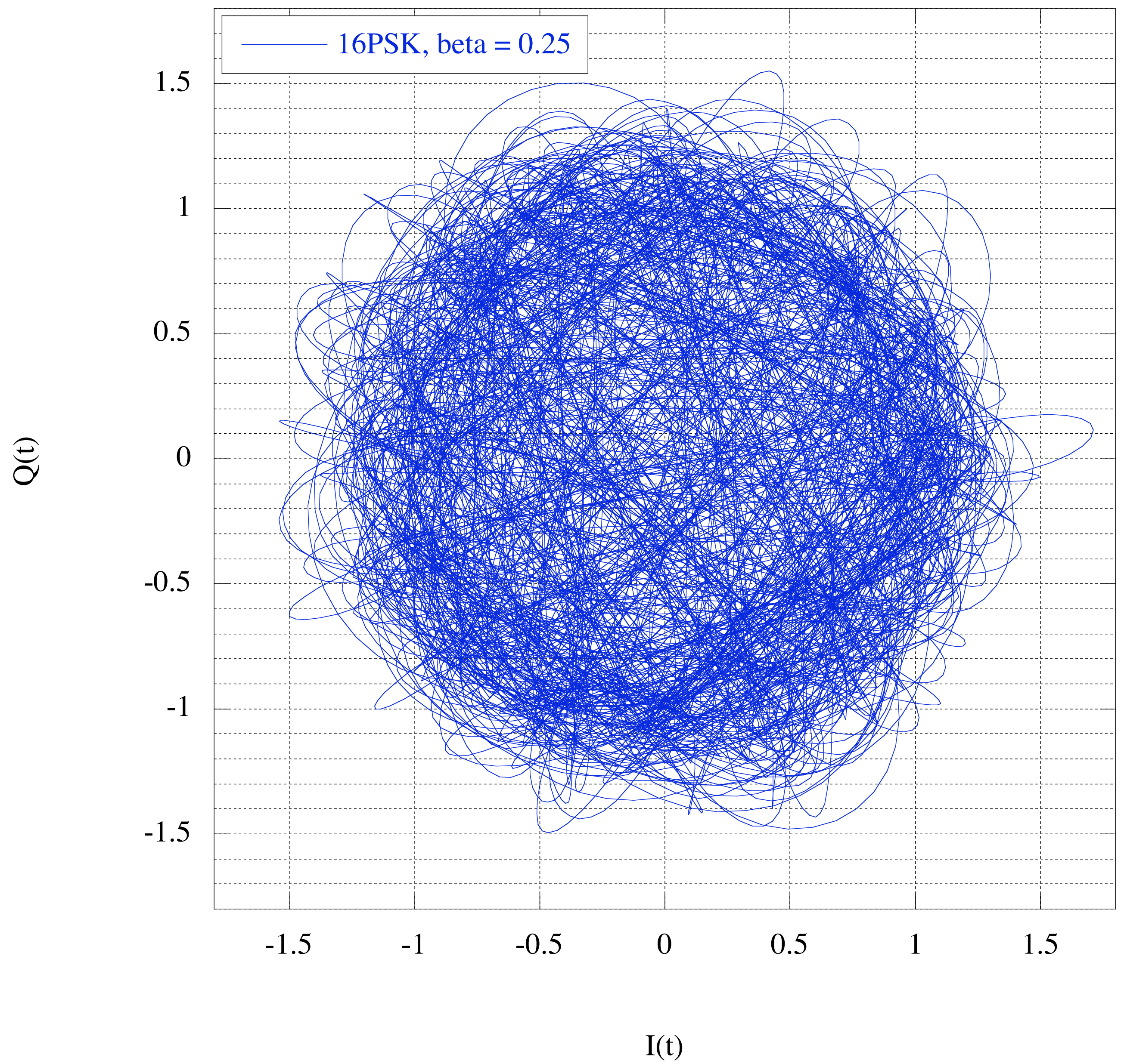


note that pulse does not pass through zero at integer multiples of T

Nyquist Pulse Shape: Raised Cosine Spectrum



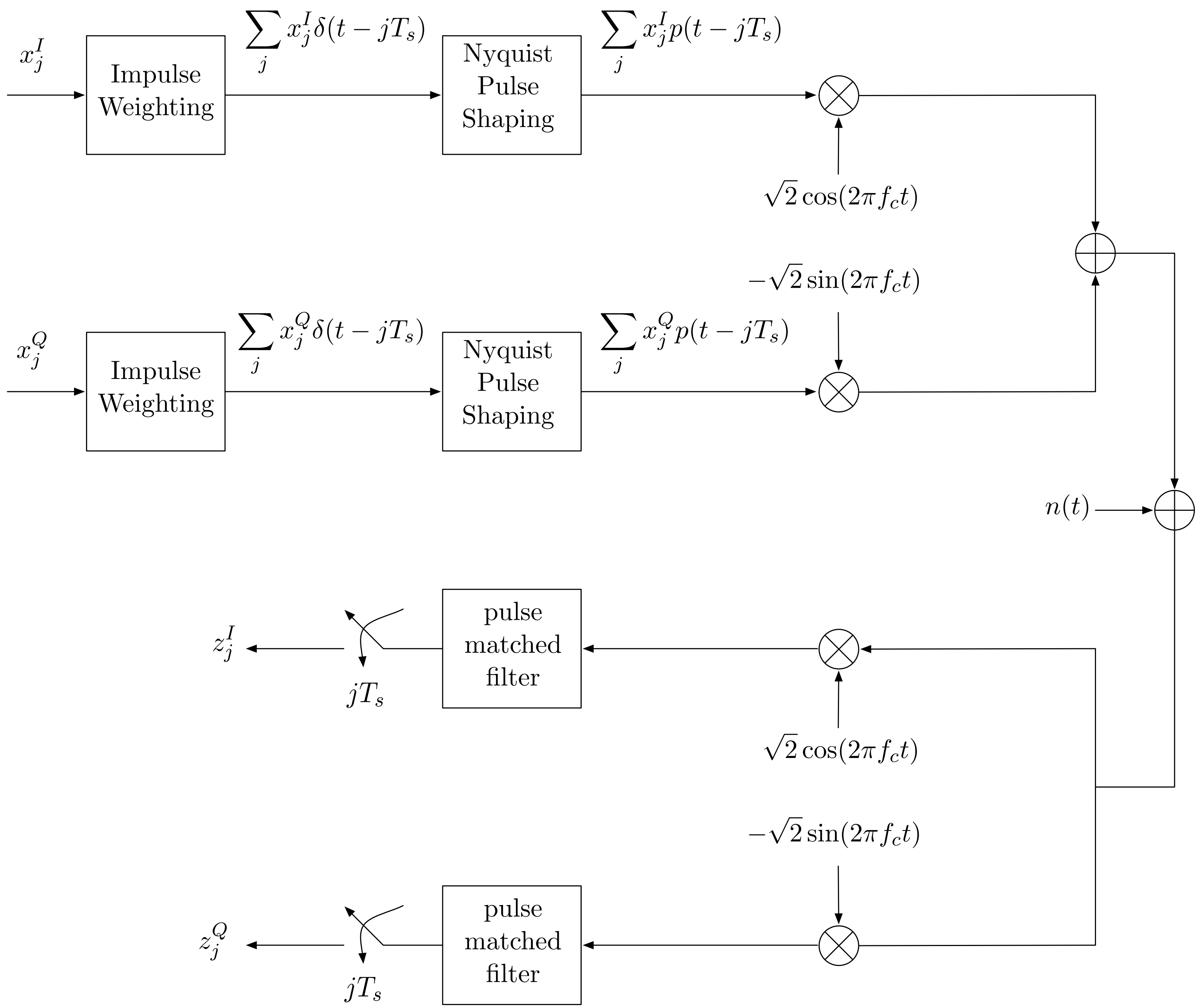
Nyquist Pulse Shape: Raised Cosine Spectrum



Signal trajectory in the I/Q plane with RRC pulse shaping

PSK with RRC has envelope variation

QASK Modulation with Nyquist Pulse Shaping



$$R_p(t) = p(t) * p(-t)$$

$$R_p(jT_s) = \delta_K(j)$$

Detection/Demod Topics

- Maximum A Posteriori decision rule for vector-AWGN channel
- Exact performance for binary modulations
- Minimum distance decision rule for M-ary modulation over AWGN
- Performance bounds
  - *Performance of common M-ary modulations*
- Continuous time model
  - Likelihood functional, sufficient statistics
- Average and generalized likelihood
  - Phase non-coherent demodulation
  - Soft-out demodulation

Composite Hypothesis Testing

- The observation model is a function of a parameter or a set of parameters
  - Nuisance parameters
- If we have a statistical model for the nuisance parameters
  - Average them out — this is called ***average likelihood***
    - Same as original likelihood, just a two step process
- If no statistical model is assumed
  - Can maximize over the parameters along with the hypothesis
    - Called ***generalized likelihood*** (joint likelihood)
      - Ad hoc in general

Composite Hypothesis Testing - Topics

- Basic concepts and definitions
- Phase noncoherent detection
- Differential encoding of PSK and differentially coherent detection
- Soft-output demappers
  - Get soft decisions out of the M-ary decision device
    - transition to coding

Composite Hypothesis Testing

Average Likelihood

$$f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m) = \int f_{\mathbf{z}(u)|\Theta(u)}(\mathbf{z}|\theta, \mathcal{H}_m) f_{\Theta(u)}(\theta|\mathcal{H}_m) d\theta$$

$$= \int f_{\mathbf{z}(u)|\Theta(u)}(\mathbf{z}|\theta, \mathcal{H}_m) f_{\Theta(u)}(\theta) d\theta \quad (\Theta(u) \text{ independent of hypothesis})$$

Generalized Likelihood

$$g_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m) = \max_{\theta} f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m; \theta)$$

$$= f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m; \hat{\theta}_m)$$

$$\hat{\theta}_m = \arg\max_{\theta} f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m; \theta)$$

Phase Noncoherent Detection

Use average likelihood with nuisance parameter being the incoming carrier phase

$$f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m) = \int_{-\infty}^{\infty} f_{\mathbf{z}(u)|\Theta_c(u)}(\mathbf{z}|\phi, \mathcal{H}_m) f_{\Theta_c(u)}(\phi) d\phi$$

$$f_{\Theta_c(u)}(\phi) = f_{\Theta_c(u)}(\phi|\mathcal{H}_m)$$

$$= \frac{1}{2\pi} \quad \phi \in [0, 2\pi)$$

Let's evaluate this for the CT-Likelihood functional in AWGN

$$L(\boldsymbol{r}|\mathcal{H}_m) = e^{-E_m/N_0} \exp\left(\frac{2}{N_0} \int_0^T \Re\left\{ \bar{r}(t) s_m^*(t) \right\} dt \right)$$

Phase Noncoherent Detection

Recall, the complex BB version of the likelihood

$$L(\boldsymbol{r}|\mathcal{H}_m) = e^{-E_m/N_0} \exp\left(\frac{2}{N_0} \int_0^T \Re\left\{ \bar{r}(t) s_m^*(t) \right\} dt \right)$$

