EE564: Problem Set – Prof. Keith M. Chugg

1 Background and Motivation

1.1. This problem addresses a method of converting two independent uniform random variable to two independent Gaussian random variables. Consider the independent uniformly distributed random variables $X_1(u)$ and $X_2(u)$

$$f_{X_1(u)}(z) = f_{X_2(u)}(z) = \begin{cases} 1 & z \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

The purpose of this problem is to demonstrate that the following are independent Gaussian random variables:

$$\begin{aligned} Y_1(u) &= \sqrt{-2\ln(X_1(u))}\cos(2\pi X_2(u)) \\ Y_2(u) &= \sqrt{-2\ln(X_1(u))}\sin(2\pi X_2(u)). \end{aligned}$$

(a) Determine the following: $f_{X_1(u)X_2(u)}(x_1, x_2)$, $\mathbb{E}\{Y_1(u)\}$ and $\mathbb{E}\{Y_2(u)\}$, and $\mathbb{E}\{Y_1(u)Y_2(u)\}$

(b) Consider the random variable $R(u) = \sqrt{-2\ln(X_1(u))}$. Determine the pdf $f_{R(u)}(r)$ and mean of this random variable.

(c) Determine the joint density of $Y_1(u)$ and $Y_2(u)$, generated as described above

(d) Answer the following questions:

- Are $X_1(u)$ and $X_2(u)$ uncorrelated?
- Are $X_1(u)$ and $X_2(u)$ orthogonal?
- Are $Y_1(u)$ and $Y_2(u)$ uncorrelated?
- Are $Y_1(u)$ and $Y_2(u)$ orthogonal?
- Are $Y_1(u)$ and $Y_2(u)$ independent?
- Are $R(u)$ and $X_2(u)$ independent?

1.2. This problem addresses the simulation of Gaussian noise. Generate N realizations of a mean zero, unit variance Gaussian random variable. From this set of realizations, compute two figures of merit:

- **A histogram of the realization values (i.e., a sample pdf):** plot the fraction of total realization between x and $x + \delta x$. Adjust the range and bin size to appropriately illustrate the pdf shape. Compare against the the pdf of a unit variance Gaussian random variable.
- **An estimate of the Q-function:** Compare against $Q(x)$, the fraction of realizations exceeding x.

Do this for $N = 100$, $N = 1000$, $N = 10^4$, and $N = 10^5$.

1.3. For complex numbers v and w, show that

$$\Re\{v\}\Re\{w\} = \frac{1}{2}\Re\{[vw + vw^*]\}$$

1.4. Let $\mathbf{n}(u)$ be a zero-mean, circular complex Gaussian random vector with covariance matrix $\mathbf{K_n} = \mathbb{E}\left\{\mathbf{n}(u)\mathbf{n}^\dagger(u)\right\}$. For a known signal vector $\mathbf{s}$, define the random variable $Z(u) = \mathbf{s}^\dagger\mathbf{n}(u)$. Determine $\text{Pr}\{\Re\{Z(u)\} > T\}$. If, $\|\mathbf{s}\|^2 = E$ and $\mathbf{K_n} = N_0\mathbf{I}$, what does this reduce to?

Hint: Use the results of problem 1.3.

1.5. Consider an event that occurs with probability p. For example, p could be the probability of error for a certain communications link or it could be the probability of "heads" for a coin flip. Suppose that p is unknown and is to be estimated by an experiment. Specifically, assume that N independent trials are conducted (e.g., coin flips) and let $K_N(u)$ denote the number of occurrences of the event. Then, take $\hat{p}(u) = K_N(u)/N$ as an estimate of p.

(a) Assuming that a large number of events are observed, approximate the pdf of $K_N(u)$ and $\hat{p}(u)$.

(b) Using this approximate pdf, find an expression for $\text{Pr}\{|\hat{p}(u) - p| > \epsilon p\}$. Note that ϵ is a measure of accuracy of the estimate and the desired probability is a measure of the statistical confidence that the estimate lies within that accuracy.

(c) Can you use this result to suggest a rule of thumb for simulation? Specifically, if one desires an estimate within 10% of p with 95% confidence, how many events (errors) should be observed? Put another way, if you run a simulation until 100 error events are observed, what can be said about the accuracy of the estimate $\hat{p}(u)$.

(d) Discuss the results of Prob. 1.2 in the context of the results of this problem.

1.6. Let $\mathbf{e}(u)$ be an $n \times 1$ binary vector with components $e_j(u)$ that are i.i.d. Bernoulli random variables, each taking the value 1 with probability ϵ and the value 0 with probability $1 - \epsilon$. Let $w(u)$ be the number of 1's in $\mathbf{e}(u)$. Determine the probability mass function for $w(u)$ – what is the name of the this pmf? Determine an expression for $\text{Pr}\{w(u) > d\}$.

1.7. **The Union Bound:** Prove the following results for arbitrary events A, B, C and A_i for $i = 1, 2 \ldots n$.

(a) $P(A \bigcup B) \leq P(A) + P(B)$. When does equality hold?

(b) $P(A \bigcup B \bigcup C) \leq P(A) + P(B) + P(C)$.

(c) $P\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i)$.

1.8. A simple method for detecting errors in a binary digital communications system is to use a parity check bit. A packet consists of $(n - 1)$ data bits and 1 parity bit. The parity bit is selected so that an even number of "1's" are contained in the transmitted packet of length n. The signal is then distorted and the receiver makes errors independently at each bit location with probability p. The number of 1's in the detected signal is then counted; if this number is even the packet is labeled good, otherwise it is labeled bad and the data is ignored.

(a) What is the probability that a packet is declared bad?

(b) Derive upper and lower bounds for the probability in (a) which can be made arbitrarily tight by including more terms.

(c) Use the family of bounds found in (b) to obtain a numerical answer for the probability of declaring the packet bad when $n = 1000$ and $p = 5 \times 10^{-3}$. Repeat for $n = 1000$ and $p = 1 \times 10^{-4}$.

1.9. Consider the expression

$$P(\alpha, \beta) = \alpha \mathrm{Q}\left(\sqrt{\frac{\beta 2E_b}{N_0}}\right)$$

Many coding systems have error probability approximated by a form similar to P. The standard way to plot such function is to plot P vs. E_b/N_0 with P on a $\log_{10}$ scale and E_b/N_0 in dB – *i.e.,* $(E_b/N_0)_{dB} = 10\log_{10}(E_b/N_0)$. This is essentially a log-log plot. Produce three plots in this format with the following curves on each plot:

(a) $P(1,1)$, $P(1,0.5)$, $P(1,2)$, $P(1,0.1)$, $P(1,10)$

(b) $P(1,1)$, $P(0.2,0.5)$, $P(0.2,2)$, $P(0.2,0.1)$, $P(0.2,10)$

(c) $P(1,1)$, $P(10^{-3},0.5)$, $P(10^{-3},2)$, $P(10^{-3},0.1)$, $P(10^{-3},10)$

Describe qualitatively the effect of α and β on the curves.

1.10. Consider the rate 1/3 repetition code on the binary symmetric channel (BSC). Determine the range of $\epsilon < 1/2$ for which this code improves performance?

1.11. Consider the rate 1/3 repetition code on the binary symmetric channel (BSC) discussed in lecture. Plot the probability of error (on a log scale) vs. E_b/N_0 in (dB) (i.e., Specifically, $X_{dB} = 10 * \log_{10} X$). Note that for such a code $E_c/N_0 = r(E_b/N_0)$ where r is the code rate.

Also, assume that the BSC is an abstraction of the BPSK-AWGN channel so that,

$$\epsilon = \mathrm{Q}\left(\sqrt{\frac{2E_c}{N_0}}\right)$$

Compare this with the case of no coding, where the error rate is $\mathrm{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right)$. Discuss this result in the context of Problem 1.10 - i.e., when does this code really help performance?

1.12. Consider the BSC as a model for the hard-decision BPSK-AWGN channel. In this case, the error probability for the coded bits is

$$\epsilon = \mathrm{Q}\left(\sqrt{\frac{2E_c}{N_0}}\right)$$

where E_c is the energy per coded bit and N_0 characterizes the noise power.

(a) Plot the capacity of the BSC as a function of E_c/N_0 in dB.

(b) Such a system can support code rates r for $r < C$. Using the relation that $E_c = rE_b$, plot the capacity of the BSC vs. E_b/N_0 (in dB) for a system operating at capacity – *i.e.,* the value of E_b/N_0 is the minimum value that can achieve the capacity.

1.13. Consider the linear block code defined by the parity check matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(a) Specify the parameters n, k, and r for this code.

(b) Draw the Tanner Graph corresponding to this parity check matrix.

2 Detection Theory

2.1. Discuss the properties of the Bayes decision rule for the following limiting cases:

(a) $\pi_0 \to 1$ with all other parameters fixed and finite.

(b) $C_{10} \to \infty$ with all other parameters fixed and finite.

2.2. Consider the binary hypothesis testing problem defined by

$$\begin{aligned} \mathcal{H}_0 : \quad & R(u) = +\sqrt{E} + N(u) \\ \mathcal{H}_1 : \quad & R(u) = -\sqrt{E} + N(u), \end{aligned}$$

where the a-priori probabilities are π_0 and π_1 and the noise is Gaussian with mean zero and variance σ^2. In class we looked in detail at this problem for $\pi_0 = \pi_1$; repeat this here for the more general case. Determine the MAP detection rule and the probability of error $P(\mathcal{E})$.

Plot $P(\mathcal{E})$ vs. E/σ^2 with a log y-axis and the SNR (E/σ^2) expressed in dB (i.e., $10\log_{10}(E/\sigma^2)$). Produce plots for $\pi_0 = 10^{-4}, 0.2, 0.5, 0.9$.

2.3. *(Spring 2001, Midterm)* A large box containing standard (fair) dice and another containing loaded (unfair) dice have been mixed together. Each loaded die is biased to roll a 1 or a 6. Specifically, let $R(u)$ denote the outcome of a roll, then $\Pr\{R(u) = 1|\text{loaded}\} = \Pr\{R(u) = 6|\text{loaded}\} = 3/10$ and $\Pr\{R(u) = r|\text{loaded}\} = 1/10$ for $r \in \{2, 3, 4, 5\}$. Of course, each fair die is equally likely to roll any integer between 1 and 6.

Consider randomly selecting a die from the box containing an equal number of fair and loaded dice. Based on one roll, you would like to determine if the selected die if fair or loaded.

(a) Find the decision rule that minimizes the probability of error in deciding whether the selected die is fair or loaded based on a given roll – i.e., given $R(u) = r$, specify either "Fair" or "Loaded" for $r = 1, 2, 3, 4, 5, 6$.

(b) What is the probability of error of the receiver derived in part (a)?

2.4. (Modified Van Trees Example) The print queue in the EE-Systems computer network observes $K(u)$ arrivals in an hour. $K(u)$ is modeled as a Poisson random variable,

$$\textsc{Pr}\left\{K(u) = k\right\} = \frac{\alpha^k}{k!}e^{-\alpha} \qquad k = 0, 1, 2 \ldots$$

If it is a weekday, $\alpha = \alpha_w$; on weekends, $\alpha = \alpha_e$. You have data from a randomly selected day (i.e., $K(u) = k$); what is the MAP rule for deciding if it is a weekday or a weekend day? Don't forget to model the a-priori probabilities appropriately.

Specialize to the example of $\alpha_w = 10$ and $\alpha_e = 1$. For this case, find the probability of error.

2.5. (Weber 4.13) Find the MAP rule for deciding between $\mathcal{H}_0$ and $\mathcal{H}_1$ based on the observation $R(u)$ with $\pi_0 = 3/4$ and

$$f_{R(u)}(r|\mathcal{H}_1) = \begin{cases} \frac{1}{5} & 0 \le r \le 5 \\ 0 & \text{otherwise} \end{cases}$$
$$f_{R(u)}(r|\mathcal{H}_0) = \frac{1}{\sqrt{2\pi}}e^{\frac{-r^2}{2}}.$$

2.6. Consider the binary hypothesis testing problem defined by

$$\begin{aligned} \mathcal{H}_0: &\quad R(u) = 0 + N(u) \\ \mathcal{H}_1: &\quad R(u) = 1 + N(u), \end{aligned}$$

where the a-priori probabilities are equal and the noise pdf is exponential with mean $1/\lambda$ under either hypothesis.

(a) Determine the decision rule which minimizes the probability of error and derive an expression for this minimum probability of error.

(b) If the a-priori probabilities are not equal, can the decision rule be implemented by

$$R(u) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} T?$$

If so, what is the range for the threshold T?

(c) Determine the minimax decision rule for this problem. **Hint:** a randomized rule is required.

2.7. You have received a high-level mathematical software package with poor documentation. There is a function called `random()` that generates random numbers – i.e., calling the function returns a realization of a random variable and successive calls are statistically independent.

You know that this function generates mean zero, unit variance random variables, but you are uncertain whether these are uniformly distributed or Gaussian (you don't have any prior bias). You make k calls of the function and from these observations, you would like to make a decision as to whether the function generates Gaussian random variables or uniformly distributed random variables.

Mathematically, let $X_1(u)$, $X_2(u)$, $\ldots X_k(u)$ model the outputs of k calls of `random()`.

(a) First consider $k = 1$ – i.e., only one call is made. Determine the conditional probability density function of $X_1(u)$ under the two possibilities – *i.e.,* $f_{X_1(u)}(x_1|\text{Gaussian})$ and $f_{X_1(u)}(x_1|\text{Uniform})$. Based on observing the realization $X_1(u) = x_1$, specify a good rule for when you would decide that `random()` generates Gaussian random variables.

(b) Find the conditional probability of error for each possibility for the good rule in part (a) – *i.e.,* $P(\mathcal{E}|\text{Gaussian})$ and $P(\mathcal{E}|\text{Uniform})$.

(c) Now specify a good rule for deciding the type of random number generator using k calls to `random()`. Specifically, given $X_1(u) = x_1$, $X_2(u) = x_2$, $\ldots X_k(u) = x_k$, state a rule for deciding that `random()` generates Gaussian random variables.

2.8. **Top Secret** *(midterm exam, K. M. Chugg, U. of Arizona, Spring 1996).* You are working as a spy! Your job is to determine if anyone is actively communicating across a given AWGN channel. Your sources have given you some reliable information: you know that communication is carried out by BPSK modulation and you also know T, N_0, along with the carrier frequency and phase. Your observation is the output of the correlator for a BPSK receiver. Your mission is to decide between the following two hypotheses:

$$\begin{aligned} \mathcal{H}_0 &: \quad R(u) = N(u) && \text{(channel not in use)} \\ \mathcal{H}_1 &: \quad R(u) = \sqrt{E}B(u) + N(u) && \text{(channel in use),} \end{aligned}$$

where the random variable $B(u)$ represents the effect of the random binary modulation:

$$\text{Pr}\{B(u) = +1\} = \text{Pr}\{B(u) = -1\} = \frac{1}{2}.$$

The noise at the output of the correlator (i.e., $N(u)$) is Gaussian with zero mean and variance $N_0/2$.

The apriori probability of someone being on the channel is assumed to be 1/2 (i.e., $\pi_0 = \pi_1 = 1/2$).

(a) Determine the probability density function of $R(u)$ under either hypothesis: $f_{R(u)}(r|\mathcal{H}_0)$ and $f_{R(u)}(r|\mathcal{H}_1)$.
Sketch these two pdfs on the same axis.

(b) Determine the MAP decision rule (based on r, the realization of $R(u)$).
Using the properties of the cosh function and its inverse, the rule can be simplified to:

$$|r| \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} T.$$

Specify the value of T which results in the minimum error probability.

(c) Your superiors want to initiate "counter-measures" when your decision device tells them the channel is being used. Not wanting to be hasty, they are concerned about the *False Alarm Probability* for your rule. Determine this probability: $P_{FA} = P(\mathcal{E}|\mathcal{H}_0)$. Also determine the detecion probability $P_D = 1 - P(\mathcal{E}|\mathcal{H}_1)$, and therefore $P(\mathcal{E})$. (Note: For the original exam, only P_{FA} was requested).

i. Plot $P(\mathcal{E})$ for this decision problem and $P(\mathcal{E})$ for the standard BPSK decision problem on the same plot (i.e., $P(\mathcal{E}) = \mathrm{Q}\left(\sqrt{\frac{2E}{N_0}}\right)$).

ii. Compare this detection problem to the one presented in lecture – i.e., where $\mathcal{H}_0$ remains the same and $\mathcal{H}_1$ is defined by $R(u) = \sqrt{E} + N(u)$. Compute the Receiver Operating Characteristic (ROC) for the problem stated above and compare it to the lecture example by plotting P_D vs. P_{FA} for various E/σ^2.

(d) Apply the method of generalized likelihood to this problem and show that this approach also yields a test of the form

$$|r| \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} T_G.$$

Determine T_G and compare $P(\mathcal{E})$ for this generalized likelihood approach and the MAP approach (i.e., average likelihood) approach used above.