Modeling the unknown incoming phase offset

$$s_m(t; \Theta_c(u)) = \Re\left\{ \bar{s}_m(t) \sqrt{2} e^{j(2\pi f_c t + \Theta_c(u))} \right\}$$

$$= \Re\left\{ \bar{s}_m(t) e^{j\Theta_c(u)} \sqrt{2} e^{j2\pi f_c t} \right\}$$

$$= \Re\left\{ \bar{s}_m(t; \Theta_c(u)) \sqrt{2} e^{j2\pi f_c t} \right\}$$

$$\bar{s}_m(t; \Theta_c(u)) = \bar{s}_m(t) e^{j\Theta_c(u)}$$

Phase Noncoherent Detection

$$L(\boldsymbol{r}|\mathcal{H}_m) = e^{-E_m/N_0} \int_0^{2\pi} \exp\left(\frac{2}{N_0}\int_0^T \Re\left\{\bar{r}(t)s_m^*(t)e^{-j\phi}\right\}dt\right)\frac{d\phi}{2\pi}$$

$$= e^{-E_m/N_0}\frac{1}{2\pi}\int_0^{2\pi} \exp\left(\frac{2}{N_0}|\bar{r}_m|\cos(\phi - \angle\bar{r}_m)\right)d\phi$$

$$= e^{-E_m/N_0}\frac{1}{2\pi}\int_{2\pi} e^{\frac{2}{N_0}|\bar{r}_m|\cos(\psi)d\psi}$$

define r_m

$$I_0(x) = \frac{1}{2\pi}\int_{2\pi} e^{x\cos\phi}d\phi$$

Average Likelihood Functional, phase noncoherent AWGN

$$L(\boldsymbol{r}|\mathcal{H}_m) = e^{-E_m/N_0} I_0\left(\frac{2}{N_0}|\bar{r}_m|\right)$$

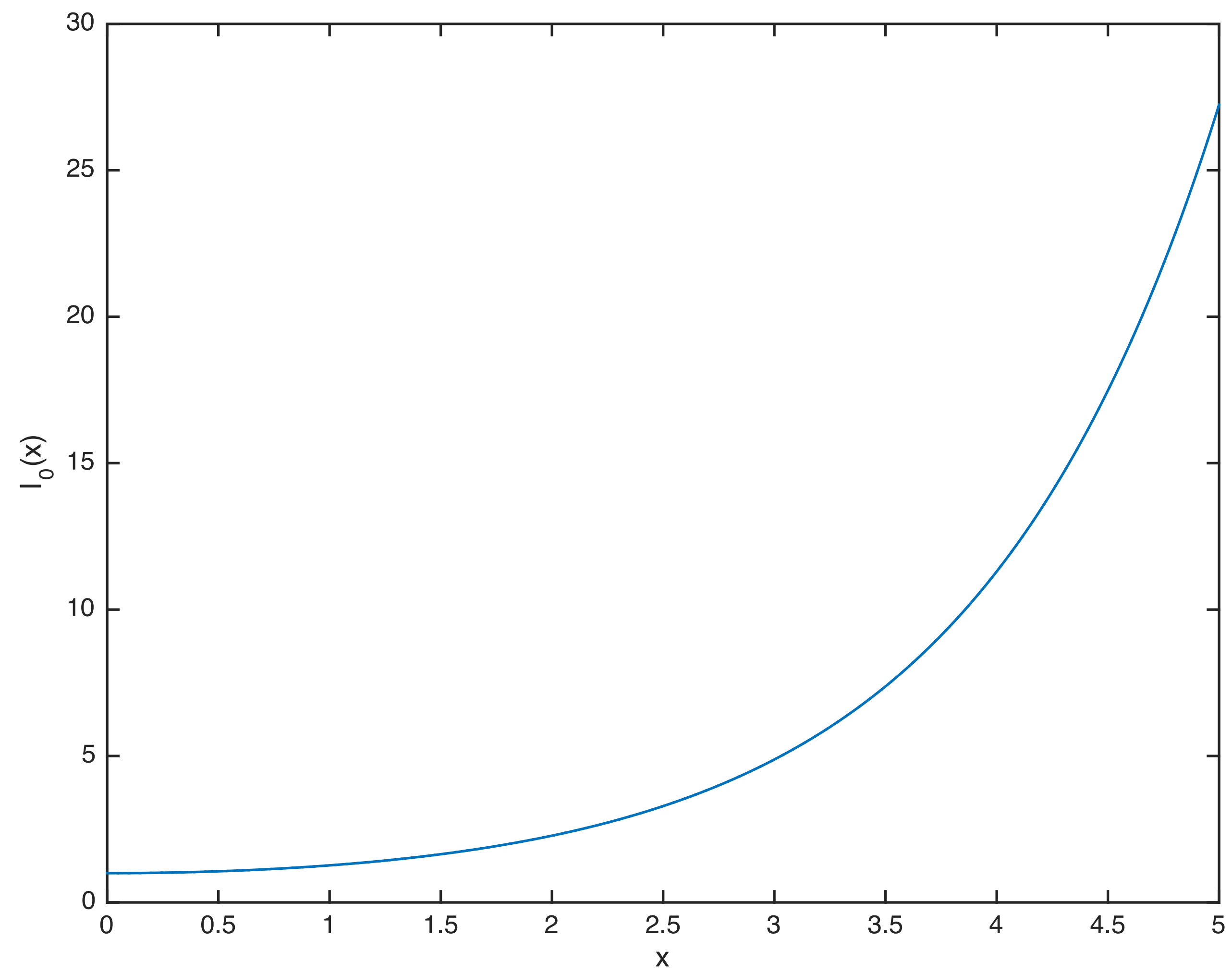
Phase Noncoherent Detection

If signals are equal energy

$$L(\boldsymbol{r}|\mathcal{H}_m) = e^{-E_m/N_0} I_0\left(\frac{2}{N_0}|\bar{r}_m|\right) \equiv I_0\left(\frac{2}{N_0}|\bar{r}_m|\right)$$

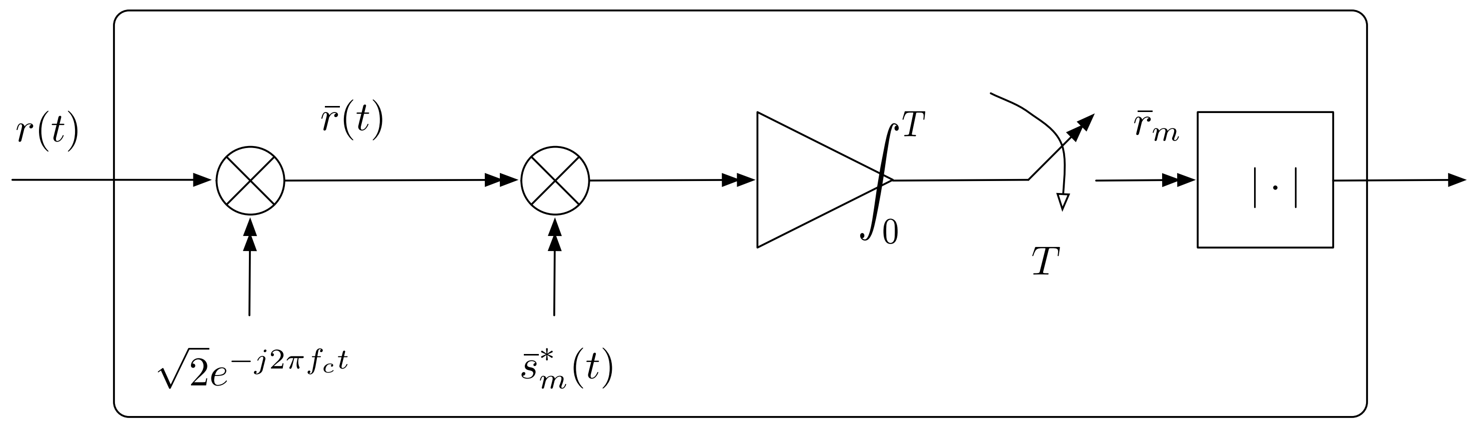
Envelope Detector (equal energy noncoherent)

$$\max_m L(\boldsymbol{r}|\mathcal{H}_m) \iff \max_m |\bar{r}_m|$$

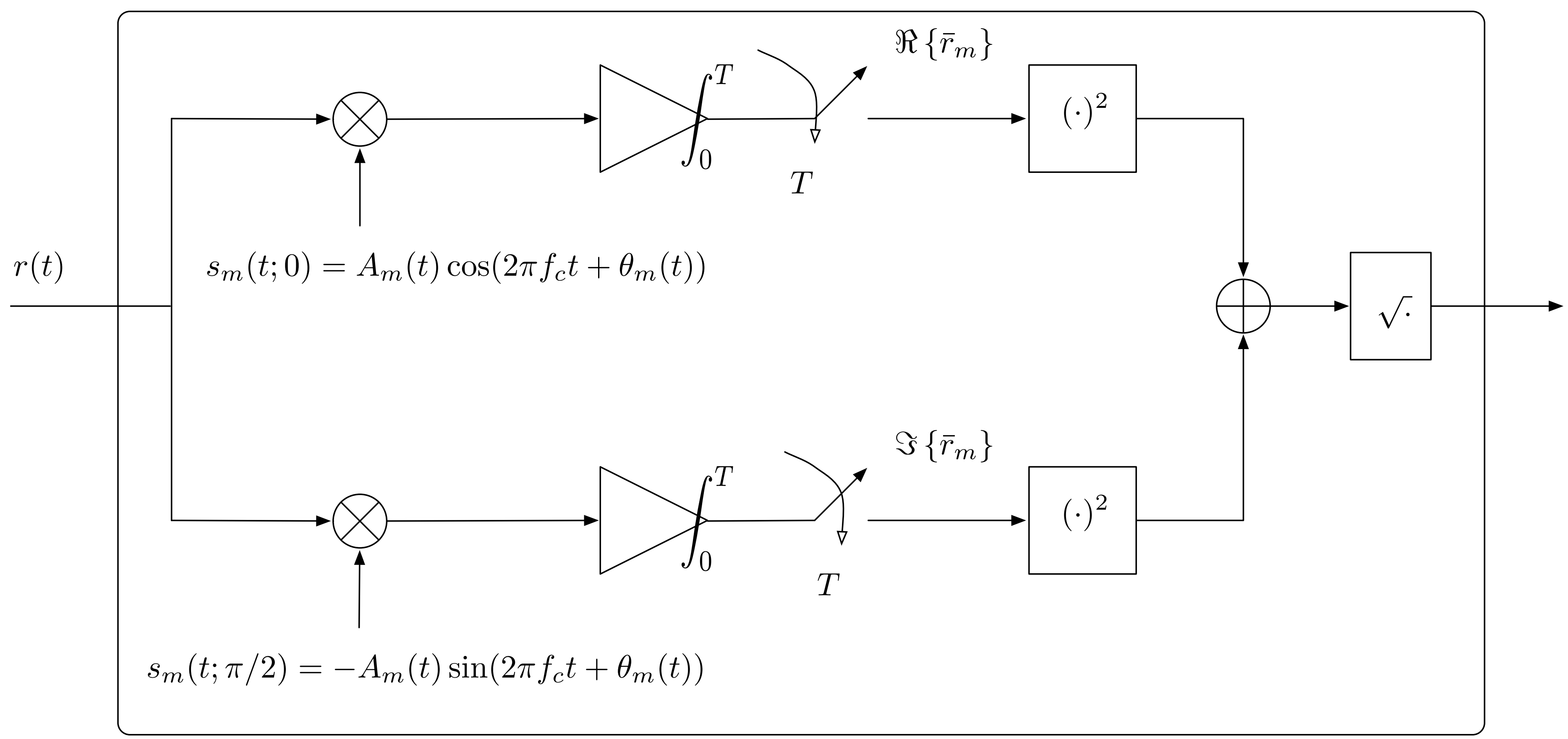


Envelope Detector Processing

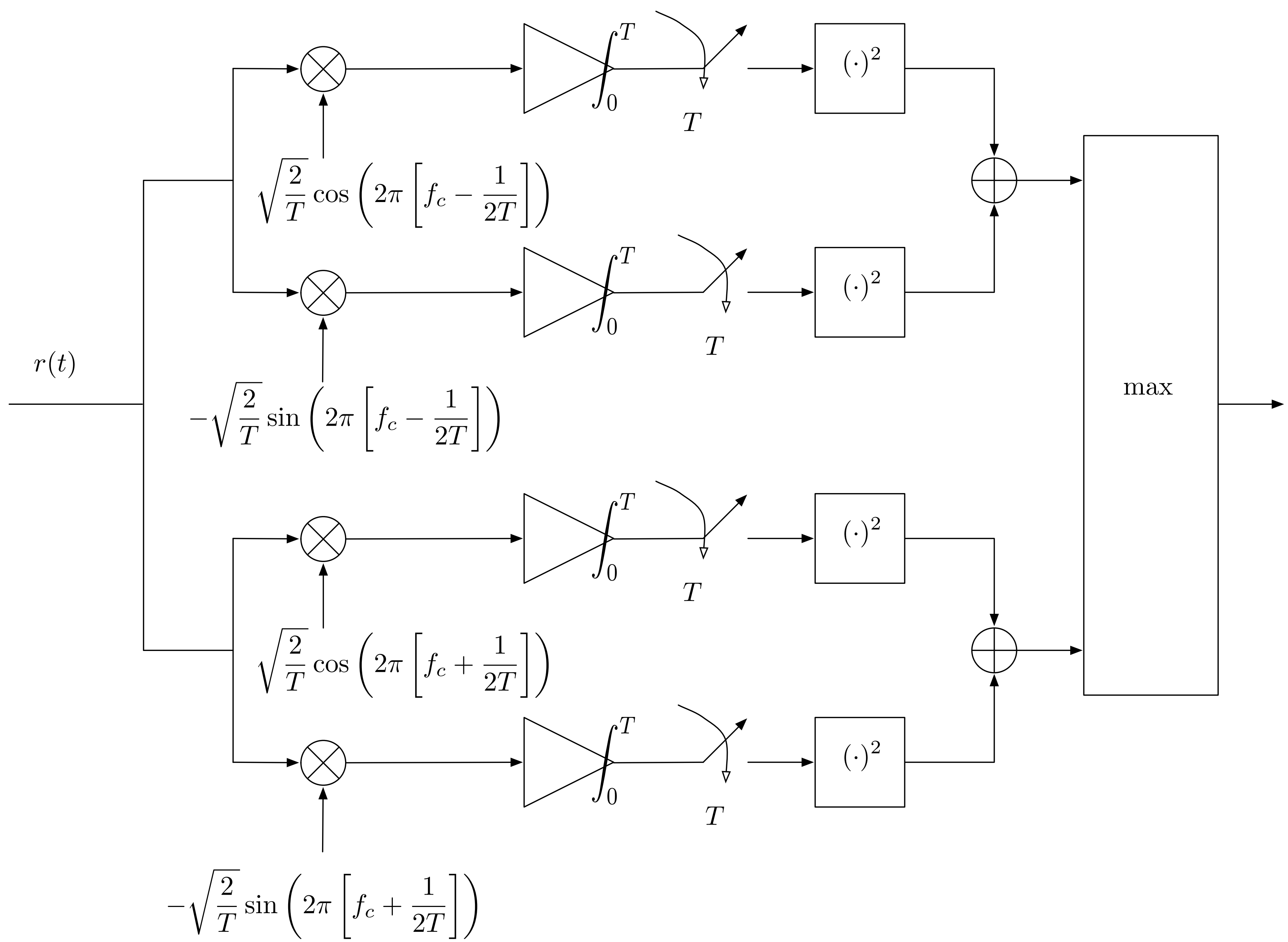
Envelope Detector (complex baseband)



Envelope Detector (narrowband processing)



Example: Non-coherent BFSK



Noncoherent Binary (equal E) Performance

Orthogonal Binary Noncoherent

$$P(\mathcal{E}) = \frac{1}{2} \exp\left(\frac{-E}{2N_0}\right)$$

$$\begin{aligned} P(\mathcal{E}|\mathcal{H}_0) &= \textsc{Pr}\left\{|\bar{r}_0(u)| < |\bar{r}_1(u)| | \mathcal{H}_0\right\} \\ &= \textsc{Pr}\left\{\text{Rayleigh rv} > \text{Rice rv}\right\} \\ &= P(\mathcal{E}|\mathcal{H}_1) \end{aligned}$$

$$P(\mathcal{E}) = \mathrm{Q}(a, b) - \frac{1}{2} \exp\left(\frac{-(a^2 + b^2)}{2}\right) I_0(ab)$$

$$a = \sqrt{\frac{E}{2N_0}\left(1 - \sqrt{1 - |\rho_c|^2}\right)}$$

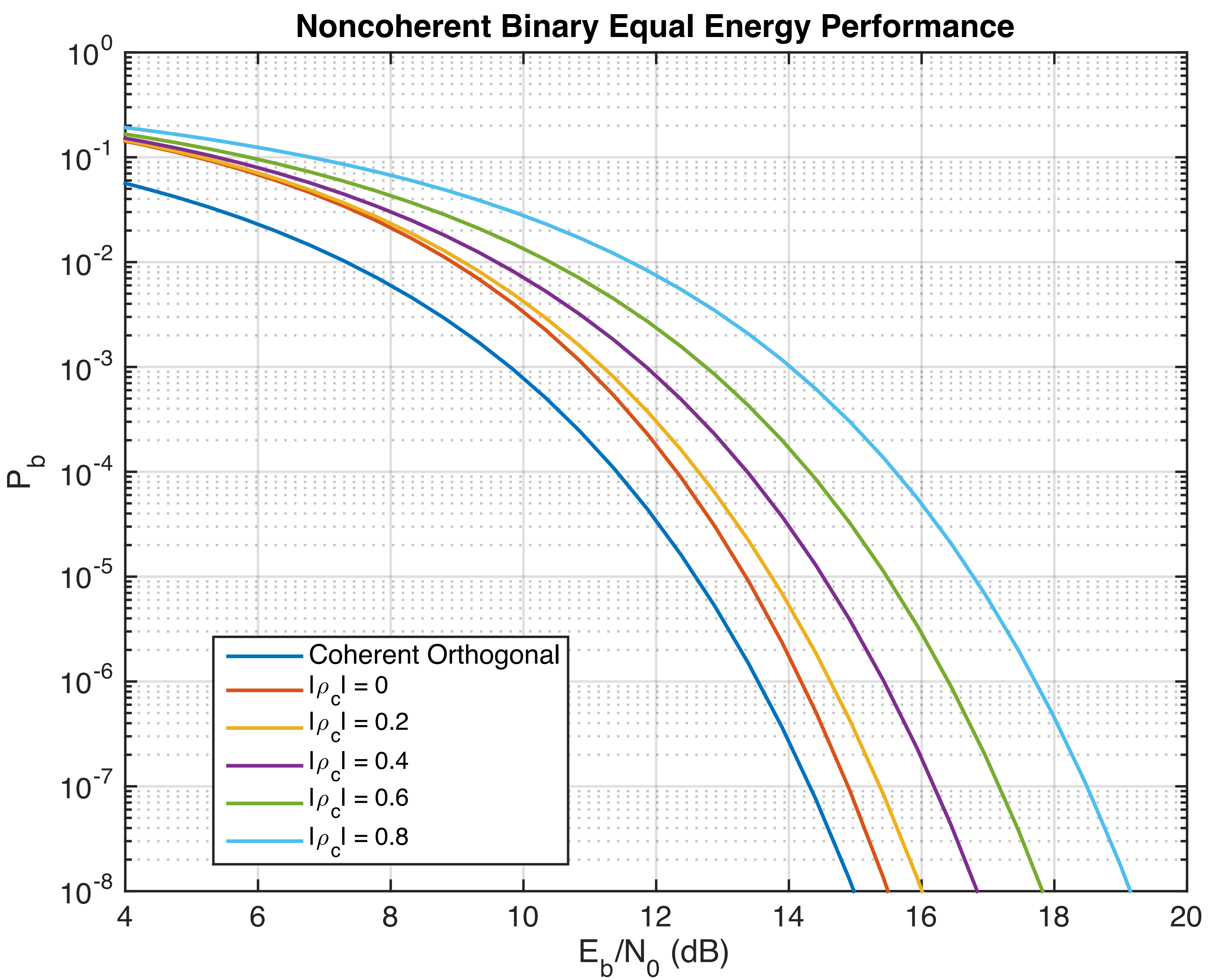
$$\rho_c = \frac{1}{E} \int_0^T \bar{s}_0(t) \bar{s}_1^*(t) dt$$

$$b = \sqrt{\frac{E}{2N_0}\left(1 + \sqrt{1 - |\rho_c|^2}\right)}$$

$$\mathrm{Q}(a, b) = \int_b^\infty x \exp\left(\frac{-(a^2 + b^2)}{2}\right) I_0(abx) dx$$

Marcum Q-function

Noncoherent Binary (equal E) Performance



Best non-coherent performance is for orthogonal

Phase Noncoherent Detector (signal basis)

complex BB correlation to basis signals are sufficient statistics

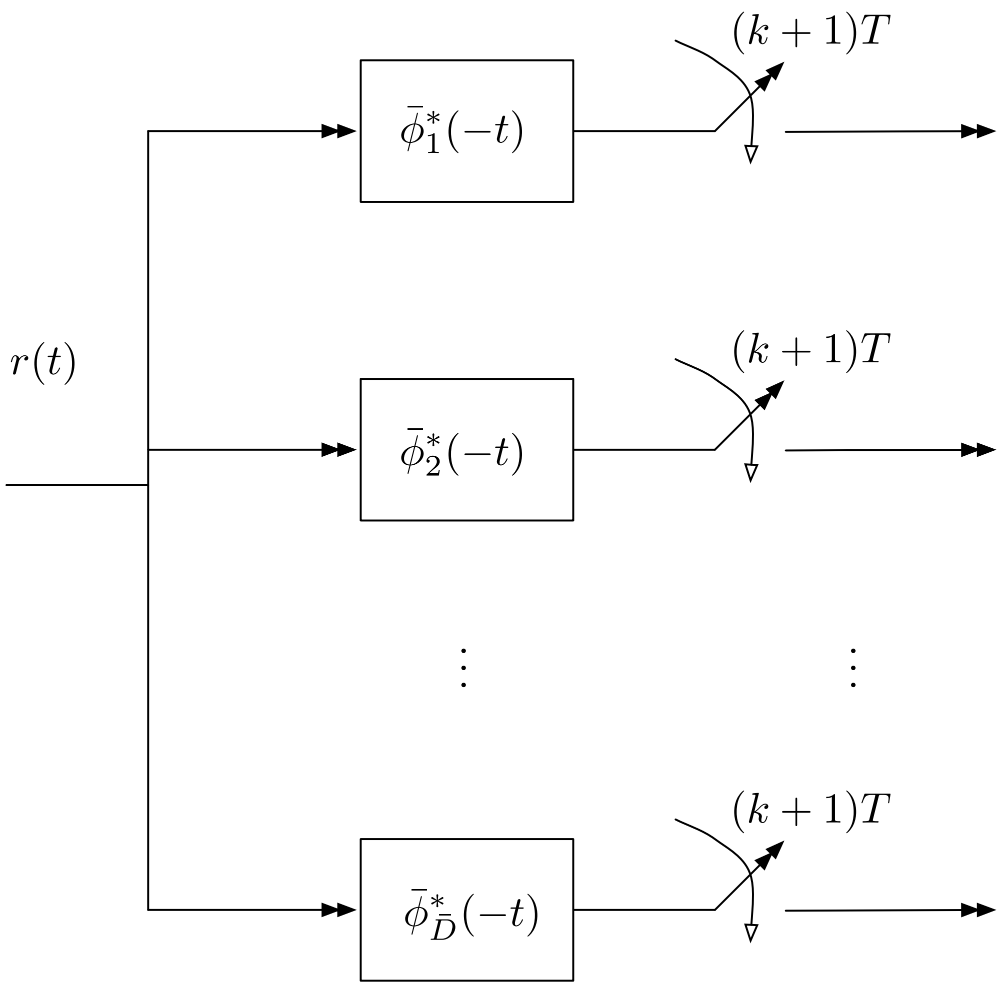
$$\{\bar{\phi}_i(t)\}_{i=1}^{\bar{D}} = \text{ Orthonomal basis for complex BB model}$$

$$\int_{\mathcal{T}} \bar{r}(t)\bar{s}_m^*(t)dt = \int_{\mathcal{T}} \bar{r}(t) \sum_{i=1}^{\bar{D}} \bar{S}_m^*(i)\bar{\phi}_i^*(t)dt$$

$$= \sum_{i=1}^{\bar{D}} \bar{S}_m^*(i) \int_{\mathcal{T}} \bar{r}(t)\bar{\phi}_i^*(t)dt$$

$$\left\{\int_{\mathcal{T}} \bar{r}(t)\bar{\phi}_i^*(t)dt\right\} = \text{ sufficient statistics}$$