2.9. Consider the binary hypothesis testing problem defined by the (2×1) observation

$$\mathcal{H}_0 : \quad \mathbf{r}(u) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \mathbf{n}(u)$$

$$\mathcal{H}_1 : \quad \mathbf{r}(u) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \mathbf{n}(u),$$

where the a-priori probabilities are equal and the noise is zero mean Gaussian with covariance

$$\mathbf{K_n} = \begin{bmatrix} 10 & 9 \\ 9 & 10 \end{bmatrix}.$$

Sketch the signals and contour plots of the conditional densities (i.e., $f_{\mathbf{r}(u)}(\mathbf{z}|\mathcal{H}_j);\ j = 0, 1$) in the $\mathcal{R}^2$–plane. Also indicate the decision boundary of the MAP decision rule.

Repeat for the white noise case with the same noise power (i.e., $\sigma^2 = 10$).

Hint: You know the eigenvectors and eigenvalues from lecture. Plot the decision boundary in the coordinate system of the eigenvectors.

2.10. **8-PSK:** Consider the problem of data detection for an 8-PSK signal. Such a signal is represented by

$$\mathcal{H}_m : \quad \mathbf{r}(u) = \sqrt{E_s} \begin{bmatrix} \cos\left(\frac{\pi}{4} m\right) \\ \sin\left(\frac{\pi}{4} m\right) \end{bmatrix} + \mathbf{n}(u) = \mathbf{s}_m + \mathbf{n}(u) \qquad m = 0, 1, 2, \ldots 7$$

where the noise is a Gaussian random vector, with zero mean and uncorrelated components, each with variance σ^2. You may assume that the a-priori probabilities are 1/8 for each hypothesis.

(a) Determine the MAP rule for minimizing the symbol error probability. Sketch in the two-dimensional plane, the signals and the associated decision regions.

(b) **(Challenging Problem)** Consider the case when each of these 8-PSK symbols corresponds uniquely to a set of three input bits. For example, assume that $\mathcal{H}_m$ corresponds to a particular possibility for $A(u)$, $B(u)$, and $C(u)$, each taking on 0 or 1 with probability 1/2 and independent of each other. Specifically, map these three binary symbols to an 8-ary symbol by $\mathbf{s}_m \iff (a, b, c)$ where

$$\begin{aligned}
\mathbf{s}_0 &\iff 000 \\
\mathbf{s}_1 &\iff 001 \\
\mathbf{s}_2 &\iff 010 \\
\mathbf{s}_3 &\iff 011 \\
\mathbf{s}_4 &\iff 100 \\
\mathbf{s}_5 &\iff 101 \\
\mathbf{s}_6 &\iff 110 \\
\mathbf{s}_7 &\iff 111
\end{aligned}$$

Consider two approaches to finding a decision rule that minimizes the probability of bit error $P_b(\mathcal{E})$.

- **Approach 1:** Find $f_{\mathbf{r}(u)}(\mathbf{z}|A(u) = 0)$ and $f_{\mathbf{r}(u)}(\mathbf{z}|A(u) = 1)$. Use this to design the MAP rule for deciding on $A(u)$. Repeat for $B(u)$ and $C(u)$.
- **Approach 2:** Determine an appropriate cost matrix for the 8-ary test so that the average number of bit errors is minimized. Determine the associated decision rule.

(c) Discuss your results. In particular, are the two decision rules from part (b) the same? If not, which is better. Under what conditions are all three decision rules equivalent? How about approximately equivalent?

2.11. Write a program to simulate the signals from problem 10 – i.e., to generate realizations of the random vector $\mathbf{r}(u)$. Plot in the 2D plane, 100 realizations of $\mathbf{r}(u)$ given that hypothesis $\mathcal{H}_0$ is true. Do this for $E_s/2\sigma^2 = 3$ dB, 8 dB and 15 dB. If the decision rule from 10(a) was used, how many of these 100 realizations would result in a symbol error?

2.12. Use the simulation program developed in problem 11 for 8-PSK to approximate the *bit* error probability for 8-PSK signaling. Plot the simulation BER results for two cases: (i) natural bit ordering and (ii) Gray mapping. This plot should be for BER (log-scale) vs. $E_s/2\sigma^2$ in dB. Be sure to use the results of problem 1.2 to ensure that you are plotting reliable estimates.

2.13. A binary communication system uses $k = 100$ dimensional signaling with each hypothesis occurring with equal a-priori probability. The hypotheses are:

$$\begin{aligned}
\mathcal{H}_0 : \quad \mathbf{r}(u) &= \mathbf{s}_0 + \mathbf{w}(u) \\
\mathcal{H}_1 : \quad \mathbf{r}(u) &= \mathbf{s}_1 + \mathbf{w}(u)
\end{aligned}$$

where $\mathbf{w}(u)$ is Gaussian, with zero mean and covariance $\mathbf{K}_\mathbf{w} = \sigma^2\mathbf{I}$. Let $s_i(n)$ be the n^{th}

component of the i^{th} signal, for $n = 0, 1, \ldots 99$. Then the two signals are given by

$$s_0(n) = \cos\left(\frac{2\pi}{4}n\right) \qquad n = 0, 1, \ldots 99$$
$$s_1(n) = \cos\left(\frac{2\pi}{5}n\right) \qquad n = 0, 1, \ldots 99$$

The two hypotheses occur with equal probability a-priori.

(a) Determine the minimum error probability receiver given the realization $r(u, n) = r(n)$ for $n = 0, 1, \ldots 99$. Sketch the preferred implementation of this receiver.

(b) Find the probability of error for the MAP receiver of part (a).

2.14. Consider the vector communication channel with

$$\begin{aligned} \mathcal{H}_0 :& \quad \mathbf{r}(u) = \mathbf{s}_0(u) + \mathbf{n}(u) \\ \mathcal{H}_1 :& \quad \mathbf{r}(u) = \mathbf{s}_1(u) + \mathbf{n}(u), \end{aligned}$$

where under either hypothesis the noise and signals are independent and the noise is zero mean Gaussian with covariance $\mathbf{K_n}$. The signals $\mathbf{s}_0(u)$, and $\mathbf{s}_1(u)$ are zero mean Gaussian with covariance matrices $\mathbf{K}_0$ and $\mathbf{K}_1$, respectively. Determine the minimum error probability rule for equal prioir probabilities. Simplify for the special case of $\mathbf{K_n} = \sigma_n^2\mathbf{I}$, $\mathbf{K}_0 = \sigma_0^2\mathbf{I}$, and $\mathbf{K}_1 = \sigma_1^2\mathbf{I}$. Assume throughout that $\mathbf{K}_0 \neq \mathbf{K}_1$ and justify your (white) rule intuitively.

2.15. *(Midterm Exam, F98)* Two recent USC graduates are assigned the problem of designing and analyzing a decision rule for the two-dimensional binary hypothesis problem:

$$\begin{aligned} \mathcal{H}_0 :& \quad \mathbf{r}(u) = +\mathbf{s} + \mathbf{n}(u) \\ \mathcal{H}_1 :& \quad \mathbf{r}(u) = -\mathbf{s} + \mathbf{n}(u) \end{aligned}$$

where the two equally-likely ($\pi_0 = \pi_1$) signals are defined by ($S > 0$)

$$\mathbf{s} = \begin{bmatrix} S & 0 \end{bmatrix}^{\mathrm{t}}$$

and the noise pdf is given by

$$f_{n(u,1),n(u,2)}(x, y) = \begin{cases} 2 & (x, y) \in \mathcal{I}_+ \iff x \in [0, \frac{1}{2}]\text{and}y \in [0, \frac{1}{2}] \\ 2 & (x, y) \in \mathcal{I}_- \iff x \in [\frac{-1}{2}, 0) and y \in [\frac{-1}{2}, 0) \\ 0 & \text{otherwise} \end{cases}$$

In other words, $\mathbf{n}(u)$ is uniformly distributed over the shaded region shown in Fig. 1

One USC graduate, Wilma, has taken EE564, while the other, Fred, has taken only EE562a.

(a) Fred designs his receiver based on the second moment description of the noise. Determine these quantities – i.e., $\mathbf{m_n}$ and $\mathbf{K_n}$.

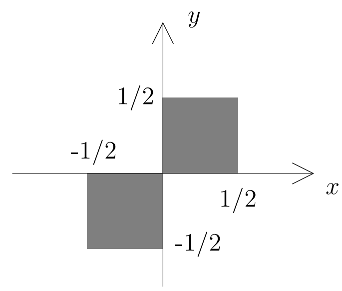


Figure 1: Probability distribution function for two-dimensional noise.

(b) Determine Fred's "colored-noise minimum distance detector," which is based on the above moments. Specifically, state the decision to be taken when $\mathbf{r}$ is observed. Sketch this decision rule in the $(r(1), r(2))$-plane.

(c) Wilma uses MAP detection theory. Derive this rule and demonstrate your understanding of it by sketching the decision regions for the special case of $S = \frac{1}{8}$.

(d) For $S > S_p$, the MAP decision rule is perfect – i.e., $P(\mathcal{E}) = 0$. Determine the minimum value of S_p having this property.

For Wilma's MAP decision rule, determine probability of error for $S < S_p$.

Determine probability of error for Fred's rule when $S = S_p$; use a sketch to indicate the region in the plane corresponding to an error for this case.

2.16. Two bits are to be communicated using four (4×1) vectors. These bits are represented by the random variables $A(u)$ and $B(u)$ which are independent and each equally likely to to be -1 or $+1$. The signal sent given $A(u) = a$ and $B(u) = b$ are

$$\mathbf{s}_{a,b} = a\sqrt{E} \begin{bmatrix} +1 \\ +1 \\ b \\ b \end{bmatrix}$$

The channel is an AWGN channel so that your decision is to be based on $\mathbf{r}(u) = \mathbf{s}_{a,b} + \mathbf{n}(u)$, where $\mathbf{n}(u)$ is a mean-zero Gaussian with $\mathbf{K_n} = \sigma^2\mathbf{I}$.

(a) Find the decision rule that minimizes the probability of error for deciding only $B(u)$ based on a realization $\mathbf{r}$ of the observation.

(b) Find the decision rule that minimizes the probability of error for deciding only $A(u)$ based on a realization $\mathbf{r}$ of the observation.

(c) Consider the detector that minimizes the probability of 4-ary error, $P(\mathcal{E})$ (i.e., the optimal symbol detector). Describe this decision rule in a simple form.

Taken together, the rules from (a) and (b) describe a 4-ary rule. Is this rule the same as the one found just above? Explain.

(d) Find a lower bound on the probability of 4-ary error for the receiver in part (c)

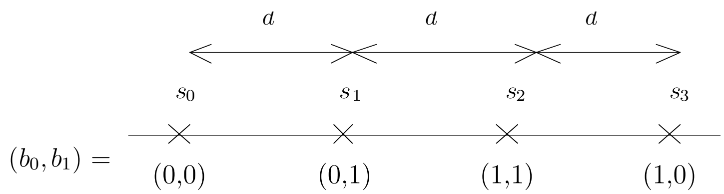


Figure 2: The 4-PAM signal set with Gray bit labeling.

2.17. Consider the 4-PAM system with signals as defined in Fig. 2.

Each signal is labeled with (b_0, b_1) which are each equal to 0 or 1 with probability 1/2 and are independent. Based on the post-matched-filter observation

$$z(u) = s(u) + w(u)$$

where $w(u)$ is mean-zero Gaussian with variance $N_0/2$ and $s(u)$ is the PAM constellation point, determine the following:

(a) The decision rule that minimizes the probability of symbol error and expressions for the following:
   i. The symbol error probability.
   ii. The probability that the best symbol decision results in an error for b_0.
   iii. The probability that the best symbol decision results in an error for b_1.

(b) The decision rule that minimizes the probability of error for b_0. Simplify as much as possible.

(c) The decision rule that minimizes the probability of error for b_1. Simplify as much as possible.

(d) To determine a lower bound on the bit error probabilities for bit-optimal receivers, consider the case when a "genie" provide side information in the form of an auxiliary observation $\mathbf{v}(u)$. The corresponding bit-optimal decision that exploits this side-information cannot be worse than the bit-optimal rule which operates without side information.
   i. State the optimal receiver for bit b_1 given the observation z and side information $\mathbf{v}$
   ii. Construct a lower bound for the optimal detector for b_1 using the side information

$$\begin{aligned}
P(\mathbf{v}(u)|s_0) = 1 \quad &\Longleftrightarrow \quad \mathbf{v} = (s_0, s_1) \\
P(\mathbf{v}(u)|s_1) = 1 \quad &\Longleftrightarrow \quad \mathbf{v} = (s_0, s_1) \\
P(\mathbf{v}(u)|s_2) = 1 \quad &\Longleftrightarrow \quad \mathbf{v} = (s_2, s_3) \\
P(\mathbf{v}(u)|s_3) = 1 \quad &\Longleftrightarrow \quad \mathbf{v} = (s_2, s_3)
\end{aligned}$$

2.18. A binary communication system uses k dimensional signaling with each hypothesis occurring with equal a-priori probability. The hypotheses are:

$$\begin{aligned}
\mathcal{H}_0: \quad \mathbf{r}(u) &= +\mathbf{s} + \mathbf{w}(u) \\
\mathcal{H}_1: \quad \mathbf{r}(u) &= -\mathbf{s} + \mathbf{w}(u)
\end{aligned}$$

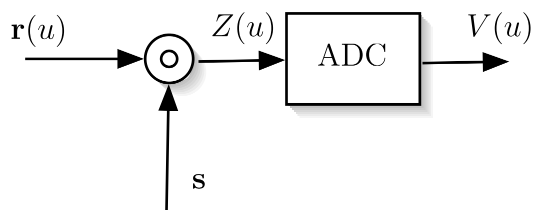


Figure 3: Receiver structure constraint.

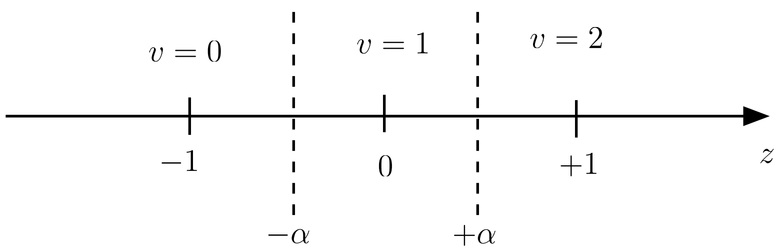


Figure 4: Receiver structure constraint.

where $\mathbf{w}(u)$ is Gaussian, with zero mean and covariance $\mathbf{K_w} = \sigma^2\mathbf{I}$, and $\mathbf{s}$ is a known vector with $\|\mathbf{s}\| = 1$.

It is your job to design a receiver for this system based on the output of a predetermined front-end processor. Specifically, the current design has a correlator to $\mathbf{s}$ and a "bit and a half" analog to digital converter (ADC). Mathematically, the random variable $Z(u) = \mathbf{r}^{\mathrm{t}}(u)\mathbf{s}$ is generated and converted to $V(u)$ via the ADC as diagrammed in Fig. 3.

The ADC has three possible outputs - i.e., $V(u) \in \{0, 1, 2\}$. The mapping from $Z(u)$ to $V(u)$ is as shown in Fig. 4 for possible realizations of $V(u) = v$ and $Z(u) = z$.

Note that α is an adjustable design parameter and $0 \leq \alpha \leq 1$ – i.e., the value will be fixed for implementation, but can be selected during design.

(a) Find the conditional probability mass function of $V(u)$ under each hypothesis

(b) Find a MAP decision rule for deciding between $\mathcal{H}_0$ and $\mathcal{H}_1$ based on $V(u) = v$.

(c) Find the probability of error for the MAP receiver. Which is the best choice for α - i.e.,α_{opt}? Compare the MAP receiver using $\mathbf{r}(u)$ and $V(u)$ – *i.e.,* is the performance MAP receiver using $V(u)$ with α_{opt} better, the same, or worse than that of the MAP receiver using $\mathbf{r}(u)$?

2.19. An $M = 4$ communication system uses the following set of two dimensional signals:

$$\mathbf{s}_0 = \sqrt{\frac{E}{2}} \begin{bmatrix} +1 \\ +1 \end{bmatrix} \qquad \mathbf{s}_1 = \sqrt{\frac{E}{2}} \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

$$\mathbf{s}_2 = \sqrt{\frac{E}{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \mathbf{s}_3 = \sqrt{\frac{E}{2}} \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

Each of the four signals is sent with equal probability. The channel is modeled as colored additive Gaussian noise such that

$$\mathcal{H}_m : \quad \mathbf{r}(u) = \mathbf{s}_m + \mathbf{n}(u), \quad m = 0, 1, 2, 3$$

where $\mathbf{m_n} = \mathbf{0}$ and the nosie covariance is

$$\mathbf{K_n} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) The rule that minimizes the probability of 4-ary error given $\mathbf{r}(u) = \mathbf{r}$ can be described as

$$\text{Decide } \mathcal{H}_m \quad \Longleftrightarrow \quad g_m(\mathbf{r}) > g_j(\mathbf{r}), \; j \in \{0, 1, 2, 3\}, \; j \neq m$$

Specify a valid set of functions $g_m(\mathbf{r})$ in the simplest form you can obtain.

(b) Determine and sketch the decision regions imposed by the minimum error probability rule in the (r_1, r_2) plane. *Clearly identify the parameters that completely identify the decision regions in the manner most convenient to you.*

(c) Find good upper and lower bounds for the error probability for the above rule.

2.20. A binary communication link is modeled as

$$\begin{aligned} \mathcal{H}_0 : \quad \mathbf{r}(u) &= +A(u)\mathbf{s} + \mathbf{w}(u) \\ \mathcal{H}_1 : \quad \mathbf{r}(u) &= -A(u)\mathbf{s} + \mathbf{w}(u) \end{aligned}$$

where $\mathbf{w}(u)$ is a $(k \times 1)$ Gaussian random vector, with zero mean and covariance $\mathbf{K_w} = \sigma^2 \mathbf{I}$, and $\mathbf{s}$ is a known vector with $\|\mathbf{s}\| = 1$. Each hypothesis occurs with probability 0.5.

The amplitude of the signal is a positive random variable, independent of the hypothesis, with probability density function

$$f_{A(u)}(a) = e^{-a}\mathrm{U}(a)$$

where $\mathrm{U}(a)$ is the unit step function.

The goal of this problem is to obtain the rule that minimizes the probability of error.

(a) Given a conditional value of $A(u) = a > 0$, what is the minimum error probability rule based on $\mathbf{r}(u) = \mathbf{r}$? Also determine the associated error probability.

(b) Determine the decision rule that minimizes the probability of error averaged over the statistical description of $A(u)$. Give a brief justification of the above rule based on decision theory.

2.21. Consider a California election with two candidates: candidate A and candidate B. Steve works for a national-level political action committee (PAC) that uses funding in an attempt to affect elections. Steve's PAC favors candidate A. Steve must decide whether to get involved in this

California election in support of candidate A or whether he should use these resources to support other candidates in elections in other states.

Steve has a MSEE degree and decides to apply decision theory to determine his best action. Steve only wants to get involved in this election if it is a very close election – *i.e.,* his PAC can help candidate A win. (If candidate A is a heavy favorite to win, he doesn't need to invest resources. If candidate B is a heavy favorite to win, he shouldn't invest resources because he cannot sway enough voters to secure the win for candidate A.) So, Steve models the problem as a 3-hypothesis test with a-prior equally likely hypotheses:

$$\begin{aligned}
\mathcal{H}_A &: \text{ candidate A is a strong favorite to win} \\
\mathcal{H}_T &: \text{ it is a virtual tie (very close) between the two candidates} \\
\mathcal{H}_B &: \text{ candidate B is a strong favorite to win}
\end{aligned}$$

The observation that Steve will use to make his decision is the result of phone-polling of subset of likely voters – *i.e.,* he has a measurement z of the percentage of polled voters who plan to vote for candidate A. Steve models this observation $Z(u)$ as Gaussian with variance σ^2. The mean of $Z(u)$ is 75 given $\mathcal{H}_A$, 50 given $\mathcal{H}_T$, and 25 given $\mathcal{H}_C$.

(a) State the decision rule based on $Z(u) = z$ that minimizes the probability of 3-ary error in the simplest form to implement and determine the probability of error for this rule.

(b) After obtaining the above rule, Steve wonders if he used the appropriate optimality criterion. Specifically, it is worse for Steve to not invest in a close election (*i.e.,* lose an election he could have won) than it is to invest in an election with a heavy favorite (*i.e.,* invest without affecting the outcome).

Steve incurs no penalty if he decides the correct hypothesis. He also incurs no cost if he decides one candidate is a heavy favorite when the other candidate is actually the heavy favorite since he would not invest in either case. If Steve decides the election is close when A or B is a heavy favorite, he pays the cost of investing without a chance of affecting the election – say $C > 0$. If the election is close, but Steve decides that there is a heavy favorite, he pays a higher cost – say $10C$.

Determine the decision rule that minimizes Steve's cost in the simplest form.

3 Modualtion and Performance on the AWGN Channel

3.1. For this problem, let $\langle \mathbf{a}, \mathbf{b} \rangle$ denote *any valid inner product* on $\mathcal{R}^n$, and $\|\mathbf{b}\| = \sqrt{\langle \mathbf{b}, \mathbf{b} \rangle}$ denote the associated norm (i.e. these are *not* necessarily the standard Euclidean inner product and norm). Consider the standard real binary hypothesis testing problem

$$\mathcal{H}_i : \quad \mathbf{x}(u) = \mathbf{s}_i + \mathbf{n}(u) \quad i = 1, 2,$$

where $\mathbf{m_n} = \mathbf{0}$ and $\mathbf{K_n}$ is invertible.

Start with the *generalized minimum distance criterion* for the binary hypothesis testing problem:

$$\|\mathbf{x}(u) - \mathbf{s}_1\| \underset{\mathcal{H}_1}{\overset{\mathcal{H}_2}{\gtrless}} \|\mathbf{x}(u) - \mathbf{s}_2\|.$$

(a) Show that an equivalent decision is

$$\langle \mathbf{s}_1 - \mathbf{s}_2, \mathbf{x}(u) \rangle \underset{\mathcal{H}_2}{\overset{\mathcal{H}_1}{\gtrless}} \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2}.$$

(b) Show that the following defines a valid inner product on $\mathcal{R}^n$

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{n}} \triangleq \mathbf{b}^{\mathrm{t}} \mathbf{K}_{\mathbf{n}}^{-1} \mathbf{a}.$$

(c) What is the decision rule corresponding to this choice of inner product? Do you recognize this rule?

(d) Sketch the locus of points which are unit distance from the origin with the distance function implied by the inner product in (1b). Consider the simple case of

$$\mathbf{K}_{\mathbf{n}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

In other words sketch the curve

$$d_{\mathbf{n}}(\mathbf{b}, \mathbf{0}) = 1,$$

where

$$d_{\mathbf{n}}(\mathbf{b}, \mathbf{0}) = \|\mathbf{b} - \mathbf{0}\|_{\mathbf{n}} = \left(\langle \mathbf{b}, \mathbf{b} \rangle_{\mathbf{n}}\right)^{1/2}.$$

3.2. (Legendre Polynomials) Consider the functions $x_n(t) = t^n$ for $n = 0, 1, 2, \ldots$ as points in $\mathcal{L}_2[-1, 1]$. Find an orthonormal basis for the subspace of $\mathcal{L}_2[-1, 1]$ which is spanned by $\{x_0(t), x_1(t), x_2(t)\}$.

3.3. (Proakis 5-18): Suppose that a BPSK modulation is used for transmitting information over an AWGN channel with $N_0/2 = 10^{-10}$ W/Hz. The transmitted signal energy is $E_b = A^2T/2$, where T is the bit interval and A is the signal amplitude. Determine the signal amplitude required to achieve an error probability of 10^{-6} when the data rate is (a) 10 kbits/sec (kbps), (b) 100 kbps, (c) 1 Mbit/sec.

3.4. Consider the binary hypothesis testing problem defined by the observation

$$\begin{aligned} \mathcal{H}_0: \quad r(u,t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_c t) + n(u,t) & t \in [0, T] \\ \mathcal{H}_1: \quad r(u,t) &= \sqrt{\frac{2E}{T}} \sin(2\pi f_c t) + n(u,t) & t \in [0, T], \end{aligned}$$

where the a-priori probabilities are equal and the noise is Gaussian with $K_n(\tau) = \frac{N_0}{2}\delta_D(\tau)$. Determine and sketch the optimal receiver. Determine the error probability and compare the performance to the BPSK example presented in class.

3.5. Consider the BPSK example developed in class:

$$\begin{aligned} \mathcal{H}_0: \quad r(u,t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \theta_c) + n(u,t) & t \in [0, T] \\ \mathcal{H}_1: \quad r(u,t) &= -\sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \theta_c) + n(u,t) & t \in [0, T], \end{aligned}$$

where the noise is the standard AWGN.

(a) Repeat the performance analysis from lecture without the narrowband assumption – i.e., do not discard the double frequency terms.

(b) Your answer should be of the form

$$P(\mathcal{E}) = Q\left(\sqrt{2\gamma_{\text{eff}}}\right),$$

where γ_{eff} is the effective $\gamma = E/N_0$ including the double frequency terms. Consider the special case of a 300 kilo-bits per second (kbps) system with a carrier frequency of $f_c = 1$ GHz $= 10^9$ Hz. Plot the $\gamma_{\text{eff}}/\gamma$ vs. θ_c for $\theta_c \in [0, 2\pi]$. What conclusions can you draw?

3.6. (R. A. Scholtz – USC Midterm, Spring 1994) Consider the binary hypothesis testing problem defined by ($\pi_0 = \pi_1$)

$$\begin{aligned} \mathcal{H}_0: &\quad r(u,t) = n(u,t) && t \in [0,T] \\ \mathcal{H}_1: &\quad r(u,t) = s(t) + n(u,t) && t \in [0,T]. \end{aligned}$$

The KL-expansion for the noise is known. The orthonormal set of eigenfunctions for $K_n(t_1, t_2)$ are $\{e_m(t)\}_{m=1}^{\infty}$. The signal and noise expansions are

$$\begin{aligned} s(t) &= \sum_{m=1}^{\infty} \frac{1}{m^2} e_m(t) \\ n(u,t) &= \sum_{m=1}^{\infty} N(u,m) e_m(t), \end{aligned}$$

where the random variables $N(u,m)$ are mean zero Gaussian, with

$$\mathbb{E}\left\{N(u,m)N(u,i)\right\} = 2^{-m}\delta_K(m-i).$$

(a) What is the signal energy – $E = \int_0^T s^2(t)\,dt$? Write down the series expansion for the noise cavariance.

(b) Consider the *suboptimal* detector which only uses a single component of the expansion. Based on only the m^{th} component, the best rule takes the form

$$\text{“decide } \mathcal{H}_1 \text{ is true”} \quad \Longleftrightarrow \quad \int_0^T r(u,t) e_m(t)\,dt > T_m.$$

Determine the best threshold T_m and the error performance of this suboptimal receiver.

(c) What is the probabbility of error of the *optimal* detector for this problem?
Hint: What happens to the error probability of the suboptimal detector as $m \to \infty$?

3.7. Consider the standard binary detection problem in AWGN:

$$\begin{aligned} \mathcal{H}_0: &\quad r(u,t) = s_0(t) + n(u,t) && t \in [0,T] \\ \mathcal{H}_1: &\quad r(u,t) = s_1(t) + n(u,t) && t \in [0,T], \end{aligned}$$

with $\pi_0 = \pi_1$. The signals are shown in Fig. 5 (note that $s_1(t)$ is parametized by x).

Sketch the optimal detector and find the probability of error. For what value of x is the performance the same as BPSK? For what value of x is the performance the same as that for the system of problem 4? Can you explain this?

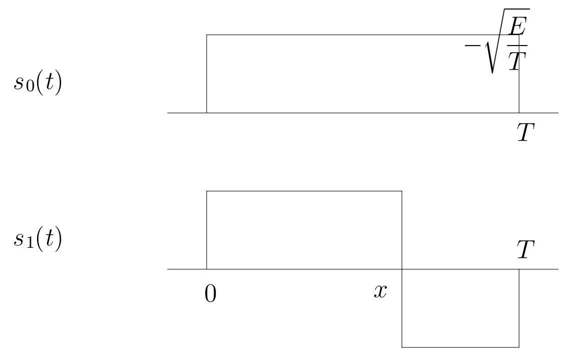


Figure 5: Two signals with $s_1(t)$ parametized by x.

3.8. Consider the complex-valued observation

$$r(u,t) = s(t) + n(u,t) \qquad t \in [0,T]$$

where $s(t)$ is a known signal with $\int_0^T |s(t)|^2 dt = E$ and $n(u,t)$ is the complex baseband equivalent of white noise (i.e., $En(u,t_1)n^*(u,t_2) = N_0\delta_D(t_1 - t_2)$). Consider the output of a correlator designed for a signal $y(t)$

$$R(u) = S_y + N_y(u) = \int_0^T r(u,t)y^*(t)dt$$

(a) Find the unit energy signal $y(t)$ (on $t \in [0,T]$) that maximizes the signal-to-noise ratio (SNR) at the output. Specifically,

$$\text{SNR} = \frac{|S_y|^2}{\mathbb{E}\left\{|N_y(u)|^2\right\}}$$

Hint: Cauchy-Schwartz Inequality

(b) Show that $R(u)$ can also be obtained by sampling the output of a filter with impulse response $y^*(-t)$ once at a particular time. Determine this time. Draw this impulse response for SNR-maximizing choice of $y(t)$ for the two signals in problem 7.

3.9. (Chugg, Final Exam, Spring 1996, Arizona) Consider the following three equally likely hypotheses

$$\begin{aligned}
\mathcal{H}_0 &: \quad r(u,t) = n(u,t) \\
\mathcal{H}_1 &: \quad r(u,t) = A\sin(2\pi f_c t) + n(u,t) \\
\mathcal{H}_2 &: \quad r(u,t) = -A\sin(2\pi f_c t) + n(u,t),
\end{aligned}$$

where in each case the observation is made for $t \in [0,T]$. The noise is AWGN with spectral level $N_0/2$ and $A > 0$.

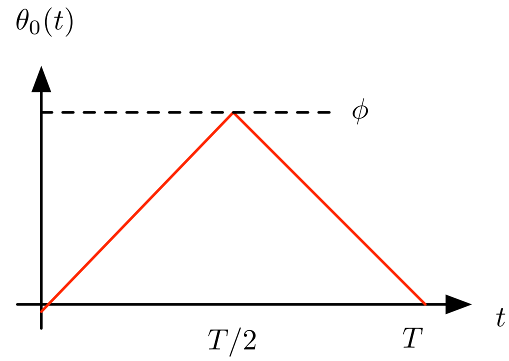


Figure 6: The phase function for Problem 3.11.

(a) Describe the receiver that minimizes the probability of error. Sketch this receiver and describe all parameters.

(b) Determine the probability of error for the optimal receiver in (a).

3.10. (Van Trees 2.2.13) We have considered the LR $\Lambda(r)$ for binary hypothesis testing. This problem deals with the statistics of this quantity – i.e., define $Z(u) = \Lambda(R(u))$. Show that the following hold

(a) $\mathbb{E}\left\{[Z(u)]^n|\mathcal{H}_1\right\} = \mathbb{E}\left\{[Z(u)]^{n+1}|\mathcal{H}_0\right\}$

(b) $\mathbb{E}\left\{Z(u)|\mathcal{H}_0\right\} = 1$

(c) $\mathbb{E}\left\{Z(u)|\mathcal{H}_1\right\} - \mathbb{E}\left\{Z(u)|\mathcal{H}_0\right\} = \mathrm{var}\left[Z(u)|\mathcal{H}_0\right]$

3.11. Consider the following binary communication system

$$\begin{aligned}
\mathcal{H}_0 \quad r(u,t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \theta_0(t)) + n(u,t) \qquad && t \in [0,T] \\
\mathcal{H}_1 \quad r(u,t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \theta_1(t)) + n(u,t) \qquad && t \in [0,T]
\end{aligned}$$

where the a-priori probability of each hypothesis is $1/2$, $n(u,t)$ is standard AWGN with power spectral level $N_0/2$, and $\theta_1(t) = -\theta_0(t)$. The phase modulating signal $\theta_0(t)$ is shown in Fig. 3.6 where ϕ is a parameter between 0 and 2π.

(a) State the decision rule that minimizes the probability of error (based on a realization $r(t)$) in the simplest form to implement; sketch this receiver.

(b) Determine the probability of error for the above decision rule.

(c) State whether it is possible or impossible to implement antipodal binary signaling using these signals and the corresponding value of ϕ. Repeat for orthogonal binary signaling.

(d) What value of ϕ minimizes the error probability? Give an approximation of this optimal value of ϕ and your reasoning. Also, give the approximate gain in performance relative to orthogonal signaling, measured in dB of E/N_0.

3.12. A system uses BFSK with fast frequency-hopping – *i.e.,* the carrier frequency is changed during the bit transmission time. Let this bit time be $2T$ and let received signal be the following for $m = 0, 1$

$$r(u,t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left(2\pi \left[f_A + \frac{(-1)^m}{2T}\right] + \theta_A(u)\right) + n(u,t) & 0 \leq t < T \\ \sqrt{\frac{2E}{T}} \cos\left(2\pi \left[f_B + \frac{(-1)^m}{2T}\right] + \theta_B(u)\right) + n(u,t) & T \leq t < 2T \end{cases}$$

where $|f_A - f_B| > 1/T$, $\theta_A(u)$ and $\theta_B(u)$ are independent random variables, each modeled as uniform on an interval of length 2π, and $n(u,t)$ is standard AWGN with spectral level $N_0/2$.

Determine and sketch the optimal receiver based on the received signal $\{r(t) : 0 \leq t \leq 2T\}$.

3.13. An equivalent complex-baseband, $(k \times 1)$ vector model for an M-ary digital communication system is

$$\mathbf{r}(u) = \mathbf{s}_m e^{j\phi_c(u)} + \mathbf{n}(u)$$

where $\mathbf{n}(u)$ is mean-zero, complex circular Gaussian with variance $\mathbb{E}\left\{|n_i(u)|^2\right\} = N_0$ and $E_m = \|\mathbf{s}_m\|^2$.