Phase Noncoherent Detector (signal basis)



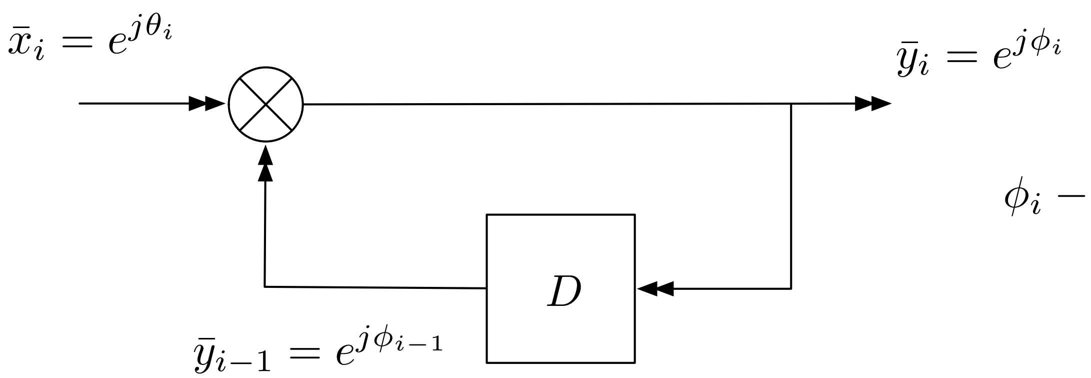
$$\mathcal{H}_m: \quad \bar{\mathbf{z}}(u) = \bar{\mathbf{s}}_m e^{j\theta_c(u)} + \bar{\mathbf{w}}(u) \qquad (\bar{D} \times 1)$$

$$\bar{\mathbf{w}}(u) \sim \mathcal{N}_{\bar{D}}^{cc}(\cdot; \mathbf{0}; N_0 \mathbf{I})$$

$$f_{\mathbf{z}(u)}(\mathbf{z}|\mathcal{H}_m) = e^{-E_m/N_0} \frac{1}{2\pi} \int_{2\pi} \exp\left(\frac{2}{N_0} \bar{\mathbf{s}}_m^\dagger \bar{\mathbf{z}} e^{-j\phi} \right) d\phi$$

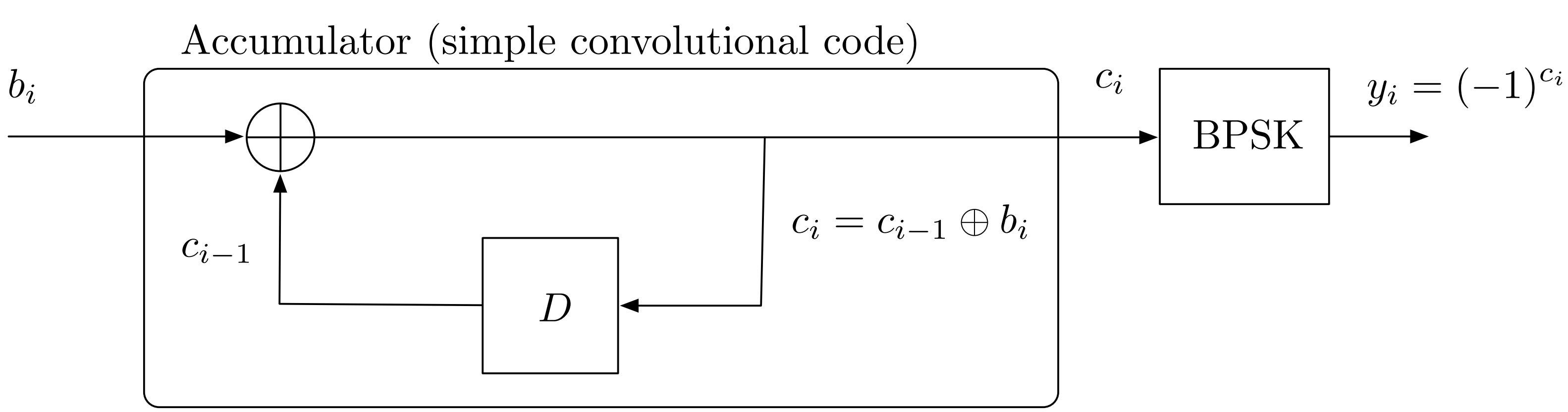
$$= e^{-E_m/N_0} I_0 \left(\frac{2}{N_0} |\bar{\mathbf{s}}_m^\dagger \bar{\mathbf{z}}| \right)$$

Differential Encoding of PSK



$$\phi_i - \phi_{i-1} = \theta_i$$

differential phase encoder



special case for M = 2
(differentially encoded BPSK)

Can detect several ways

Differential PSK

- **Coherent detection with differential decoding**
  - First do coherent MSK detection, then put hard symbol decisions through inverse of differential encoder
- ***"Differentially Coherent" detection (DPSK)***
  - Do phase noncoherent detection over two symbol times
- **Optimal MAP detection**
  - Optimal processing decides by processing entire sequence
    - Viterbi or Forward-Backward Algorithm

Differentially Coherent Detection of DE-PSK

noncoherent based on two symbols

$$\mathcal{H}_m: \quad \begin{bmatrix} \bar{z}_k(u) \\ \bar{z}_{k-1}(u) \end{bmatrix} = \sqrt{E_s} \begin{bmatrix} e^{j\phi_k} \\ e^{j\phi_{k-1}} \end{bmatrix} e^{j\theta_c(u)} + \begin{bmatrix} \bar{w}_k(u) \\ \bar{w}_{k-1}(u) \end{bmatrix} \qquad \theta_k = \phi_k - \phi_{k-1} = \frac{2\pi}{M} m$$

$$\mathcal{H}_m: \quad \bar{\mathbf{z}}(u) = \bar{\mathbf{y}}_m e^{j\theta_c(u)} + \bar{\mathbf{w}}(u)$$

$$\max_m \left| \bar{\mathbf{y}}_m^\dagger \bar{\mathbf{z}} \right|^2 \iff \max_m \left| e^{-j\phi_{k-1}} \begin{bmatrix} e^{-\theta_k} & 1 \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \bar{z}_{k-1} \end{bmatrix} \right|^2$$

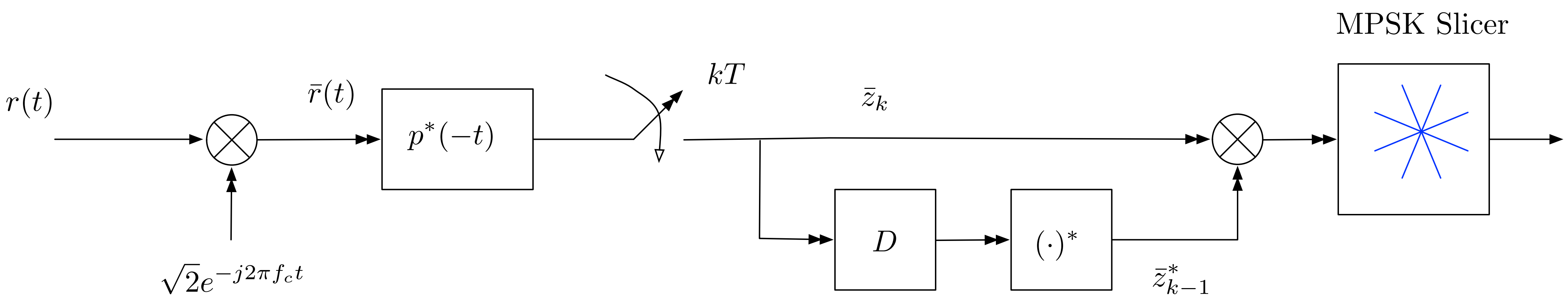
$$\iff \max_m \left| \bar{z}_k e^{-j\theta_k} + \bar{z}_{k-1} \right|^2$$

Differentially-Coherent PSK Demod

$$\iff \max_{\theta_k \in \left\{ \frac{2\pi}{M} m \right\}} \Re\left\{ \bar{z}_k \bar{z}_{k-1}^* e^{-\theta_k} \right\}$$

$$\iff \min_{\theta_k \in \left\{ \frac{2\pi}{M} m \right\}} \left| \angle(\bar{z}_k \bar{z}_{k-1}^*) - \theta_k \right|$$

Differentially Coherent Detection of DE-PSK



$$\mathcal{H}_m : \quad \begin{bmatrix} \bar{z}_k(u) \\ \bar{z}_{k-1}(u) \end{bmatrix} = \sqrt{E_s} \begin{bmatrix} e^{j\phi_k} \\ e^{j\phi_{k-1}} \end{bmatrix} e^{j\theta_c(u)} + \begin{bmatrix} \bar{w}_k(u) \\ \bar{w}_{k-1}(u) \end{bmatrix} \qquad \theta_k = \phi_k - \phi_{k-1} = \frac{2\pi}{M} m$$

Performance of DC-BPSK

$$\mathcal{H}_0 : \quad \begin{bmatrix} \bar{z}_k(u) \\ \bar{z}_{k-1}(u) \end{bmatrix} = \sqrt{E_s} \begin{bmatrix} +1 \\ +1 \end{bmatrix} e^{j\theta_c(u)} + \begin{bmatrix} \bar{w}_k(u) \\ \bar{w}_{k-1}(u) \end{bmatrix}$$

$$\mathcal{H}_1 : \quad \begin{bmatrix} \bar{z}_k(u) \\ \bar{z}_{k-1}(u) \end{bmatrix} = \sqrt{E_s} \begin{bmatrix} -1 \\ +1 \end{bmatrix} e^{j\theta_c(u)} + \begin{bmatrix} \bar{w}_k(u) \\ \bar{w}_{k-1}(u) \end{bmatrix}$$