The above model assumes that the receiver has performed a complex correlation with phase reference $\hat{\Theta}_c(u)$ and $\phi_c(u)$ models the residual phase error. For the specific phase tracking loop used, the residual phase error is well modeled by the following probability density function

$$f_{\phi_c(u)}(\phi) = \frac{1}{2\pi I_0(\gamma_L)} e^{\gamma_L \cos(\phi)} \qquad \phi \in [-\pi, +\pi)$$

with the pdf being zero for $|\phi| > \pi$. The parameter γ_L is the tracking loop SNR and the probability mass concentrates near $\phi = 0$ as γ_L increases.

(a) Determine the (equivalent) average likelihood given the above model in the simplest form.

(b) Suppose that $E_m = E$ for all m and the signals have equal a-priori probability. Determine the decision rule, in simplest form, that minimizes the probability of symbol error.

(c) Discuss the limiting forms of the rule from part (b) as γ_L tends towards zero and as γ_L tends towards infinity.

3.14. A satellite link uses BPSK with a standard rectangular pulse shape. It is desired to improve the spectral side-lobe roll-off by replacing the T-second rectangular pulse with the following T-second pulse:

$$p(t) = \sqrt{\frac{2}{T}} \sin(\pi t/T) \quad t \in [0, T]$$

in the transmitter. The satellite receiver cannot be reprogrammed, however, so it will still correlate to the rectangular pulse – *e.g.,* integrate and dump. This will result in a degradation in performance of X dB of E_b/N_0. Determine X.

3.15. Consider a modulation format based on frequency domain multiplexing (FDM) of QASK signals. Specifically, the transmitted signal is

$$x(t) = \sum_{k=0}^{N-1} x_k(t)$$

where $x_k(t)$ is a QASK-modulated signal at carrier frequency

$$f_k = f_c + k\Delta$$

Specifically,

$$x_k(t) = \Re\left\{\sqrt{E_k}\bar{x}_k(t)\sqrt{2}e^{j2\pi f_c t}\right\}$$
$$\bar{x}_k(t) = \left(\sum_i \bar{X}_i(k)p(t-iT)\right)e^{j2\pi\Delta kt}$$

where $\bar{X}_i(k) = \left(X_i^I(k) + jX_i^Q(k)\right)$ is and independent, identically distributed, data symbol sequence, uniformly distributed over a QASK constellation with $\mathbb{E}\left\{|\bar{X}_i(u,k)|^2\right\} = 1$, and $\mathbb{E}\left\{\bar{X}_i(u,k)\right\} = 0$. The data sequences on different carriers are also independent – *i.e.,* $\{\bar{X}_i(u,k)\}_i$ and $\{\bar{X}_i(u,l)\}_i$ are independent for $k \neq l$.

The symbol time on each carrier is T. The pulse $p(t)$ is the same for each carrier and is a real-valued, unit energy rectangular pulse

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0,T) \\ & t \notin [0,T) \end{cases}$$

(a) What is the minimum carrier frequency separation Δ that makes the signals $x_k(t)$ and $x_l(t)$ orthogonal for $k,l \in \{0,1,2,\dots N-1\}$ and $k \neq l$. In other words, what is the minimum carrier separation that ensures that the FDM is orthogonal frequency division multiplexing (OFDM)? Consider one symbol time for all carriers – *i.e.,* $x(t)$ for $t \in [0,T]$. What is the centroid $c(t)$ of this signal set for $\Delta = \Delta_\perp$?

(b) For $\Delta = \Delta_\perp$ from part (a) and $E_k = E_s$ for all k, what is the power spectral density of $x(t)$? Alternatively you may provide the PSD of the complex baseband equivalent of $x(t)$ – circle which you are providing. Sketch this for $N = 8$.

(c) Sketch the optimal (minimum symbol error probability, phase-coherent) receiver when $x(t)$ is observed in standard additive white Gaussian noise (AWGN) with spectral level $N_0/2$. Your diagram may include complex baseband components or you may draw it in terms of passband processing only.

3.16. Consider MAP detection for M equally likely hypotheses $\mathcal{H}_0, \dots \mathcal{H}_{M-1}$. There are two observations available, $\mathbf{r}_a(u)$ and $\mathbf{r}_b(u)$. Show that, if $\mathbf{r}_a(u)$ and $\mathbf{r}_b(u)$ are independent, that the decision rule based on both observations can be implemented by setting $\pi_m = f_{\mathbf{r}_b(u)}(\mathbf{r}|\mathcal{H}_m)$ in a MAP detector designed to use only $\mathbf{r}_a(u)$. If the two observations are not independent, how would you modify this?

Hint: Suppose you have a box that takes as it's input $\mathbf{r}_a$ and π_m for $m = 0, 1, \ldots M-1$ and outputs the decision based on the MAP rule for the observation $\mathbf{r}_a$. Now, you want to use this box to make a decision based on the observation $(\mathbf{r}_a, \mathbf{r}_b)$ and you know that $\pi_m = 1/M$ – what would you input into the box for "$\mathbf{r}_a$ and π_m"?

3.17. Consider a Gray-labeled MPSK system over an AWGN channel. Compare the approximate expressions for the probability of bit error against Monte Carlo simulation results. Specifically, plot the BER vs. E_b/N_0 and against E_s/N_0 for $M = 8$ and $M = 16$.

3.18. (Modified version of "Hexagony" by R.A. Scholtz) At the output of an I-Q correlator for a particular QASK format, the two-dimensional (real) signal model is

$$\mathcal{H}_{mn}: \quad \mathbf{R}(u) = \mathbf{S}_{mn} + \mathbf{N}(u) \qquad\qquad m,\ n \in \{0, \pm 1, \ldots \pm 5\},$$

where the constellation is defined by

$$\mathbf{S}_{mn} = A\left(m \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{n}{2} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right).$$

The noise vector is Gaussian with zero mean and an identity covariance matrix ($\mathbf{K}_{\mathbf{N}} = \mathbf{I}$). The parameter A is fixed, positive, and available at the receiver. Each of these 121 hypotheses are equally likely.

(a) Sketch the portion of the constellation surrounding $\mathbf{S}_{00}$ – i.e., this point and its nearest-neighbors. On this sketch, also draw the decision region for deciding $\mathcal{H}_{00}$ and indicate the parameter A and label the signals.

(b) State a good upper and lower bound to the probability of error given that $\mathcal{H}_{00}$ is true (in terms of the Q-function and A):

$$\leq P(\mathcal{E}|\mathcal{H}_{00}) \leq$$

(c) By inscribing circumscribing circles about the decision region, develop explicit upper and lower bounds on $P(\mathcal{E}|\mathcal{H}_{00})$ (these bounds will differ from those in (b). **HINT:** $\mathrm{Pr}\{\|\mathbf{N}(u)\| > r\} = \exp(-r^2/2)$

$$\leq P(\mathcal{E}|\mathcal{H}_{00}) \leq$$

(d) Numerically evaluate your bounds for the following values of A: 1,4, 8, 16.

- What would you use for an upper and lower bound? For this choice, plot the upper and lower bounds vs. $20\log_{10}(A)$.
- Is $P(\mathcal{E}|\mathcal{H}_{mn})$ the same for all possible choices of m and n? Explain.

3.19. (Modified Weber Problem) Consider two binary communication signaling schemes. Technique A uses the signals (for $t \in [0, T]$)

$$s_0(t) = 0$$
$$s_1(t) = \sqrt{\frac{4E}{T}}\cos(2\pi f_c t),$$

and technique B uses

$$s_0(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t)$$
$$s_1(t) = -\sqrt{\frac{2E}{T}}\cos(2\pi f_c t).$$

These two techniques are to be compared for communication over an AWGN channel with $\pi_0 = \pi_1$.

(a) What is the performance of the optimal receiver for Technique A? — for technique B?

(b) What is the average amount of energy transmitted using either technique?

(c) Which technique is preferred? Why? Can you draw a general conclusion from this example?

3.20. Consider a PSK system with fading amplitude

$$\mathcal{H}_m: \quad r(u,t) = A(u)\cos\left[2\pi f_c t + \frac{2\pi m}{M} + \phi(u)\right] + n(u,t) \qquad t \in [0,T],\ m = 0,\ldots M-1$$

where $A(u)$ and $\phi(u)$ represent the effects, in polar coordinates, of zero-mean, equal variance Gaussian I and Q amplitudes. Specifically,

$$A(u) = \sqrt{B_I^2(u) + B_Q^2(u)}$$

with $\mathbb{E}\{B_I(u)\} = \mathbb{E}\{B_Q(u)\} = 0$ and $\mathbb{E}\{B_I^2(u)\} = \mathbb{E}\left\{B_Q^2(u)\right\} = \sigma_B^2$. Furthermore, $B_I(u)$ and $B_Q(u)$ are independent and also mutually independent of the AWGN $n(u,t)$. It follows that $A(u)$ is Rayleigh distributed. Assume that the phase of this random complex amplitude $\phi(u)$ can be perfectly estimated at the receiver, so that the mixer phase can be set to this value.

(a) Find the optimal receiver conditioned on $A(u) = a$.

(b) Find the approximate performance for this receiver conditioned on $A(u) = a$, $P(\mathcal{E}|a)$.

(c) Average $P(\mathcal{E}|a)$ over the statistics of $A(u)$ to obtained $P(\mathcal{E})$ Plot this vs. E/N_0, where E is the average signal energy.

(d) Does the optimal receiver need to estimate $A(u)$? Would this be the same for a QAM format? Explain.

3.21. Consider an MPSK system in AWGN. Suppose the the receiver's estimate of the carrier phase is $\hat{\theta}_c$. Determine the approximate performance of the receiver as a function of E/N_o and $\phi = \theta_c - \hat{\theta}_c$. Discuss the effects of this phase estimation error on performance.

3.22. Consider the communication of one of four equally-likely signals in AWGN, where the signals are defined by

$$s_m(t) = A_m p(t)\sqrt{2}\cos(2\pi f_c t)$$

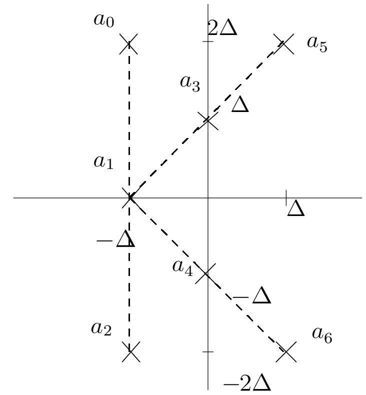


Figure 7: The 'K' constellation.

with $A_0 = +\Delta$, $A_1 = -\Delta$, $A_2 = +3\Delta$, and $A_3 = -3\Delta$, for some positive signal shift Δ. The waveform $p(t)$ has unit energy ($\int_0^T p^2(t)dt = 1$) and has frequency content low enough to make the narrowband assumption valid – i.e., you may neglect all $2f_c$ terms.

The objective is to decide between these four hypotheses based on the observation $r(u,t) = s_m(t) + n(u,t) \quad t \in [0,T]$, where $n(u,t)$ is standard AWGN (i.e., 2-sided spectral level $N_0/2$).

(a) Determine the dimension of the signal space k and define the equivalent vector model, based on an set of functions $\{e_i(t)\}_{i=1}^k$:

$$\mathcal{H}_m : \qquad \mathbf{R}(u) = \mathbf{S}_m + \mathbf{N}(u) \quad (k \times 1)$$

where $f_{\mathbf{N}(u)}(\mathbf{z}) = \mathcal{N}_k(\mathbf{z}; \mathbf{0}; \frac{N_0}{2}\mathbf{I})$.

(b) Find the decision rule that minimizes the symbol error probability. This rule may be stated as a partition of the observation space – i.e., Decide $\mathcal{H}_m \iff \mathbf{R} \in \mathcal{X}_m$. State the optimal rule by defining the four decision regions.
Sketch the optimal receiver.

(c) Find the probability of error for this optimal receiver.
Hint: First find $P(\mathcal{E}|\mathcal{H}_0)$ and $P(\mathcal{E}|\mathcal{H}_3)$.

3.23. (*Spring 1998 Final Exam*) I need to send one of seven messages across an AWGN channel every 0.001 second with a carrier frequency of 2 GHz. I decide to use the "K" constellation shown in Fig. 7 because it's my initial.

More precisely, I send the signal with complex baseband representation

$$s(u,t) = \sum_i A_i(u)p(t - iT)$$

where $T = 0.001$ second, $p(t)$ is a unit energy pulse satisfying the Nyquist condition for no intersymbol interference, and $A_i(u)$ is a sequence of i.i.d. random variables, each taking on values in $\{a_m\}_{m=0}^6$ with equal probability. This signal is observed AWGN, with two-sided spectral level $N_0/2$.

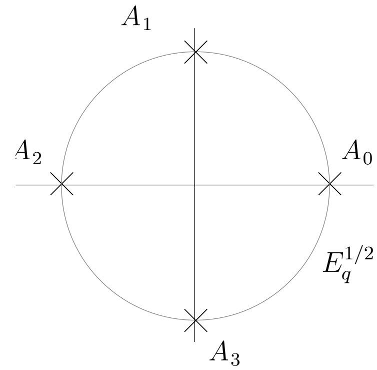


Figure 8: A QPSK constellation.

(a) The output of the bandpass matched filter sampled at any symbol time is represented by

$$\mathcal{H}_m : \quad R(u) = a_m + N(u), \qquad\qquad m = 0, 1 \ldots 6$$

Clearly indicate the decision regions, along with
in the complex plane for the decision rule operating on $R(u)$ which minimizes the message error probability.

(b) Using the method described in lecture, determine a lower bound and upperbound for the symbol error probability

(c) The exact performance of this modulation format can be obtained by using a constellation $\{b_m\}_{m=0}^{6}$ which has a lower average signal energy. Determine the constellation $\{b_m\}$ and the ratio of the average energies for the two constellations E_b/E_a:

3.24. **Emergency Signal (Fall 1998 Final Exam)** A particular communication system uses QPSK and each of the four messgages are equally likely. It is desired to add another point to the QPSK constellation to send a fifth special (emergency) message.

This fifth signal will be added to the existing QPSK constellation which has energy E_q. It is determined that adding the fifth point should not increase the average energy – i.e., the average energy in the new constellation must be no greater than E_q

(a) Determine where the fifth signal should be added in order to maximize the minimum distance of the 5-point constellation. Indicate you answer by marking the location of the new point A_4 on the QPSK constellation shown in Fig. 8

(b) Assuming that each of the five signals in the new costellation are transmitted with probability 1/5 and the channel is AWGN, skecth the decision regions for the MAP symbol detector below (add your A_5 answer too) – labal the boundaries.

Under the above assumptions, determine the average energy transmitted E and give good upper and lower bounds for the performance of the 5-ary system:

(c) A co-worker points out to you that the emegency message is not expected to be used frequently; she estimates that it occurs with probability $\pi_5 = p < 1/5$. The original 4 QPSK signals are well-modeled as equally likely, each occuring with probability $\pi_m = (1-p)/4$ for $m = 0, 1, 2, 3$.

i. Based on the assumption of these a-priori probabilities, reconsider your answers above: Skecth the decision regions for the MAP symbol detector below (add your A_5 answer too) – label the boundaries:

ii. Note that, for suffiently small p, A_4 is never selected. Give the minimum value of p, p_{min} for which the decision region for A_4 is not empty

iii. Assuming that $p \in (p_{min}, 1/5)$, determine the average energy transmitted E and give good upper and lower bounds for the performance of the 5-ary system.

3.25. (Final Exam, Spring 1998) You are working on a BPSK data link which is not meeting the required BER specifications. The current system uses standard rectangular pulse shaping so that the received signal is

$$r(u,t) = \sqrt{\frac{2E_b}{T_b}} B(u) \cos(2\pi f_c t) + n(u,t) \quad t \in [0, T_b)$$

where $n(u,t)$ is the standard AWGN process. The transmitted bit is $B(u) = +1$ under $\mathcal{H}_0$ and $B(u) = -1$ under $\mathcal{H}_1$, which are equally likely a-priori.

Your colleague, who has no experience in digital communications, suggests that the reliability will be improved if a "send-negate-repeat" (SNR) signaling scheme is used. In his SNR-signaling format, for each value of $B(u)$, the random variables $C_1(u) = B(u)$, $C_2(u) = -B(u)$, and $C_3(u) = B(u)$ are constructed. The data rate of the system must be maintained so all three of these pulses are sent during the bit time. Specifically, the following signal is sent during $[0, T_b]$:

$$\begin{aligned}
r(u,t) &= \sqrt{\frac{2E_b}{T_b}} C_1(u) \cos(2\pi f_c t) + n(u,t) & t \in [0, T_c) \\
r(u,t) &= \sqrt{\frac{2E_b}{T_b}} C_2(u) \cos(2\pi f_c t) + n(u,t) & t \in [T_c, 2T_c) \\
r(u,t) &= \sqrt{\frac{2E_b}{T_b}} C_3(u) \cos(2\pi f_c t) + n(u,t) & t \in [2T_c, 3T_c),
\end{aligned}$$

where $T_c = T_b/3$. After looking at the standard BPSK detector, your colleague suggests the receiver shown in Fig. 9 for his SNR signaling format

Since your colleague has never had EE564, he asks you to determine the MAP decision rule for deciding between $\mathcal{H}_0$ and $\mathcal{H}_1$ based on $\hat{C}_1(u)$, $\hat{C}_2(u)$, and $\hat{C}_3(u)$.

(a) You point out that $\hat{\mathbf{c}}(u) = [\ \hat{C}_1(u)\ \hat{C}_2(u)\ \hat{C}_3(u)\ ]^{\mathrm{t}}$ can take on only 8 possible values – i.e., $\hat{C}_n(u) = \pm 1$, so $\hat{\mathbf{c}}(u)$ takes on the values $\mathbf{g}(k) = [\ G_1(k)\ G_2(k)\ G_3(k)\ ]^{\mathrm{t}}$ for $k = 0, 1, \ldots 7$ as shown below.

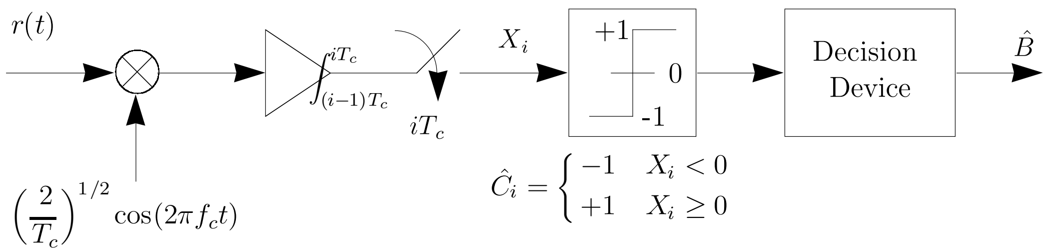


Figure 9: The suggested receiver for the SNR signal format.

k	$G_1(k)$	$G_2(k)$	$G_3(k)$	$\Pr\{\hat{\mathbf{c}}(u) = \mathbf{g}(k)\|\mathcal{H}_0\}$	$\Pr\{\hat{\mathbf{c}}(u) = \mathbf{g}(k)\|\mathcal{H}_1\}$	Rule: $\hat{B}(u) =$
0	+1	+1	+1			
1	+1	+1	-1			
2	+1	-1	+1			
3	+1	-1	-1			
4	-1	+1	+1			
5	-1	+1	-1			
6	-1	-1	+1			
7	-1	-1	-1			

For each possible value of $\hat{\mathbf{c}}(u)$, determine the likelihood of each hypothesis on $B(u)$. Finally, determine the rule that minimizes the BER based on observing $\hat{\mathbf{c}}(u)$. List your answers in the above table. Feel free to use any shorthand notation you like as long as you clearly define it.

(b) Determine the probability of error for the SNR signaling scheme with the optimal decision rule based on observing $\hat{\mathbf{c}}(u)$.

Based on your experience in EE564 you tell your colleague that his SNR signaling scheme and the above decision rule will never achieve the performance of the current, standard BPSK system. Furthermore, his scheme introduces at least one clear undesirable property. *Rigorously* support this claim below and indicate the undesirable property.

3.26. *(CSI Qualifying Exam, Fall 1998)* Consider the M-ary vector hypothesis-testing problem with

$$\mathcal{H}_m: \quad \mathbf{z}(u) = \mathbf{s}_m + \mathbf{w}(u) \qquad\qquad m = 0, 1 \ldots M-1$$

where $\mathbf{w}(u)$ is a complex-circular white Gaussian vector (i.e., zero mean and covariance $\mathbf{K}_{\mathbf{w}} = \frac{N_0}{2}\mathbf{I}$) and $P(\mathcal{H}_m) = \pi_m$.

(a) Prove that the performance of the MAP decision rule for the above problem is the same as the MAP decision rule for the problem

$$\mathcal{H}_m: \quad \mathbf{z}(u) = \mathbf{x}_m + \mathbf{w}(u) \qquad\qquad m = 0, 1 \ldots M-1$$

where $\mathbf{w}(u)$ is as above and the set of signals is given by $\mathbf{x}_m = \mathbf{U}\mathbf{s}_m + \mathbf{b}$, where $\mathbf{U}$ is any unitary matrix and $\mathbf{b}$ is an arbitrary constant vector.

(b) Is this true if the noise covariance $\mathbf{K_w} \neq \frac{N_0}{2}\mathbf{I}$? If so, prove it; if not, state why and given an analogous condition for $\mathbf{U}$ and $\mathbf{b}$ such that the performance of the two systems is the same.

(c) State a practical consequence of the fact proved in the first part of this problem.

3.27. *(Spring 1998 Midterm)* Consider an M-ary digital modulation technique where information is conveyed via discrete shifts in the carrier's frequency. The received signal is modeled by ($\pi_m = 1/M$)

$$\mathcal{H}_m: \qquad r(u,t) = \sqrt{\frac{2E}{T}}\cos(2\pi[f_c + \Delta_m]t) + n(u,t) \qquad t \in [0,T]; \quad m = 0,1,\ldots M-1$$

where $n(u,t)$ is the standard AWGN (i.e., $K_n(\tau) = (N_0/2)\delta_D(\tau)$). You should assume that $0 < \Delta_m \ll f_c$ so that the narrowband assumption is valid.

(a) Find a set of frequency shifts $\{\Delta_m\}_{m=0}^{M-1}$ for which the signals are orthogonal – i.e.,

$$i \neq j \quad \Longrightarrow \quad \int_0^T s_i(t)s_j(t)\,dt = 0.$$

Determine the dimensionality of the signal space k and give a set of orthonormal basis functions $\{\phi_i(t)\}_{i=0}^{k-1}$.

(b) Determine the equivalent $(k \times 1)$ vector model for the decision problem in terms of

$$R(u,i) = \int_0^T r(u,t)\phi_i(t) \qquad i = 0,1,\ldots k-1$$

Specifically, the equivalent model is of the form

$$\mathcal{H}_m: \qquad \mathbf{R}(u) = \mathbf{S}_m + \mathbf{N}(u) \qquad (k \times 1); \quad m = 0,1,\ldots M-1$$

Determine the following: $\mathbf{S}_m$ and $f_{\mathbf{N}(u)}(\mathbf{z})$.

(c) Determine the decision rule the minimizes the probability of error for the M-ary decision problem conditioned on the observation $\mathbf{R}(u) = \mathbf{r}$. Simplify this result to the extent possible.

(d) In this part, the performance of the optimal decision rule is to be determined. To do so, first determine the following densities: $f_{\mathbf{R}(u)}(\mathbf{r}|\mathcal{H}_0)$ and

$$f_{R(u,1),R(u,2)\cdots R(u,k-1)|R(u,0)}(r(1),r(1),\ldots r(k-1)|\mathcal{H}_0,r(0))$$

(e) For $M=2$, compute the probability of a correct decision given $\mathcal{H}_0$ and $R(u,0) = r(0)$.

(f) Generalize the above to an arbitrary M.

(g) Write an expression down for the probability of error given $\mathcal{H}_0$ by completely defining an integral.

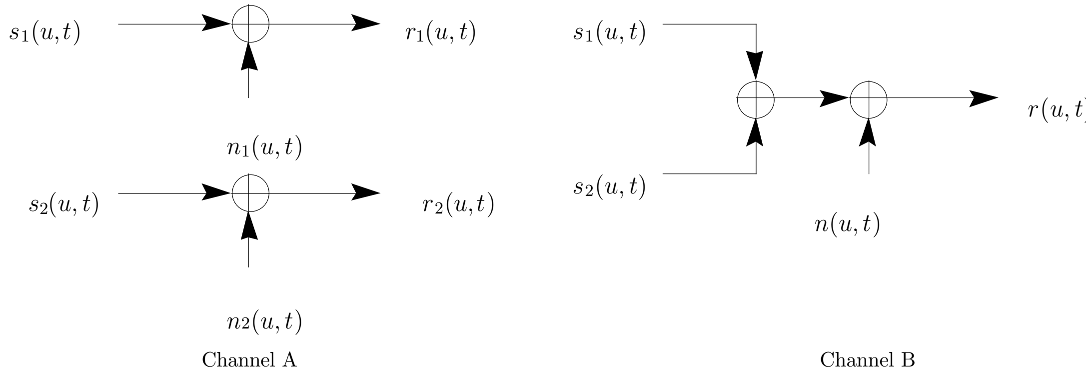


Figure 10: Two channels with two users. Channel B has multiuser interference and channel A does not.

(h) Suppose that the it is learned that the noise is actually colored Gaussian noise. The eigenfunctions and eigenvalues of the noise covariance operator are known and denoted by $e_i(t)$, λ_i for $i = 0, 1, 2, \ldots$. Determine a set of signals $s_m^{'}(t)$ that could be used for this colored Gaussian noise channel that would result in the same performance as the white noise system investigated above:
Determine the specifics of this model: $s_m^{'}(t)$ as a function of t, E,f_c, Δ, T, $\{e_i(t)\}$, $\{\lambda\}$, and N_0.

3.28. Consider two different channels for providing communications to two users. In channel A, there is a separate AWGN channel available for each of the users. In channel B, both users communicate across the same AWGN channel. This is diagrammed in Fig. 10.

In either case assume that the (real) signals used by each user are

$$s_1(u,t) = A_1(u)\sqrt{E_1}\phi_1(t)$$
$$s_2(u,t) = A_2(u)\sqrt{E_2}\phi_2(t)$$

where $\phi_k(t)$ is nonzero only for $t \in [0,T]$, $A_1(u)$ and $A_2(u)$ are i.i.d. and both equally likely to be -1 or $+1$, and

$$\int_0^T \phi_i^2(t)dt = 1 \quad (i = 1, 2) \qquad \int_0^T \phi_1(t)\phi_2(t)dt = \rho,$$

assume that $|\rho| < 1$. For channel B, the two noise signals are independent AWGN processes with two-sided PSD-level $N_0/2$, as is $n(u,t)$. All noise signals are independent of the user signals.

(a) Determine and sketch the receiver processing which minimizes the joint error probability for channel A based on the observation $r_1(t)$ and $r_2(t)$ for $t \in [0,T]$.

(b) Determine and sketch the receiver processing which minimizes the joint error probability for channel B based on the observation $r(t)$ $t \in [0,T]$.

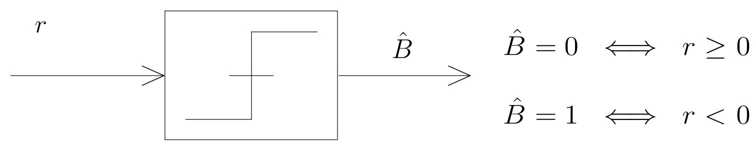


Figure 11: The decision device available for deciding between antipodal signals.

(c) Compute and compare the probability of joint error for the two systems.

(d) For channel B, how does this performance vary with ρ? If you could design $\phi_1(t)$ and $\phi_2(t)$, what would you choose?

3.29. Space-time coding is a method where signals are sent from multiple transmit antennas to multiple receive antennas. Consider the case of two antennas at both the receiver and transmitter. Then, the observation at the receiver is a vector $\mathbf{r}(u)$. Consider the case where one of two equally likely signals is sent so that

$$\begin{aligned} \mathcal{H}_0: \quad \mathbf{r}(u) &= \mathbf{H}\mathbf{s}_0 + \mathbf{w}(u) \\ \mathcal{H}_1: \quad \mathbf{r}(u) &= \mathbf{H}\mathbf{s}_1 + \mathbf{w}(u) \end{aligned}$$

where $\mathbf{H}$ is a known channel matrix defining the the gain for each transmit-receive antenna pair. The noise is AWGN with mean zero and $\mathbf{K_n} = \sigma^2\mathbf{I}$.

(a) Find the decision rule that minimizes the probability of error.

(b) What is the probability of error of the receiver derived in part (a)?

(c) What is the best space-time code for this problem? Specifically, with an energy constraint $\|\mathbf{s}_0\|^2 = \|\mathbf{s}_1\|^2 = E$, describe the best choice for the signal vectors.

3.30. *(Spring 2000, midterm exam)* Consider the problem of detecting antipodal signaling in the additive Gaussian noise

$$\begin{aligned} \mathcal{H}_0: \quad R(u) &= +\sqrt{E} + N(u) \\ \mathcal{H}_1: \quad R(u) &= -\sqrt{E} + N(u) \end{aligned}$$

where $N(u)$ is mean-zero, Gaussian with variance $N_0/2$. You have access to a decision device as illustrated in Fig. 11 where $\hat{B} = 0$ corresponds to deciding $\mathcal{H}_0$ and $\hat{B} = 1$ corresponds to $\mathcal{H}_1$.

However, you now are informed that $\pi_0 = 0.9$ and $\pi_1 = 0.1$. This problem deals with modifying the above detector to account for this knowledge.

(a) What is the performance the receiver shown above? Explain why this receiver is not the minimum error probability receiver.

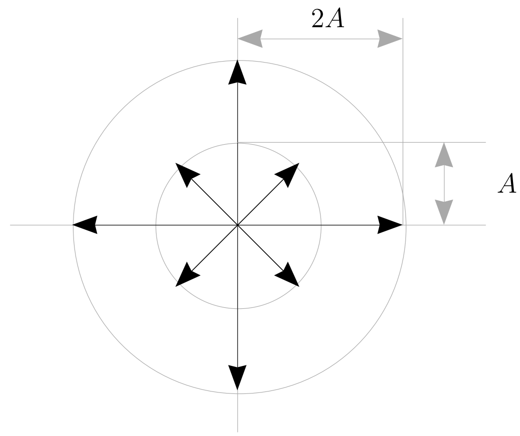


Figure 12: An 8-ary signal set based on two concentric circles.

(b) If you have access to the realization of $R(u)$, r, what is the minimum error probability rule?

What is the probability of error for this receiver?

(c) Suppose that only you have access to the decisions at the output of the receiver from (a). Then, the channel can be modeled as a Binary Symmetric channel. Specifically, let $\hat{B}(u)$ be the decision provided by the receiver in (a)

Find the minimum error probability receiver based on observing $\hat{B}(u)$. Let your final decision be denoted by $d \in \{0, 1\}$. Describe your rule by giving the condition for changing the decisions $\hat{B}$.

$$\text{Change } \hat{B} = 0 \text{ to } d = 1 \iff$$
$$\text{Change } \hat{B} = 1 \text{ to } d = 0 \iff$$

(d) What is the probability of error for the receiver in (c)? Note that you will have to describe this function separately on different regions of E/N_0.

3.31. *(Spring 2001, Midterm)* An 8-ary signal set in two-dimensional real space has been defined as a mixture of two 4-ary PSK signal sets at two different amplitudes. The 8 signal points are shown in Fig. 12.

Each of the 8 signals are equally likely to be transmitted. Under hypothesis $\mathcal{H}_m$, the received waveform is $\mathbf{r}(u) = \mathbf{s}_m + \mathbf{w}(u)$, where $\mathbf{w}(u)$ is a mean-zero Gaussian vector with covariance matrix $\mathbf{K_w} = \sigma^2\mathbf{I}$.

(a) Determine the decision rule that minimizes the probability of error. Specify this decision rule by *carefully and accurately* drawing the decision regions on the the signal constellation.

(b) Let $d(i, j) = \|\mathbf{s}_i - \mathbf{s}_j\|$ be the Euclidian distance between two signals. In constructing a good upper and lower bound on $P(\mathcal{E})$, state which values of $d^2(i, j)$ are used. State good lower and upper bounds on the probability of error for the receiver described in part (a).

3.32. Consider the 8-ary modulation by the following two dimensional signals

$$\mathbf{s}_{i,j} = \begin{bmatrix} i \\ 2j \end{bmatrix}$$

where (i, j) takes on the values $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, $(-1, -1)$, $(+2, +2)$, $(+2, -2)$, $(-2, +2)$, and $(-2, -2)$.

The detection problem is then

$$\mathcal{H}_{i,j} \quad \mathbf{r}(u) = \mathbf{s}_{i,j} + \mathbf{w}(u)$$

where the pdf of $\mathbf{w}(u)$ is $\mathcal{N}_2(\cdot; \mathbf{0}; \sigma^2\mathbf{I})$. All hypotheses occur with equal a-priori probability.

(a) Carefully draw and label the signals. Specify the MAP 8-ary receiver for this problem by carefully drawing the decision regions. Be sure to label the diagram so as to completely define the rule.

(b) Find a good lower and upper bound for the probability of error for the MAP receiver of part (a).

3.33. *(Spring 2001, Midterm)* A particular practical binary modulation format has the following equivalent (2×1) vector model:

$$\mathcal{H}_0: \quad \mathbf{r}(u) = \sqrt{E}\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} + \mathbf{w}(u)$$

$$\mathcal{H}_1: \quad \mathbf{r}(u) = \sqrt{E}\begin{bmatrix} \mathrm{sinc}(2\Delta) \\ \\ \sqrt{1 - \mathrm{sinc}^2(2\Delta)} \end{bmatrix} + \mathbf{w}(u)$$

The parameter Δ is a real number that defines the precise form of the waveforms used. The sinc function is $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. The noise is AWGN with mean zero and $\mathbf{K}_\mathbf{w} = \sigma^2\mathbf{I}$. The a-priori probabilities are equal.

(a) Find the decision rule, based on the realization $\mathbf{r}(u) = \mathbf{r}$, that minimizes the probability of error. Sketch the *preferred* receiver processing to implement this rule.

(b) What is the probability of error of the receiver derived in part (a)?