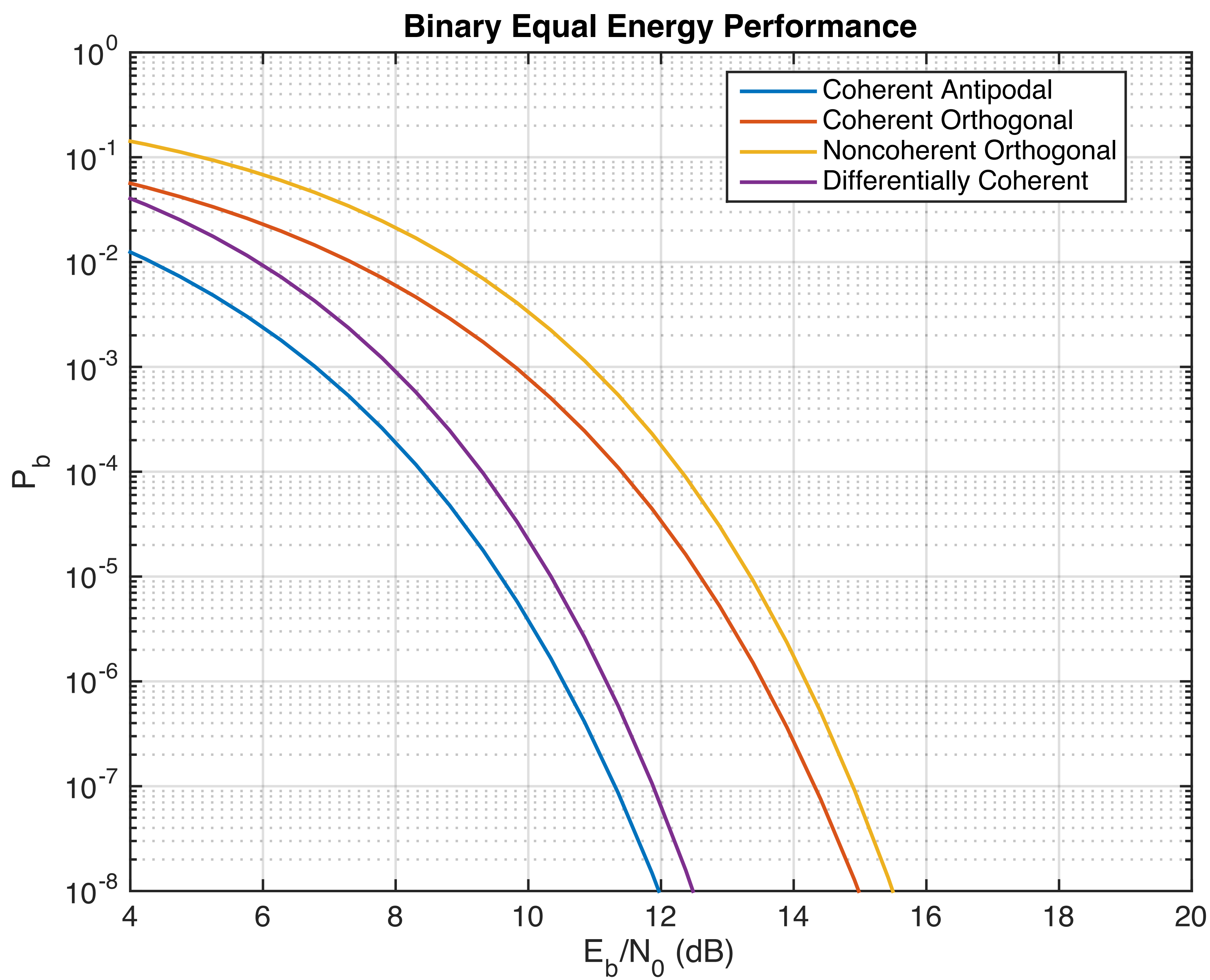
Binary, orthogonal, noncoherent:

$$P(\mathcal{E}) = \frac{1}{2} \exp\left(\frac{-(2E)}{2N_0} \right) = \frac{1}{2} \exp\left(\frac{-E_b}{N_0} \right)$$

DBPSK Performance

DC-MPSK Performance can be bounded with PW-error given by non-orthogonal, binary noncoherent

Comparison of Binary Signaling/Detection Methods

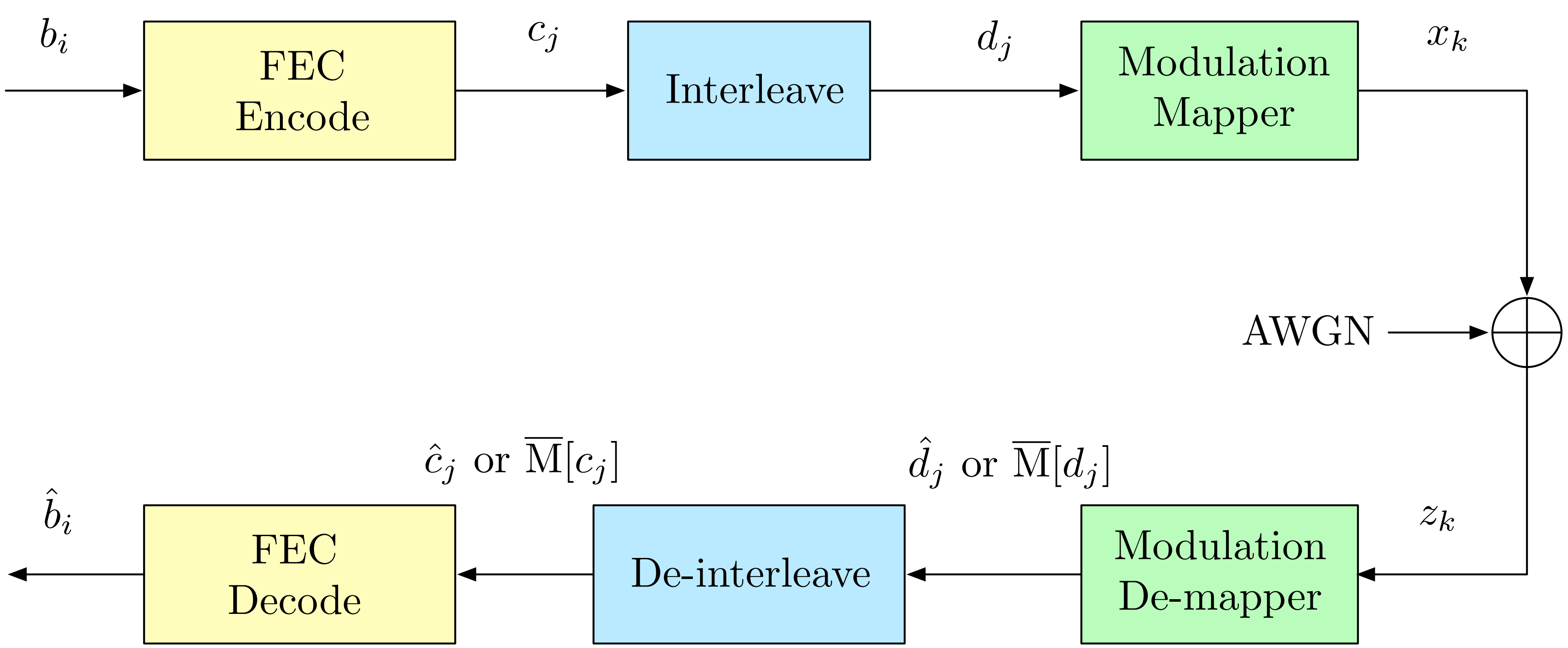


DBPSK is a simple way to approach coherent BPSK without a phase reference

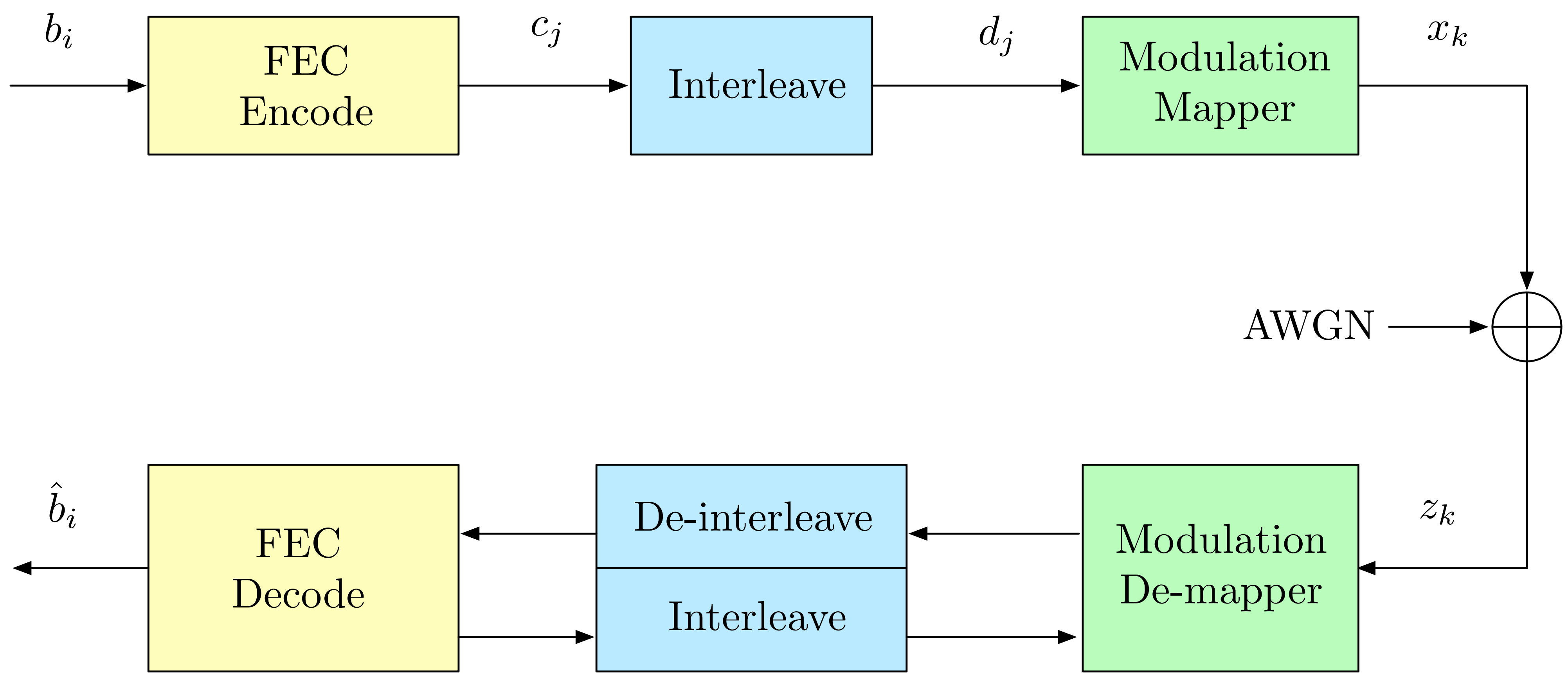
Soft-out Demapper (SOMAP, Soft-dempper)

- Using M-ary modulation with q bits labeling each symbol
- Have focused on MAP symbol detection
  - Selecting the MAP symbol implied a decision on the q bit labels
- We will now consider the MAP rule for deciding each bit
  - other bits are viewed as nuisance parameters and form the average likelihood

Motivation: Bit-Interleaved Coded Modulation (BICM)



BICM with Iterative Decoding/Demod



In general, soft-demapper should take in a priori soft decision information on dj as well as channel likelihoods