(c) What is the best choice for the parameter Δ – i.e., the value Δ_{opt} that minimizes the error probability? Please use sketches to describe this best choice and give an approximate numerical value for this optimal choice and the corresponding error probability.

3.34. *(Spring 2000, Final Exam)* A potential improvement to PSK has been suggested which shifts the phase at the midpoint of the symbol time. Specifically, M-ary *"double PSK"* is defined by the signal set

$$s_m(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \frac{2\pi m}{M}\right) & 0 \leq t \leq T/2 \\ -\sqrt{\frac{2E}{T}} \sin\left(2\pi f_c t + \frac{2\pi m}{M}\right) & T/2 < t \leq T \end{cases}$$

for $m = 0, 1, \ldots M-1$. Consider the case where the above is the received signal in AWGN

$$\mathcal{H}_m: \quad r(u,t) = s_m(t) + n(u,t) \quad t \in [0,T]$$

where $n(u,t)$ is AWGN with spectral level $N_0/2$. Assume that $f_c \gg 1/T$ and that each of the M signals are equally likely to be transmitted.

(a) There is an equivalent vector model that contains a set of sufficient statistics for detecting the double-PSK signal from a realization $\{r(t),\ t \in [0,T]\}$ of the form

$$\mathbf{Z}(u) = \mathbf{S}_m + \mathbf{N}(u) \qquad (k \times 1)$$

where $\mathbf{N}(u)$ is a real, mean-zero Gaussian vector with covariance matrix $\mathbf{K}_\mathbf{N} = \frac{N_0}{2}\mathbf{I}$. Assume that $M > 2$ and specify k and $\mathbf{S}_m$.

Sketch and label the receiver processing that yields this model.

(b) Consider the special case where $M = 4$. Specify explicitly the 4 signal vectors for the equivalent model obtained in (a) and give upper and lower bounds on the probability of error.

(c) Can double-PSK be used as a direct substitute in a system that currently uses PSK with rectangular pulse shaping? Explain briefly.

3.35. *(Spring 2000, Final Exam)* SquareComm is a hot new wireless communications company. According to their webpage, their key patented technology if *Epoch Shift Keying (ESK)* which is the concept of PSK extended to *square-wave carriers* in place of sinusoidal carriers. According to their webpage "the square-wave provides the most efficient mapping from binary data to 2-level waveforms."

From their patent, you learn that the carrier is the based on the basic square wave $q(t)$ with period 1 as shown in Fig. 13.

The ESK signal is the direct analogy of PSK applied to the square-wave carrier:

$$s_m(t) = \sqrt{\frac{E}{T}} q(K_c t + \epsilon_m) \qquad t \in [0,T]$$

where $\epsilon_m = \frac{m}{M}$ for $m = 0, 1 \ldots M-1$. You can assume that $T \gg 1/K_c$ and T/K_c is an integer.

Assume that the ESK signal is observed in AWGN

$$\mathcal{H}_m: \quad r(u,t) = s_m(t) + n(u,t) \quad t \in [0,T]$$

where $n(u,t)$ is AWGN with spectral level $N_0/2$. Assume that each of the M signals are equally likely to be transmitted.

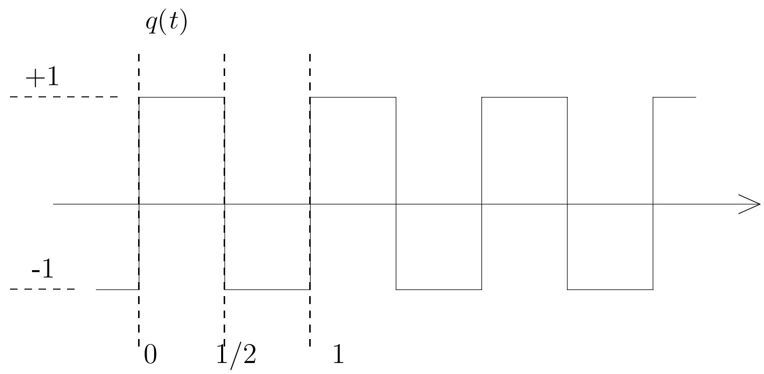


Figure 13: Square wave used as a carrier waveform.

(a) Consider the special case of Binary ESK – i.e., $M = 2$. Sketch and label the optimal B-ESK receiver.

What is the probability of error for B-ESK?

(b) Each of the M ESK signals has energy E. In order to characterize the ESK signal set you should compute the correlation coefficient for a epoch shift ϵ of the square-wave carrier:

$$\beta(\epsilon) = \frac{1}{T}\int_0^T q(K_c t)q(K_c t + \epsilon)dt$$

Evaluate this correlation coefficient for $0 \leq \epsilon \leq 1$. Sketch and label $\beta(\epsilon)$:

Hint: $\beta(\epsilon)$ is symmetric around $\epsilon = 1/2$.

Define the correlation coefficient between signals as usual:

$$\rho(m, n) = \frac{1}{E}\int_0^T s_m(t)s_n(t)dt$$

Give $\rho(m, n)$ for M-ESK signals in terms of $\beta(\cdot)$.

What is the squared-distance between two ESK signals in terms of $\rho(\cdot)$

(c) Consider the special case of $M = 4$. Fill in the following table for $\rho(m, n)$

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$m = 0$				
$m = 1$				
$m = 2$				
$m = 3$				

What is the dimensionality of the $M = 4$ ESK signal set?

What is the probability of symbol error for the optimal 4-ary detector?

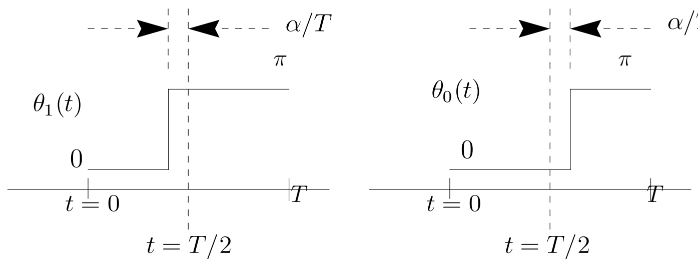


Figure 14: The phase pulses used in the suggested BPSK modification.

(d) Consider the $M = 8$ case. Again, fill in a table for $\rho(m, n)$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$m=0$								
$m=1$								
$m=2$								
$m=3$								
$m=4$								
$m=5$								
$m=6$								
$m=7$								

What is the dimensionality of the $M = 8$ ESK signal set?
Give simple bounds for the probability of symbol error for the optimal 8-ary detector?

(e) Compare the performance of ESK with the corresponding PSK technique:

$M = 2$: ESK performs *Worse* *Same* *Better* than PSK
$M = 4$: ESK performs *Worse* *Same* *Better* than PSK
$M = 8$: ESK performs *Worse* *Same* *Better* than PSK

3.36. A variation on BPSK with phase shaping has been suggested. Given a binary information sequence $b_i \in \{0, 1\}$, the modulated waveform is

$$s(t; \{b_i\}) = \sum_i s_{b_i}(t - iT)$$

where

$$s_0(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + \theta_0(t)) \quad t \in [0, T)$$
$$s_1(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + \theta_1(t)) \quad t \in [0, T)$$

with the phase shaping functions for a 0 and a 1 data bit shown in Fig. 14.

Specifically, $\theta_1(t)$ transitions from 0 to π at $t = T(1/2 - \alpha)$ and $\theta_0(t)$ transitions from 0 to π at $t = T(1/2 + \alpha)$ – where $0 < \alpha < 1/2$.

Assume that an iid information sequence with $\pi_0 = \pi_1$ is transmitted and that the above signals are received in standard AWGN (spectral level $N_0/2$).

(a) Based on the observation $r(t)$, state the decision rule for the bit $b_i(u)$. Sketch the *simplest* form of this optimal receiver.

(b) Determine the probability of error for this system. What is the difference in performance between this modification and standard BPSK as measured as a change in the effective E_b/N_0?

(c) The proponents of this modulation format claim that it can achieve a bandwidth efficiency of 90 bits-per-second/Hz when $\alpha = 1/8$. What is the minimum value of E_b/N_0 required of *any* signaling scheme which achieves this spectral efficiency on the AWGN channel?

(d) Show that, if the centroid of the signal set is subtracted from $s_0(t)$ and $s_1(t)$, that this modified BPSK format can be viewed as standard BPSK with *amplitude* pulse shaping. Sketch and label this effective pulse shape, $p(t)$. Based on this interpretation, do you expect this signal to achieve a bandwidth efficiency of 90 bps/Hz? Explain.

3.37. (*Matching Robustness*) Consider a binary communication system with

$$\begin{aligned} \mathcal{H}_0: \quad r(u,t) &= +\sqrt{\frac{E}{T}} + n(u,t) \qquad && t \in [0,T] \\ \mathcal{H}_1: \quad r(u,t) &= -\sqrt{\frac{E}{T}} + n(u,t) \qquad && t \in [0,T] \end{aligned}$$

where $n(u,t)$ is mean-zero Gaussian noise. Assume that $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2$.

(a) For the case with $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$, specify the following:

   i. The processing of the receiver that minimizes the probability of error based on a realization $r(t)$, $t \in [0,T]$.

   ii. The probability of error for this receiver

(b) Consider the case when

$$R_n(\tau) = \frac{N_0}{4\alpha T} e^{-\frac{|\tau|}{\alpha T}}$$

where $\alpha > 0$ is a constant.

   i. What is the performance of the receiver designed in part 1 of this problem when this noise is present at the receiver front-end?

   ii. Is the receiver for part 1 optimal in the sense of minimum error probability in the presence of this Gaussian noise process observed on $[0,T]$? Explain.

   iii. Discuss and interpret your result as a function of α.

3.38. (Midterm Exam, Spring 1998) A simple binary communication system is intended to send a $+1$ or -1 across a Gaussian channel. However, distortion occurs in the channel resulting in intersymbol interference (ISI). Specifically, the observation is

$$R(u) = A(u) + \alpha B(u) + N(u),$$

where $A(u)$ is the signal of interest, $B(u)$ is the previous signal sent across the channel, α is a real number with $|\alpha| < 1$, and $N(u)$ is a mean-zero, Gaussian random variable with variance of σ^2. You may assume that $A(u)$, $B(u)$, and $N(u)$ are mutually independent. Each of $A(u)$ and $B(u)$ are take on $+1$ and -1 with probability $1/2$.

The objective is to decide whether $A(u) = +1$ (i.e., $\mathcal{H}_0$) or $A(u) = -1$ (i.e., $\mathcal{H}_1$). Hence the problem may be formulated as

$$\begin{aligned}\mathcal{H}_0 : \quad & R(u) = +1 + \alpha B(u) + N(u)\\ \mathcal{H}_1 : \quad & R(u) = -1 + \alpha B(u) + N(u).\end{aligned}$$

Three methods of making a decision for $A(u)$ are considered:

Method A: Ignore the ISI. The decision device is designed assuming that $\alpha = 0$

Method B: Joint detection. The problem is reformulated as a four hypothesis test

$$\mathcal{H}^{(4)}_{m,n} : \quad R(u) = (-1)^m + \alpha(-1)^n + N(u)$$

A MAP decision is made on $\mathcal{H}_{m,n}$. If $\mathcal{H}^{(4)}_{0,0}$ or $\mathcal{H}^{(4)}_{0,1}$ is decided, then the decision is made that $\mathcal{H}_0$ is true (i.e., $A(u)$ is detected as a $+1$). Otherwise $\mathcal{H}_1$ is decided.

Method C: The MAP rule for deciding on $\mathcal{H}_m$ is applied.

For each of these three techniques, determine the decision rule based on $R(u) = r$, draw the decision region (label all quantities), and determine the error probability. Specifically, provide the following for each method:

(a) Rule: Decide $\mathcal{H}_0 \iff$ and athe derivation/reasoning.
(b) Sketch the decision region.
(c) Determine the error probability $P(\mathcal{E})$.

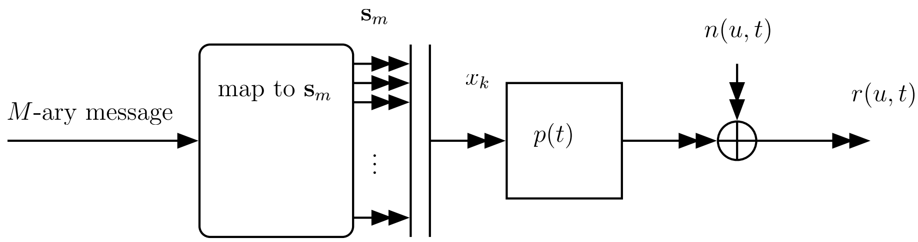
3.39. Consider a system that uses the vectors associated with the Discrete Fourier Transform (DFT) as signal vectors. Specifically, consider the M-ary modulation format with model

$$\mathcal{H}_m : \quad \mathbf{z}(u) = \mathbf{s}_m + \mathbf{w}(u)$$

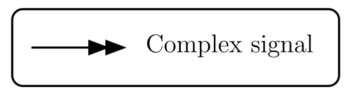
where $\mathbf{w}(u)$ is $(M \times 1)$ complex circular Gaussian noise with $\mathbf{K}_{\mathbf{w}} = N_0\mathbf{I}$ and the signals are

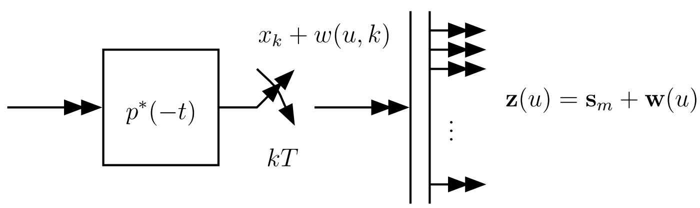
$$\mathbf{s}_m = \frac{\sqrt{E}}{\sqrt{M}} \begin{bmatrix} \exp\left[j\frac{2\pi}{M}m(0)\right] \\ \exp\left[j\frac{2\pi}{M}m(1)\right] \\ \exp\left[j\frac{2\pi}{M}m(2)\right] \\ \vdots \\ \exp\left[j\frac{2\pi}{M}m(M-1)\right] \end{bmatrix} = \frac{\sqrt{E}}{\sqrt{M}} \begin{bmatrix} 1 \\ \exp\left[j\frac{2\pi}{M}m(1)\right] \\ \exp\left[j\frac{2\pi}{M}m(2)\right] \\ \vdots \\ \exp\left[j\frac{2\pi}{M}m(M-1)\right] \end{bmatrix}$$

This system is realized by sending the M consecutive complex symbols that define $\mathbf{s}_m$ using a standard QASK systems with Nyquist pulse shaping. This is illustrated below.



Transmitter/Channel Model (Complex BB)





Receiver Front-End Processing Model (Complex BB)

In other words, the signal $\mathbf{s}_m$ can be viewed as a sequence of M complex symbols

$$s_m(n) = \frac{1}{M} \exp\left[j\frac{2\pi}{M} m(n) \right] \qquad n = 0, 1, \ldots M-1$$

and these are sent using a standard QASK link – i.e., it takes MT seconds to send all M components of $\mathbf{s}_m$. Notice that because of the memoryless channel, the optimal receiver makes decisions on the $M \times 1$ blocks, each modeled by the vector model above.

(a) Based on the vector model, determine and state the Maximum Likelihood M-ary rule. Sketch the processing for this rule.

(b) Find good upper and lower bounds for the the M-ary probability of error for the receiver obtained in (a).

(c) Suppose that you have access to a processor that computes an M-point FFT, can you use this to implement the receiver derived in (a)? If so, illustrate how this can be used.

Compare this format to standard QPSK. Specifically, if exactly the same Nyquist-pulse QASK link is used to send QPSK, how will it compare to the Fourier modulation in terms of power and bandwidth efficiency?

For bandwidth efficiency, define η as the bandwidth effiecny in bits per second per Hz. State the relative bandwidth efficiency of the Fourier modulation to that of QPSK.

For power efficiency, the error probability for each modulation requires a certain minimum value of E_b/N_0 – i.e., . $\left(\frac{E_b}{N_0}\right)_{req}$. Define β as the ratio of E_b/N_0 required for the two modulations:

$$\beta = \frac{\left(\frac{E_b}{N_0}\right)_{req,Fourier}}{\left(\frac{E_b}{N_0}\right)_{req,QPSK}}$$

Using this definition, is the Fourier modulation ($M > 4$) more or less power efficient than QPSK?

3.40. *(Spring 1998 Final Exam)* You have just started at LEO.com, working on their mobile satellite system. They have decided to use 8PSK modulation with rectangular pulses. Due to the rapid velocity of the satellites, the frequency tracking loops cannot maintain accurate carrier frequency lock. However, perfect phase lock and symbol timing can be assumed.

The receiver correlation processing is normalized so that, for a given output sample (i.e., every symbol time), *if no frequency error was present*, the output of the complex baseband equivalent matched filter is

$$\mathcal{H}_m : \quad R(u) = \sqrt{E_s} e^{j\frac{\pi}{4}m} + N(u), \qquad\qquad m = 0, 1 \ldots 7$$

where $N(u)$ is a circular complex Gaussian random variable with $\mathbb{E}\{N(u)\} = 0$ and $\mathbb{E}\left\{|N(u)|^2\right\} = N_0$.