Soft-out Demapper (SOMAP, Soft-demapper)

$$f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|d_j) = \sum_{\bar{\mathbf{d}}_j} f_{\mathbf{z}(u)|\mathbf{d}(u)}(\mathbf{z}|\bar{\mathbf{d}}_j, d_j) p_{\bar{\mathbf{d}}_j(u)}(\bar{\mathbf{d}}_j)$$

$$= \sum_{\bar{\mathbf{d}}_j} \left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{x}(\mathbf{d})) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]$$

$$\equiv \sum_{\bar{\mathbf{d}}_j} \left[\exp\left(\frac{-\|\mathbf{z} - \mathbf{x}(\mathbf{d})\|^2}{N_0} \right) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]$$

if AWGN channel

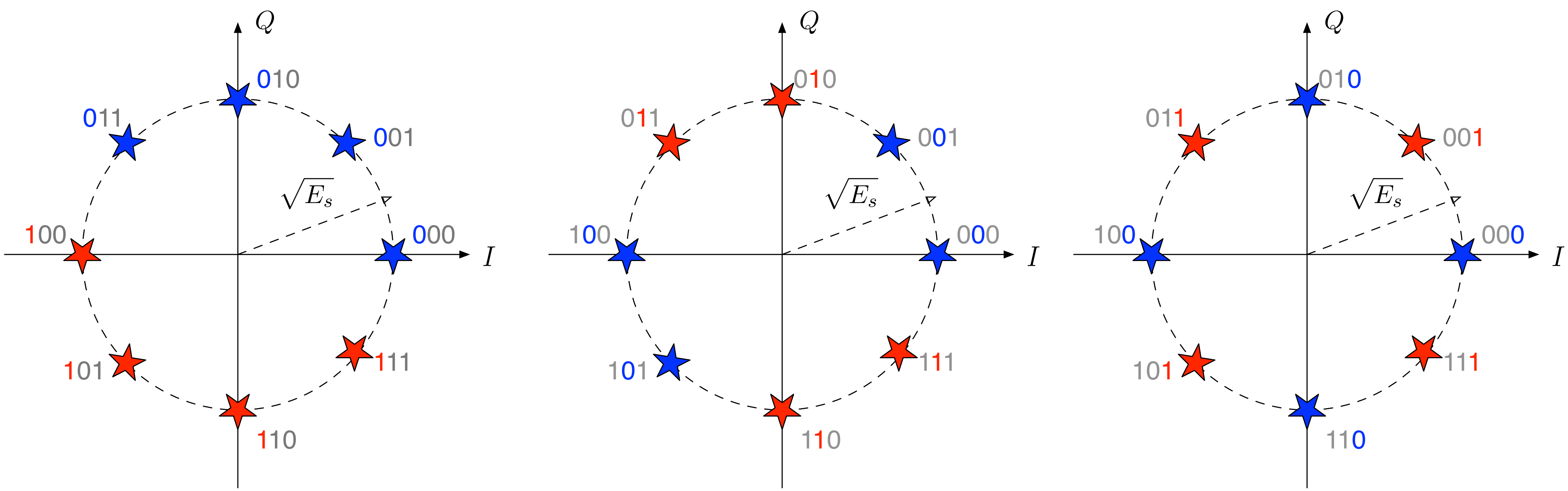
$$\equiv \sum_{\bar{\mathbf{d}}_j} \exp\left(\frac{-\|\mathbf{z} - \mathbf{x}(\mathbf{d})\|^2}{N_0} \right)$$

if d's are a priori uniform

$$\bar{\mathbf{d}}_j = \{d_i\}_{i \neq j}$$

nuisance parameters in this context

Soft-out Demapper (SOMAP, Soft-demapper)



for each bit location, we average over the the subset of signals with a 0 in location j, then over all points with 1 in location j

Soft-out Demapper (SOMAP, Soft-demapper)

$$\frac{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)} = \frac{\sum_{\bar{\mathbf{d}}_j, d_j=1}\left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0}\right)\prod_{i\neq j} p_{d_i(u)}(d_i)\right]}{\sum_{\bar{\mathbf{d}}_j, d_j=0}\left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0}\right)\prod_{i\neq j} p_{d_i(u)}(d_i)\right]}$$

average likelihood ratio for bit dj — soft-decision sent to decoder

$$f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)p_{d_j(u)}(1) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)p_{d_j(u)}(0)$$

This is the MAP bit decision rule for bit dj

Decision Format vs. Optimality Criterion

$$\left\{ f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1) p_{d_j(u)}(1) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0) p_{d_j(u)}(0) \right\}_{j=0,1,\ldots q-1}$$

The q MAP bit decision rules imply an M-ary decision rule

This is the M-ary Bayes rule with C(i,j) = number of bit label differences

minimizes the average number of bit errors, or Pb

$$\max_{m \in \{0,1,\ldots M-1\}} \left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{s}_m) \prod_{i=0}^{q-1} p_{d_i(u)}(d_i(m)) \right]$$

The one M-ary MAP symbol decision rule implies q bit decision rules — **what are these?**

Decision Format vs. Optimality Criterion

$$g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|d_j) = \max_{\bar{\mathbf{d}}_j} f_{\mathbf{z}(u)|\mathbf{d}(u)}(\mathbf{z}|\bar{\mathbf{d}}_j, d_j) p_{\bar{\mathbf{d}}_j(u)}(\bar{\mathbf{d}}_j)$$

$$= \max_{\bar{\mathbf{d}}_j} \left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{x}(\mathbf{d})) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]$$

$$\equiv \max_{\bar{\mathbf{d}}_j} \left[\exp\left(\frac{-\|\mathbf{z} - \mathbf{x}(\mathbf{d})\|^2}{N_0} \right) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]$$

$$\equiv \max_{\bar{\mathbf{d}}_j} \exp\left(\frac{-\|\mathbf{z} - \mathbf{x}(\mathbf{d})\|^2}{N_0} \right)$$

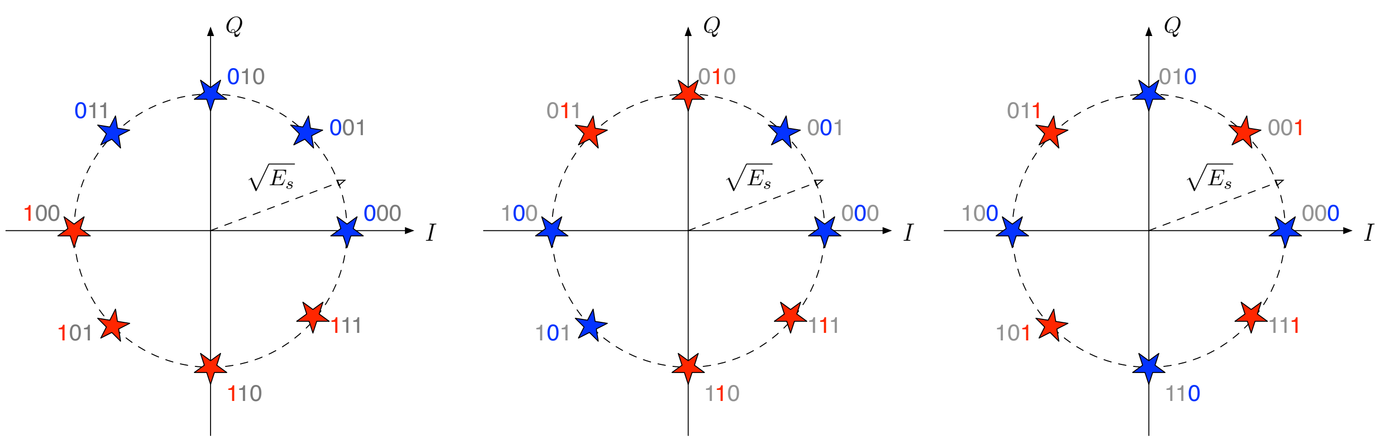
$$\left\{ g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1) p_{d_j(u)}(1) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0) p_{d_j(u)}(0) \right\}_{j=0,1,\ldots q-1}$$

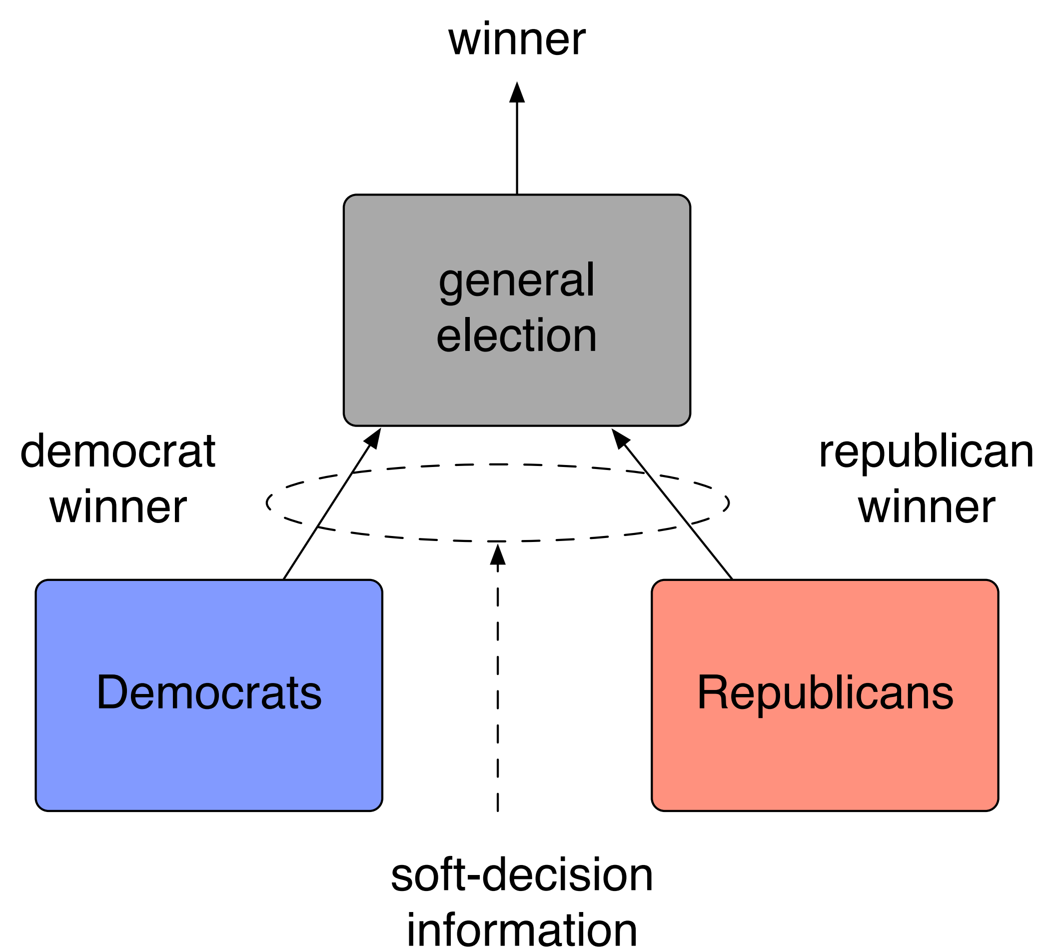
MAP M-ary symbol decision rule expressed as q bit-level decisions

Decision Format vs. Optimality Criterion

$$\frac{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)} = \frac{\max_{\bar{\mathbf{d}}_j, d_j=1}\left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0}\right)\prod_{i\neq j} p_{d_i(u)}(d_i)\right]}{\max_{\bar{\mathbf{d}}_j, d_j=0}\left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0}\right)\prod_{i\neq j} p_{d_i(u)}(d_i)\right]}$$

generalized likelihood ratio for bit dj — soft-decision sent to decoder





SOMAP processing in AWGN

sum-product SOMAP (AWGN)

$$\frac{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)} = \frac{\sum_{\bar{\mathbf{d}}_j, d_j=1} \left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \right) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]}{\sum_{\bar{\mathbf{d}}_j, d_j=0} \left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \right) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]}$$

max-product SOMAP (AWGN)

$$\frac{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)} = \frac{\max_{\bar{\mathbf{d}}_j, d_j=1} \left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \right) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]}{\max_{\bar{\mathbf{d}}_j, d_j=0} \left[\exp\left(\frac{-\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \right) \prod_{i \neq j} p_{d_i(u)}(d_i) \right]}$$

both of these can be implemented in the metric domain (-ln(.))

Metric Domain Processing for Max-Product

$$-\ln\left(\max_m p_m\right) = -\max_m\left[-\ln(p_m)\right] = \min_m\left[-\ln(p_m)\right]$$

$$-\ln\left(\frac{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)}\right) = \min_{\bar{\mathbf{d}}_j, d_j=1}\left(\mathrm{MI}[\mathbf{x}(\mathbf{d})] + \sum_{i\neq j}\mathrm{MI}[d_i]\right) - \min_{\bar{\mathbf{d}}_j, d_j=0}\left(\mathrm{MI}[\mathbf{x}(\mathbf{d})] + \sum_{i\neq j}\mathrm{MI}[d_i]\right)$$

min-sum SOMAP
(metric domain implementation of max-product)

$$\mathrm{MI}[\mathbf{x}(\mathbf{d})] = -\ln\left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{x}(\mathbf{d}))\right]$$

$$= \frac{\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \quad \text{(AWGN)}$$

$$\mathrm{MI}[d_i] = -\ln\left(p_{d_i(u)}(d_i)\right)$$

Metric Domain Processing for Sum-Product

$$-\ln\left(\sum_m p_m\right) = \min_m{}^* \left[-\ln(p_m)\right]$$

$$\min{}^*(m_1, m_2) = -\ln\left(e^{-m_1} + e^{-m_2}\right)$$

metric domain averaging

$$\min_i{}^* \; m_i = -\ln\left(\sum_i e^{-m_i}\right)$$

simple implantation as a pairwise operation:

$$\min{}^*(m_1, m_2) = -\ln\left(e^{-m_1} + e^{-m_2}\right)$$

$$= \min(m_1, m_2) - \ln\left(1 + e^{-|m_1 - m_2|}\right)$$

$$\min{}^*(m_1, m_2, m_3) = \min{}^*(\min{}^*(m_1, m_2), m_3)$$

Metric Domain Processing for Max-Product

$$-\ln\left(\frac{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)}\right) = \min_{\bar{\mathbf{d}}_j, d_j=1}{}^{*}\left(\mathrm{MI}[\mathbf{x}(\mathbf{d})] + \sum_{i\neq j}\mathrm{MI}[d_i]\right) - \min_{\bar{\mathbf{d}}_j, d_j=0}{}^{*}\left(\mathrm{MI}[\mathbf{x}(\mathbf{d})] + \sum_{i\neq j}\mathrm{MI}[d_i]\right)$$

min*-sum SOMAP
(metric domain implementation of sum-product)

$$\mathrm{MI}[\mathbf{x}(\mathbf{d})] = -\ln\left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{x}(\mathbf{d}))\right]$$

$$= \frac{\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \quad \text{(AWGN)}$$

(replace min with min*)

$$\mathrm{MI}[d_i] = -\ln\left(p_{d_i(u)}(d_i)\right)$$

SOMAP Processing — Alternative View

$$\mathrm{MI}[\mathbf{x}(\mathbf{d})] = -\ln\left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{x}(\mathbf{d}))\right]$$

$$= \frac{\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \quad \text{(AWGN)}$$

$$\mathrm{MI}[d_i] = -\ln\left(p_{d_i(u)}(d_i)\right)$$

1. Compute the configure metrics by ***combining*** incoming metrics

$$\mathrm{M}[\mathrm{Config}=m] = \mathrm{MI}[\mathbf{s}_m] + \sum_{i=0}^{q-1}\mathrm{MI}[d_i^{(m)}]$$

2. ***Marginalize*** the configuration metric to get the marginal soft decision information

"Intrinsic" (soft) information

$$\mathrm{MSM}[d_i=1] = \min_{m:d_i=1}\mathrm{M}[\mathrm{Config}=m]$$

$$\mathrm{MSM}[d_i=0] = \min_{m:d_i=0}\mathrm{M}[\mathrm{Config}=m]$$

threshold these for best local decisions — i.e., MAP symbol/bit-sequence

3. Convert to "extrinsic format" — i.e., likelihoods

"Extrinsic" (soft) information

$$\mathrm{MO}[d_i=1] = \mathrm{MSM}[d_i=1] - \mathrm{MI}[d_i=1]$$

$$\mathrm{MO}[d_i=0] = \mathrm{MSM}[d_i=0] - \mathrm{MI}[d_i=0]$$

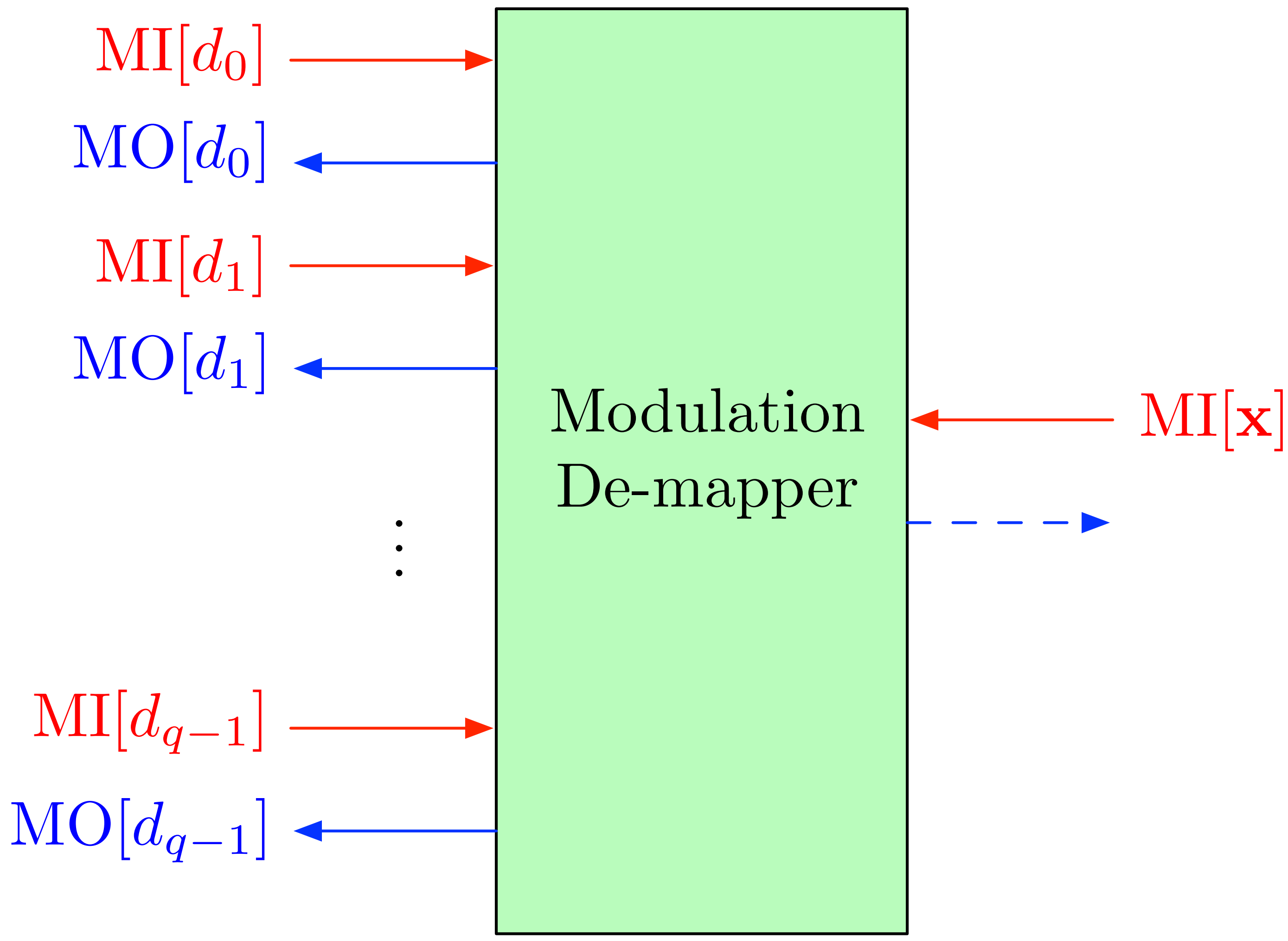
pass these to the decoder as soft decisions

SOMAP Processing — Alternative View

$$\mathrm{MI}[\mathbf{x}(\mathbf{d})] = -\ln\left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{x}(\mathbf{d}))\right]$$

$$= \frac{\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \quad \text{(AWGN)}$$

$$\mathrm{MI}[d_i] = -\ln\left(p_{d_i(u)}(d_i)\right)$$



SOMAP Processing — Alternative View

$$\mathrm{MI}[\mathbf{x}(\mathbf{d})] = -\ln\left[f_{\mathbf{z}(u)|\mathbf{x}(u)}(\mathbf{z}|\mathbf{x}(\mathbf{d}))\right]$$

$$= \frac{\|\mathbf{z}-\mathbf{x}(\mathbf{d})\|^2}{N_0} \quad \text{(AWGN)}$$

$$\mathrm{MI}[d_i] = -\ln\left(p_{d_i(u)}(d_i)\right)$$

1. Compute the configure metrics by ***combining*** incoming metrics

$$\mathrm{M}[\text{Config} = m] = \mathrm{MI}[\mathbf{s}_m] + \sum_{i=0}^{q-1} \mathrm{MI}[d_i^{(m)}]$$

2. ***Marginalize*** the configuration metric to get the marginal soft decision information

"Intrinsic" (soft) information

$$\mathrm{MS}^*\mathrm{M}[d_i = 1] = \min_{m:d_i=1}{}^*\mathrm{M}[\text{Config} = m]$$

$$\mathrm{MS}^*\mathrm{M}[d_i = 0] = \min_{m:d_i=0}{}^*\mathrm{M}[\text{Config} = m]$$

threshold these for best local decisions — i.e., MAP bit

3. Convert to "extrinsic format" — i.e., likelihoods

"Extrinsic" (soft) information

$$\mathrm{MO}[d_i = 1] = \mathrm{MS}^*\mathrm{M}[d_i = 1] - \mathrm{MI}[d_i = 1]$$

$$\mathrm{MO}[d_i = 0] = \mathrm{MS}^*\mathrm{M}[d_i = 0] - \mathrm{MI}[d_i = 0]$$