In this problem, we are concerned with the case where the 8PSK receiver uses the carrier frequency estimate $\hat{f}_c$ assuming that $\hat{f}_c = f_c$, when in actuality $\alpha = \hat{f}_c - f_c \neq 0$. Since sophisticated offset compensation and frequency tracking loops are used, you may assume that $|\alpha T| < 1$.

(a) Under hypothesis $\mathcal{H}_m$, the output of the complex-baseband correlator operating with frequency error α can be written as

$$R(u; \alpha) = S_m(\alpha) + N(u; \alpha)$$

Determine the signal parameters of this model – i.e., $S_m(\alpha)$ and $f_{N(u;\alpha)}(z)$. Below, the noise free signals for no frequency offset are shown in the complex plane. Indicate on this same sketch where each $S_m(\alpha)$ is for a fixed value of $\alpha > 0$.

(b) Determine a good upper and lower bound for the probability of symbol error given the frequency error α: $P(\mathcal{E}; \alpha)$.

(c) Your boss hasn't worked with frequency offsets before, but has studied the effect of phase error on the detection of MPSK. He suggests that the effect of the frequency error (with perfect θ_c estimation) should be about the same as an effective phase error $\phi_{eff} = \theta_c - \hat{\theta}_c$ (with perfect f_c estimation). His intuition suggests that ϕ_{eff} is the total integrated offset – i.e., $\phi_{eff} = 2\pi\alpha T$. Do you agree?

(d) Provide justification to the above answer. If the concept of an approximately equivalent phase offset is invalid, explain why. If the concept is valid, explain why your boss' intuition is correct, or identify and justify the value of ϕ_{eff}.

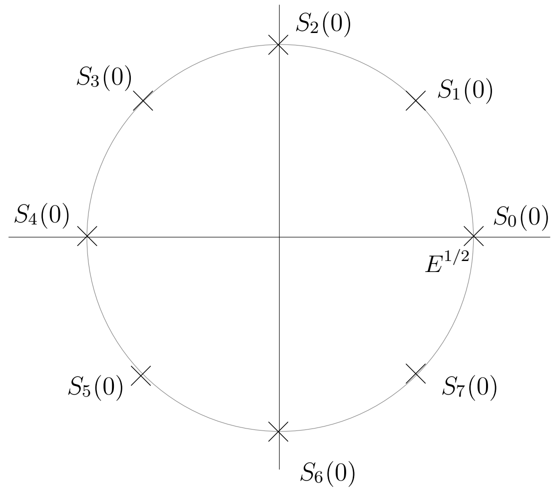


Figure 15: The 8PSK constellation used by LEO.com.

3.41. This problem considers the effects of amplitude mismatch in an 8-ary PAM system. Specifically, the post matched-filter model for a given time index is given by

$$\mathcal{H}_m : \quad Z(u) = A(S_m + N(u))$$

where $S_m \in \{\pm 1, \pm 3, \pm 5, \pm 7\}$ is the 8-ary PAM signal that takes on all 8 levels with equal probability and $N(u)$ is real Gaussian noise. The average energy per symbol E_s to noise spectral level is specified via

$$\frac{2E_s}{N_0} = \frac{\frac{1}{8}\sum_{m=0}^{7} S_m^2}{\mathbb{E}\left\{N^2(u)\right\}}$$

The positive constant, A is set by an automatic gain control (AGC) circuit. The intent is to have the AGC maintain the condition $A = 1$. Decisions on the 8-ary message are made assuming that $A = 1$.

(a) For this part assume that the AGC is working perfectly (i.e., $A = 1$) and illustrate the decision rule that minimizes the probability of 8-ary error by indicating the decisions regions.

What is the probability of error for this system with perfect AGC?

(b) Consider the case when $1 \leq A \leq \frac{6}{5}$ and the slicer described in (a) is used – i.e., the decision rule is based on perfect AGC. Find the error probability for this receiver $P(\mathcal{E}; A)$.

(c) For $1 \leq A \leq \frac{6}{5}$, the degradation can be approximately characterized as an effective loss in E_s/N_0 for moderate to high E_s/N_0. Specify this loss.

Determine the performance for $A = \frac{6.1}{5}$ as E_s/N_0 tends to infinity.

For $6/7 < A \leq 1$, the degradation can be approximately characterized as an effective loss in E_s/N_0 for moderate to high E_s/N_0. Specify this loss.

3.42. *(Spring 2000, Final Exam)* Consider a binary FSK modulation format with

$$s_0(t) = \sqrt{\frac{2E_b}{T}} \cos\left(2\pi \left[f_c + \frac{1}{2T}\right] t\right) \qquad t \in [0, T]$$
$$s_1(t) = \sqrt{\frac{2E_b}{T}} \cos\left(2\pi \left[f_c - \frac{1}{2T}\right] t\right) \qquad t \in [0, T]$$

and equally likely hypotheses. The channel corrupts these signals by introducing a carrier phase rotation and standard AWGN (spectral level $N_0/2$). The receiver uses a phase tracking loop, but with probability p it fails. Thus, the received signal can be modeled as

$$\mathcal{H}_0 : \quad r(u,t) = \sqrt{\frac{2E_b}{T}} \cos\left(2\pi \left[f_c + \frac{1}{2T}\right] t + \theta_c(u)\right) + n(u,t) \qquad t \in [0, T]$$
$$\mathcal{H}_1 : \quad r(u,t) = \sqrt{\frac{2E_b}{T}} \cos\left(2\pi \left[f_c - \frac{1}{2T}\right] t + \theta_c(u)\right) + n(u,t) \qquad t \in [0, T]$$

where, as motivated above, the phase pdf is modeled by

$$f_{\theta_c(u)}(\phi) = p\mathrm{U}(\phi) + (1-p)\delta_D(\phi)$$

where $\mathrm{U}(\phi) = 1/(2\pi)$ for $|\phi| < \pi$ and zero otherwise.

(a) Determine and state the receiver that minimizes the probability of error based on the observation $r(t)$ or, if you prefer, the complex baseband equivalent $\bar{r}(t)$. Sketch a block diagram showing the operation of this receiver.

(b) State the probability of error for this system (with the above optimal receiver) in the following two limiting cases.

3.43. In this problem, M-ary orthogonal signaling is used. Due to the method used to obtain estimates of the synchronization parameters, there is a sign uncertainty at the receiver. The resulting model is

$$\mathcal{H}_m : \quad \mathbf{z}(u) = F(u)\mathbf{s}_m + \mathbf{w}(u)$$

where $\|\mathbf{s}_m\|^2 = E$, and $\mathbf{w}(u)$ is a zero mean, real Gaussian random vector with covariance $\mathbf{K}_\mathbf{w} = \frac{N_0}{2}\mathbf{I}$. The sign ambiguity is modeled by the random variable $F(u)$ which takes on the values $+1$ and -1, each with probability 0.5, and is statistically independent of both the noise and signal.

(a) Assuming that the M hypotheses are equally likely a-priori, find the receiver that minimizes the probability of M-ary error based on the observation $\mathbf{z}(u) = \mathbf{z}$. First give the rule without

(b) Simplify the rule in (a) as much as possible and state this simplified rule. Sketch the receiver processing for this receiver.

3.44. *Symbol Synchronization Errors (Fall 1998, Final Exam)* Consider a basedband transmission system that sends a signal based on the information sequence $\{A_i(u)\}$, which is a sequence of

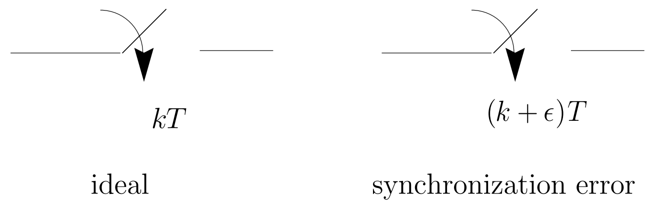


Figure 16: An ideal sampler and one with timing error.

independent, identically distributed random random varioables, each taking -1 and $+1$ with probablity 1/2. Specifically, for a given realization of $\{A_i(u)\}$ (i.e., $\{a_i\}$), the received signal is

$$r(u,t) = s(t;\mathbf{a}) + n(u,t)$$

where $n(u,t)$ is additive white Gaussian noise with (two-sided) power spectral height $N_0/2$. The signal is defined by

$$s(t;\mathbf{a}) = \sqrt{E}\sum_i a_i p(t-iT)$$

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0,T] \\ 0 & \text{otherwise} \end{cases}$$

(a) Given a realization of the received signal $r(t)$, state the decision rule that minimizes the probability of bit errors. Sketch the processing of the optimal receiver. What is the probability of error for this receiver?

(b) The optimal receiver in (a) requires symbol synchronization – i.e., sampling at $t = kT$. Consider the effects of a symbol synchronization error. Specifically, consider the receiver that samples at $(k+\epsilon)T$, where $\epsilon \in [0,1]$. In other words, the ideal sampler is replaced by the one with sampling error as illustrated in Fig. 16.

There is an equivalent (i.e., post-correlation) symbol spaced signal model for the receiver from (a) using the imperfect sampler. This model is

$$z_k = x_k + w_k$$

where w_k is an iid sequence of Gaussian random variables with mean zero and variance $N_0/2$. Determine x_k for this model.

Determine the performance of the receiver with symbol synchorization error Discuss the performance as ϵ approaches 1. Is this the worst value of ϵ?

3.45. (CSI Qualifying Exam Fall 1998) Consider an 8PSK system in AWGN. Suppose the the receiver's estimate of the carrier phase is $\hat{\theta}_c$. Due to a cheap oscillator component in the receiver, this phase estimate is know to have three "lock points" – i.e., $\hat{\theta}_c$ takes on the values θ_c, $\theta_c+\phi_g$, and $\theta_c-\phi_g$, where $\phi_g \in [0, \frac{\pi}{8})$ is the "glitch" angle. Based on this fact, the received signal is modeled as

$$\mathcal{H}_m: \quad r(u,t) = s_m(t;\Theta_c(u)) + n(u,t) \qquad t \in [0,T] \quad m = 0,1\ldots M-1$$
$$s_m(t;\Theta_c(u)) = \sqrt{\frac{2E_s}{T}} \cos\left(2\pi f_c t + \Theta_c(u) + \frac{m\pi}{4}\right),$$

with the associated model for the random phase being

$$\Theta_c(u) = \begin{cases} -\phi_g & \text{with probability } p \\ 0 & \text{with probability } 1-2p \\ +\phi_g & \text{with probability } p \end{cases}$$

Assume that all 8 signals are equally likely.

(a) Suppose that a coherent receiver is utilized based on the phase estimate $\hat{\theta}_c$ (i.e., the assumption that $\theta_c = 0$ is the above signal model is made for the purpose of designing the receiver). Answer the following:

  i. Sketch the processing.
  ii. What is an upper and lower bound for the performance (SER) of this receiver if $p = 0$?
  iii. What is an upper and lower bound for the performance (SER) of this receiver when $p \neq 0$?
  iv. What is an approximate expression for the SER for this receiver for given values of p and ϕ_g? Can you characterize the effects of the phase glitch as a rough SNR degradation? If so, provide such a characterization; if not, explain why.

(b) Suppose that the model for $\Theta_c(u)$ described above is taken into account during the design of the receiver. Determine and sketch the optimal (SER) receiver which is based on this model. What can you say regarding the performance of this receiver? How about relative to the coherent receiver discussed above?

3.46. *(Fall 1998 Midterm)* The phase noise produced by a particular phase estimator is modeled as additive with pdf

$$f_{\Phi(u)}(\phi) = \frac{1}{2\pi I_0(\gamma_L)} e^{\gamma_L \cos\phi} \qquad \phi \in [-\pi, +\pi)$$

where $\gamma_L > 0$ is the loop SNR and $I_0(\cdot)$ is the modified Bessel function of the first kind. When using an MPSK modulation, the phase estimator has M lock points, so that the overall phase error may be modeled as

$$\mathcal{H}_m: \quad \Theta(u) = \frac{2\pi}{M} m \oplus_{2\pi} \Phi(u) \qquad m = 0,1,2,\ldots M-1$$

where $\oplus_{2\pi}$ indicates modulo 2π addition.

(a) Determine the ML decision rule for determining the lock point based on observing a realization θ of $\Theta(u)$; simplify as much as possible.

(b) Suppose that instead of one phase estimate, you have access to a sequence of phase estimates, each locked to the same point, but observed in i.i.d. phase noise:

$$\mathcal{H}_m : \quad \Theta(u,n) = \frac{2\pi}{M} m \oplus_{2\pi} \Phi(u,n) \qquad n = 1, 2, \ldots N$$

Specifically, $\Phi(u,n)$ is an i.i.d. sequence, each with pdf as above.
Determine the ML decision rule based on the observation $\{\theta(n)\}_{n=1}^{N}$. Sketch the operation of the lock-detector below for the special case of $M = 4$.

3.47. The field signal of a binary optical communication system must be detected in the intensity domain. If the system is dominated by field noise, resulting in "speckle" at the intensity detector, then the the model at the output of the detector is

$$\begin{aligned} \mathcal{H}_0 : &\quad R(u) = \|\mathbf{w}(u)\| \\ \mathcal{H}_1 : &\quad R(u) = \|\mathbf{s} + \mathbf{w}(u)\| \end{aligned}$$

where $\mathbf{w}(u)$ is a (2×1) Gaussian random vector with zero mean and $\mathbf{K_w} = \sigma^2 \mathbf{I}$ and $\|\mathbf{s}\|^2 = E$. It can be shown that the pdf of $R(u)$ is Rayleigh under $\mathcal{H}_0$ and Rician under $\mathcal{H}_1$ – i.e.,

$$\begin{aligned} f_{R(u)}(r|\mathcal{H}_0) &= \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \mathrm{U}(r) \\ f_{R(u)}(r|\mathcal{H}_1) &= \frac{r}{\sigma^2} e^{-\left(\frac{r^2+E}{2\sigma^2}\right)} I_0\left(\frac{r\sqrt{E}}{\sigma^2}\right) \mathrm{U}(r) \end{aligned}$$

where $\mathrm{U}(r)$ is the unit step function and $I_0(\cdot)$ is the zero-order modified Bessel function of the first kind.

The a-priori probabilities are $\pi_0 = \pi_1 = 0.5$.

Determine the probability of error as a function of E/σ^2 and plot this result on a log-scale with E/σ^2 in dB.

Note: The CDF of the Rician random variable is

$$F_{R(u)}(r|\mathcal{H}_1) = 1 - Q_1\left(\frac{\sqrt{E}}{\sigma}, \frac{r}{\sigma}\right),$$

where $Q_1(a,b)$ is the Marcum Q-function

$$Q_1(a,b) = \int_b^{\infty} x e^{-\left(\frac{x^2+a^2}{2}\right)} I_0(ax)\, dx$$

For numerical evaluation, use (see the letter by M. K. Simon in Feb. 1998 *IEEE Communications Letters*)

$$Q_1(\zeta b, b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1 + \zeta \sin\theta}{1 + 2\zeta \sin\theta + \zeta^2}\right] \exp\left(-\frac{b^2}{2}[1 + 2\zeta\sin\theta + \zeta^2]\right) d\theta \qquad (0 \le \zeta < 1)$$

$$Q_1(a, \zeta a) = 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{\zeta^2 + \zeta \sin\theta}{1 + 2\zeta \sin\theta + \zeta^2}\right] \exp\left(-\frac{a^2}{2}[1 + 2\zeta\sin\theta + \zeta^2]\right) d\theta \qquad (0 \le \zeta < 1)$$

Note that ζ is a/b in the first expression and b/a in the second expression. You'll have to evaluate $I_0^{-1}(\cdot)$ to plot this.

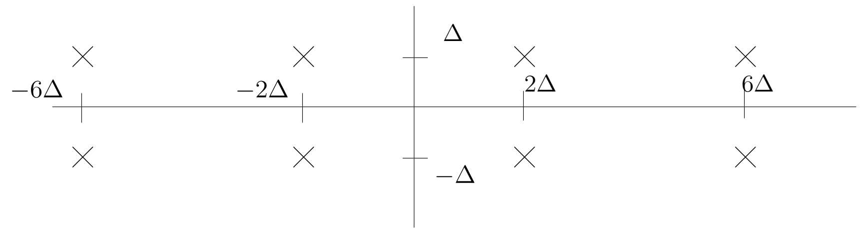


Figure 17: The 8-ary QASK constellation considered.

3.48. A current wireless point-to-point communication system operates using 8-PSK with an uncoded bit error rate (BER) of $P_b = 10^{-4}$. The system operates between the roof-tops of two buildings (linking LANs) and the channel is well-approximated by an AWGN channel. Each transmitter has access to the power supply of the building. The FCC strictly regulates the bandwidth and the system currently operates at 19.2 kbps and occupies the maximum bandwidth.

The company just got some multimedia computers and would like to double the data rate of the link without degrading the BER. What would you suggest? Justify your answer – specifically, describe how your design affects the bandwidth and power requirements.

3.49. A new start-up company has been started started by a recent MBA graduate, Eddie K. Ayear. Eddie wants to bid for an FCC license in an up-coming auction, but needs investors. Eddie wants you to invest. He shows you a business plan which shows that he plans to provide compressed video services at a data rate of 1.5 Megabits/sec across his 100 KHz of licensed bandwidth. You read in the IEEE Spectrum that this type of system operates at $P_{ave}/N_0 = 10^7$ Hz. Do you invest? Explain in detail.