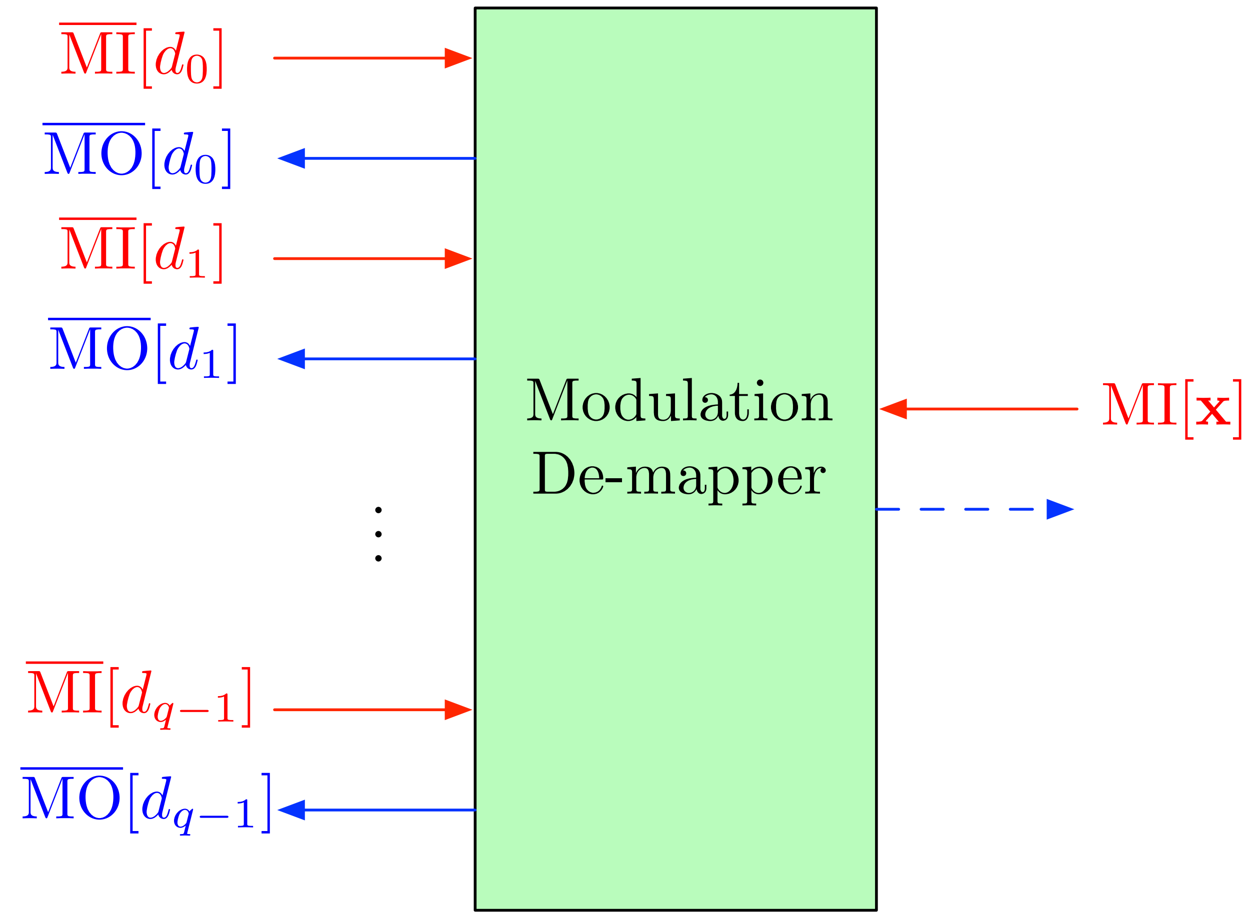
pass these to the decoder as soft decisions

SOMAP Processing — Alternative View

Can work with the Negative Log-Likelihood Ratios (NLLRs) instead

$$\overline{\mathrm{MI}}[d_i] = \mathrm{MI}[d_i] - \mathrm{MI}[d_i = 0]$$

$$\overline{\mathrm{MO}}[d_i] = \mathrm{MO}[d_i] - \mathrm{MO}[d_i = 0]$$



Can subtract any constant from metrics

SOMAP Processing — Normalized Metrics

Can always represent metrics/probabilities on M-ary variables by M-1 numbers through normalization

$$\overline{\mathrm{MI}}[d_i] = \mathrm{MI}[d_i] - \mathrm{MI}[d_i = 0]$$

$$\overline{\mathrm{MI}}[d_i = 1] = \mathrm{MI}[d_i = 1] - \mathrm{MI}[d_i = 0]$$

$$= -\ln\left[\frac{p(d_i = 1)}{p(d_i = 0)}\right]$$

$$\overline{\mathrm{MI}}[d_i = 0] = 0$$ "zeros are free"

(see spreadsheet example)

$$\overline{\mathrm{MO}}[d_i] = \mathrm{MO}[d_i] - \mathrm{MO}[d_i = 0]$$

$$\overline{\mathrm{MO}}[d_i = 1] = \mathrm{MO}[d_i = 1] - \mathrm{MO}[d_i = 0]$$

$$= -\ln\left(\frac{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{f_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)}\right) \quad (\text{min}^*\text{-sum})$$

$$= -\ln\left(\frac{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|1)}{g_{\mathbf{z}(u)|d_j(u)}(\mathbf{z}|0)}\right) \quad (\text{min-sum})$$

$$\overline{\mathrm{MI}}[d_i = 0] = 0$$ "zeros are free"

SOMAP Processing — Alternative View

$$\overline{\mathrm{MI}}[d_i] = \mathrm{MI}[d_i] - \mathrm{MI}[d_i = 0]$$

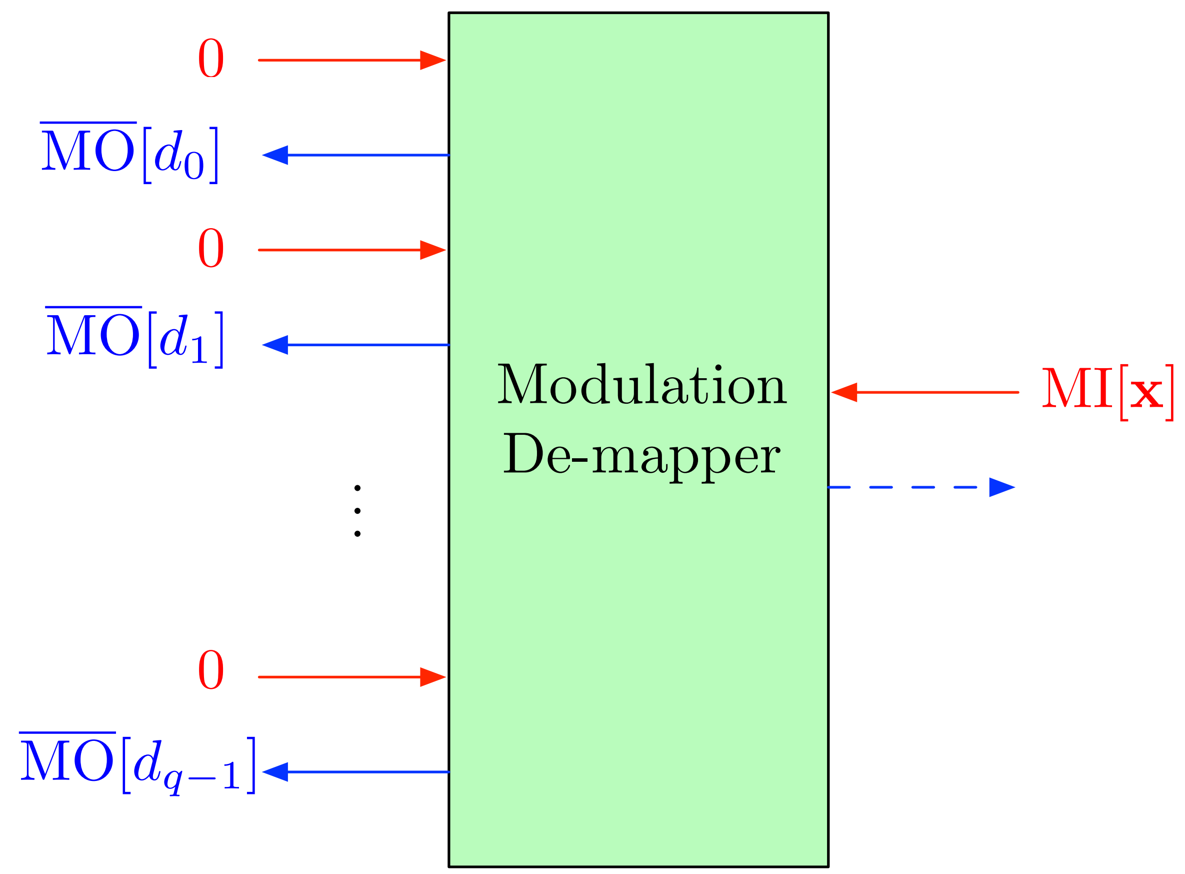
$$\overline{\mathrm{MI}}[d_i = 1] = \mathrm{MI}[d_i = 1] - \mathrm{MI}[d_i = 0]$$

I often abuse this notation and use the first to represent the second — i.e., since once of the two normalized metrics is zero by definition

$$\overline{\mathrm{MO}}[d_i] = \mathrm{MO}[d_i] - \mathrm{MO}[d_i = 0]$$

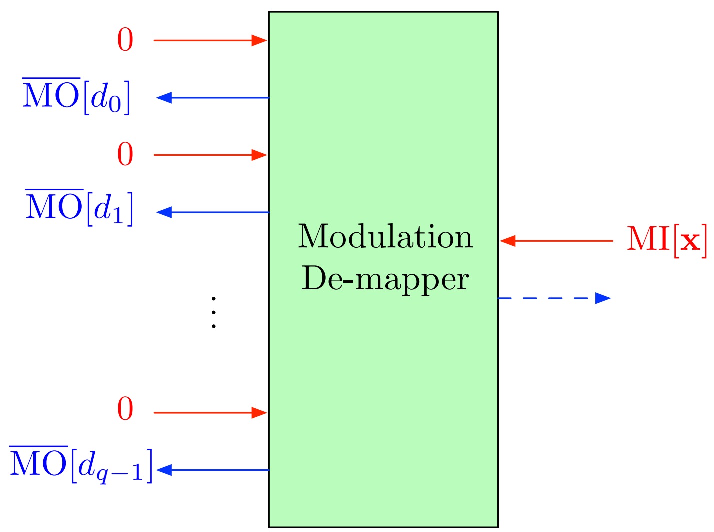
$$\overline{\mathrm{MO}}[d_i = 1] = \mathrm{MO}[d_i = 1] - \mathrm{MO}[d_i = 0]$$

SOMAP Processing — Alternative View



for equal a priori probability on the bits — e.g., first activation and/
or non-iterative BICM

SOMAP Processing — Alternative View



min-sum:

$$\overline{\text{MO}}[d_j] = \min_{m:d_j=1} \frac{\|\mathbf{z} - \mathbf{s}_m\|^2}{N_0} - \min_{m:d_j=0} \frac{\|\mathbf{z} - \mathbf{s}_m\|^2}{N_0}$$

min*-sum:

$$\overline{\text{MO}}[d_j] = \min_{m:d_j=1}{}^{*} \frac{\|\mathbf{z} - \mathbf{s}_m\|^2}{N_0} - \min_{m:d_j=0}{}^{*} \frac{\|\mathbf{z} - \mathbf{s}_m\|^2}{N_0}$$

for equal a priori probability on the bits — e.g., first activation and/or non-iterative BICM

SOMAP Processing is Special Case of SISO

- General Soft-in/Soft-out (SISO) processing
  - Digital variables (e.g., inputs/outputs) associated with a local system/constraint/code
    - Finite number of configurations
  - Combine incoming marginal soft information (e.g., sum MI's) to compute a configuration metric for each configuration
  - Marginalize over configuration metrics consistent with each value of each digital variable to produce updated marginal soft information (MO's)
- This forms the basis of all modern coding — i.e., it is the basis of iterative decoding
  - Modern coding: decode local codes in SISO manner, exchange soft information, and iterate