3.50. A typical radio link is designed to operate in the region of

$$10 \text{ dB-Hz} \leq (P_{ave}/N_0)_{dB} \text{ (dB-Hz)} = 10\log_{10}(P_{ave}/N_0) \leq 120 \text{ dB-Hz}$$

The entire library of congress is about 5×10^{12} bits. Taking the upper-end of the range (i.e., 120 dB-Hz) and assuming a very large bandwidth is available, how quickly could the Library of Congress be transmitted across such a link – i.e., how many seconds? What is the minimum E_b/N_0 (in dB) required to achieve this transfer reliably?

3.51. *(Spring 2001, Final Exam)* Consider the modulated signal of the form

$$s(t;\bar{\mathbf{a}}) = \Re\left\{\sum_i \bar{a}_i p(t - iT)\sqrt{2}e^{j2\pi f_c t}\right\}$$

where the modulating sequence $\bar{a}_i(u)$ is an independent, identically distributed sequence, with each symbol uniformly distributed over the 8-ary constellation shown in Fig. 17 and the pulse $p(t)$ is real-valued and is as shown in Fig. 18.

Assume that, for a given realization of the data sequence, the signal is observed in standard AWGN – i.e., under the hypothesis that $\bar{\mathbf{a}}$ was sent

$$r(u,t) = s(t;\bar{\mathbf{a}}) + n(u,t)$$

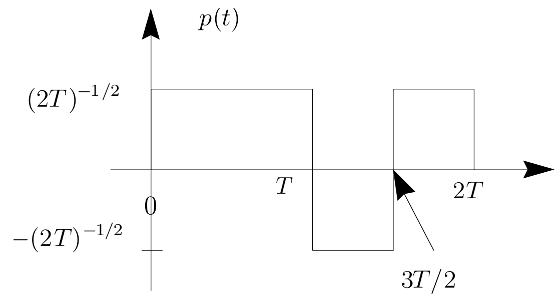


Figure 18: The pulse used for the 8-ary QASK modulation considered.

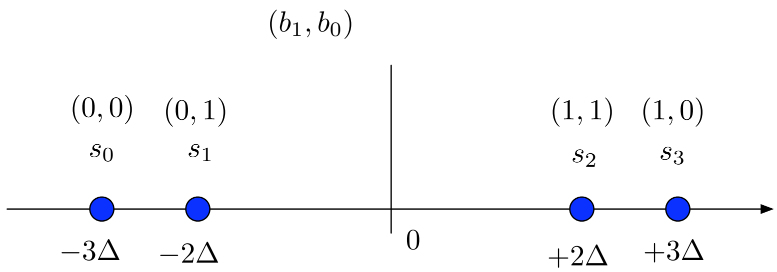


Figure 19: A multilevel PAM constellation.

where $n(u,t)$ is Gaussian with PSD equal to $N_0/2$ at all frequencies.

(a) Determine the optimal receiver for detecting the *sequence* $\{\bar{a}_i\}$. Sketch the receiver that minimizes the probability of sequence error (carefully describe the operation of each "box"). Justify this receiver and state the equivalent discrete-time statistical model for this problem.

(b) What is the probability of error for this system, $P(\mathcal{E}; \Delta)$? Restate this result in terms of the standard measure of E_b/N_0, $P(\mathcal{E}; E_b/N_0)$.

(c) Consider bit-labeling this constellation with three bits for a voice system. Suppose that making errors in the most significant bit location causes more distortion than making errors in the other two bit locations. Suggest a good bit labeling for moderate to high SNR.

3.52. The following is a model for one sample at the output of a matched-filter for a "multilevel PAM" type of modulation

$$\mathcal{H}_m : \quad z(u) = s_m + w(u)$$

where $w(u)$ is zero mean, Gaussian with variance $N_0/2$ and s_m is defined in Fig. 19.

Each of the 4 signals is sent by the transmitter with equal probability.

(a) Determine the average energy per symbol and average energy per bit.

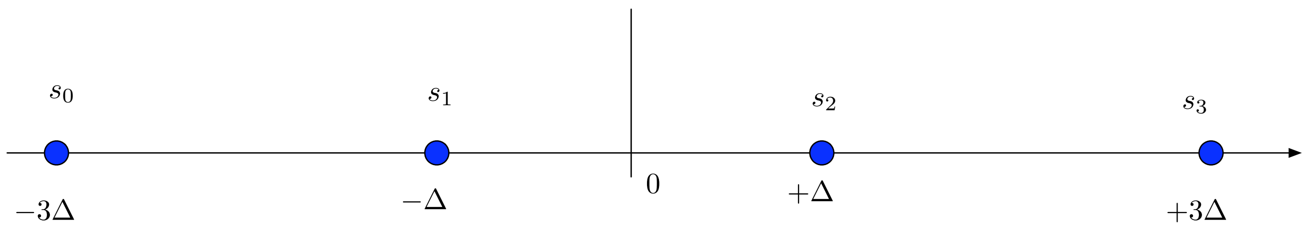


Figure 20: A 4-PAM constellation.

(b) Carefully diagram and label the decision regions for the rule that minimizes the probability of symbol error. Determine the probability of symbol error for this decision rule.

(c) Determine the probability of deciding b_1 incorrectly, P_{b_1}, and the probability of deciding b_0 incorrectly, P_{b_0}, for the decision rule above.

(d) Suppose you have side information that the source produces two classes of bits – *i.e.,* high priority bits and low priority bits – such that it is desired to have a low error rate on the high priority bits. Would you associate the high priority bits with the label b_0 or b_1?

3.53. The following is a model for one sample at the output of a matched-filter for standard 4-PAM modulation

$$\mathcal{H}_m : \quad z(u) = A(u)s_m + w(u)$$

with some amplitude estimation error. Specifically, $w(u)$ is zero mean, Gaussian with variance $N_0/2$ and s_m is defined in Fig. 20.

The amplitude $A(u)$ is modeled as uniform on the interval $[3/4, 5/4]$ – *i.e.,* the amplitude is known only within $\pm 25\%$.

Each of the 4 signals is sent by the transmitter with equal probability.

(a) Let $L(z|\mathcal{H}_m)$ denote the average likelihood for when signal s_m is present. Determine this quantity.

(b) Order the average likelihoods using assuming $\Delta = 1$ and $N_0 = 2$ for $z = 0$. Repeat for $z = 2\Delta$.

(c) Based on the results of part b, do you expect that maximizing the average likelihood will yield significantly lower error probability than simply assuming that $A(u) = 1$?

3.54. Consider the 8-ary, two dimensional constellation obtained by taking one QPSK constellation with energy E_o as an "outer ring" and another QPSK constellation with energy E_i as an "inner ring". The inner ring energy is $E_i = \alpha E_o$ where $0 < \alpha < 1$. In addition, the inner QPSK constellation is rotated counterclockwise by an angle ϕ. This constellation is shown in Fig. 21.

In this problem, the parameters α and ϕ are to be designed. Note that, by symmetry arguments, ϕ need only be considered on the range $0 \leq \phi \leq \pi/2$. The signal is observed in 2-dimensional AWGN, with variance per dimension of $N_0/2$. The 8 signals have equal a-priori probability.

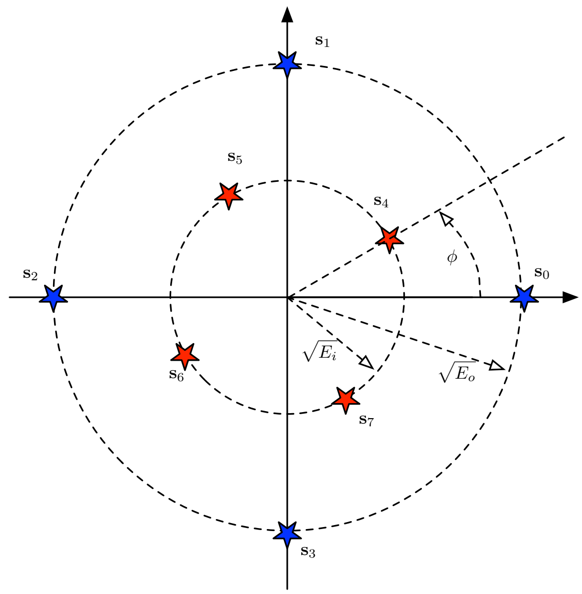


Figure 21: The constellation for Problem 3.54.

(a) Find the minimum squared distance for $\mathbf{s}_0$ – *i.e.,* the smallest squared distance between $\mathbf{s}_0$ and any other signal. Sketch $d^2_{\text{min}}(0)$ as a function of ϕ for $\alpha = 1/4$. Find the minimum squared distance between $\mathbf{s}_4$ and any inner-ring signal – denote this $d^2_i(4)$.

(b) Based on your results above, determine the value of ϕ that maximizes the minimum distance properties for the constellation. For this choice of ϕ, determine the value of α that maximizes the d^2_{min} for the constellation.

(c) For the this part of the problem, consider the values of α and ϕ to be fixed to those found in the previous part. Sketch the constellation with the decision boundaries that minimizes the 8-ary error probability.

(d) Find good upper and lower bounds for the probability of 8-ary error for this rule.

3.55. Consider the 16-ary, two dimensional constellation shown in Fig. 3.22.

In this problem, the parameter Δ is fixed and positive. The signal is observed in 2-dimensional AWGN, with variance per dimension of $N_0/2$. The 16 signals have equal a-priori probability.

(a) What is the average energy per symbol E for the above constellation? Sketch and label the decision boundaries corresponding on to the MAP symbol decision rule

(b) Determine the error probability for the above rule. Using the above expression, find a good approximation for the error probability for moderate to high E/N_0

3.56. A modulation format comprises $M/2$, equal energy (E), orthogonal signals $\{\mathbf{s}_0, \mathbf{s}_1, \ldots \mathbf{s}_{M/2-1}\}$ and the corresponding antipodal complements $\{\mathbf{s}_{M/2} = -\mathbf{s}_0, \mathbf{s}_{M/2+1} = -\mathbf{s}_1, \ldots \mathbf{s}_{M-1} = -\mathbf{s}_{M/2-1}\}$. Here $\mathbf{s}_i$ is an $(M \times 1)$ real-valued vector equivalent signal.

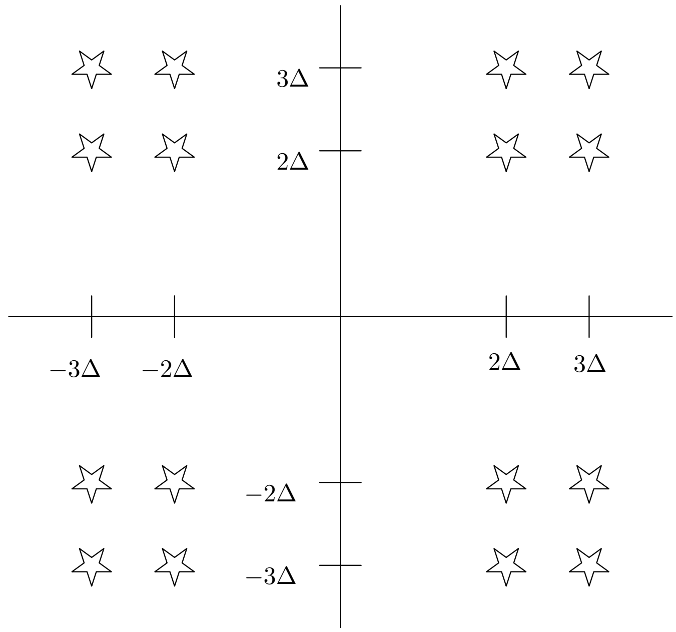


Figure 22: The constellation for Problem 3.55.

(a) Find the optimal decision rule when this modulation is observed in AWGN and the a-priori probability of transmission is uniform. Give the simplest form of the rule in terms of the observation $\mathbf{r}$.

(b) For the specific example of $M = 8$, consider implementing these signals using Hadamard sequences. Give the 8 vector signals for this case.

(c) Consider the general case of M being a power of 2 and this modulation format implemented using Hadamard sequences as above. Assuming that the components of the signal vector are sent sequentially using root-raised cosine pulse shaping with 50% excess bandwidth, what is the spectral efficiency of the transmitted waveform in bps/Hz?

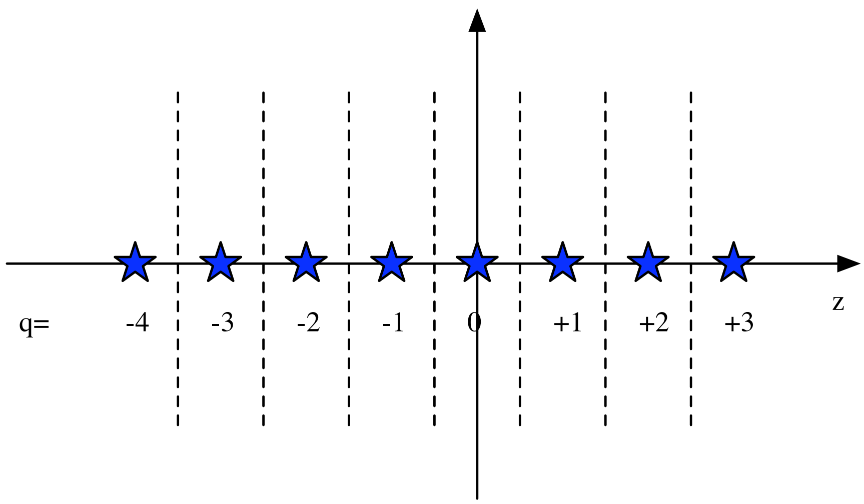
3.57. For each of the following signals, plot the complimentary CDF for the envelope and the I/Q signal trajectory:

- 16-PSK signal with a root-raised cosine pulse shape with 25% excess bandwidth
- 16-PSK signal with a root-raised cosine pulse shape with 75% excess bandwidth
- 16-QAM signal with a root-raised cosine pulse shape with 25% excess bandwidth
- 16-QAM signal with a root-raised cosine pulse shape with 75% excess bandwidth

Note that you will need to generate the signal of sufficient time duration and at a sufficient number of samples per symbol to obtain accurate results.

4 Performance Metrics and Channel Models for Coding

4.1. The goal of this problem is to extract a discrete memoryless channel model for a quantized BPSK-AWGN channel. Consider the standard BPSK-AWGN channel with signal gain adjusted to the level ± 2 – i.e., $z(u) = 2(-1)^{c(u)} + w(u)$. We can consider the following 3-bit (8-level) quantizer mapping which maps z to an integer $q \in \{-4, -3, -2, -1, 0, +1, +2, +3\}$:



Note that the labels are offset to account for simple two's complement digital logic.

Let E_c/N_0 be 2 dB and determine the corresponding abstracted DMC.

4.2. Using the results of Problem 1.12 and the `FEC_limits` program (or BPSK-AWGN capacity curve), characterize the degradation in E_b/N_0 for using hard-in decoding rather than soft-in decoding on the BPSK-AWGN channel. Specifically, plot the difference in required E_b/N_0 (in dB) for the two cases against the maximum rate in bits per channel use.

4.3. Determine the ML decoding rule for the BSC channel when the channel error rate is greater than 0.5.

4.4. Consider a standard BPSK-AWGN channel with real-valued observation

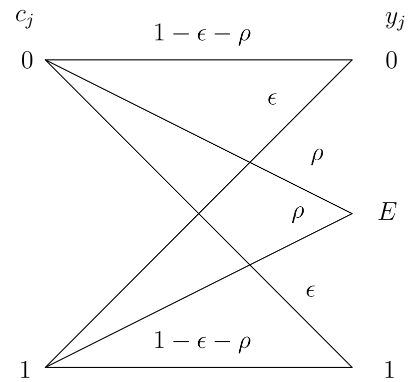
$$z_j(u) = \sqrt{E_c}(-1)^{c_j(u)} + w_j(u)$$

where E_c is the energy per coded bit, $c_j(u)$ is the j^{th} coded bit, and $w_j(u)$ is an i.i.d. sequence of Gaussian random variables with mean zero and variance $N_0/2$.

The receiver converts the real-valued observation $z_j(u)$ into a discrete observation $y_j(u) = h(z_j(u))$ where the function $h(z)$ is defined as

$$h(z) = \begin{cases} 0 & z \geq +\sqrt{E_c}/2 \\ E & |z| < \sqrt{E_c}/2 \\ 1 & z \leq -\sqrt{E_c}/2 \end{cases}$$

This results in the discrete memoryless channel (DMC) shown below



where ϵ is the error probability, ρ is the erasure probability, and E is the channel output corresponding to an erasure.

(a) Determine ρ and ϵ for the above DMC.

(b) Consider performing maximum likelihood codeword decoding (assuming a purely random source of information bits) on this channel. Determine the normalized channel metrics for each of the three values of y_j – *i.e.,* If $y_j = 0$, what is $\overline{\text{MI}}[c_j] = L_0$; if $y_j = E$, what is $\overline{\text{MI}}[c_j] = L_E$; if $y_j = 1$, what is $\overline{\text{MI}}[c_j] = L_1$?

(c) Consider ML codeword decoding the $(5, 1)$ repetition code over the above channel with $E_b/N_0 = 3$ dB. First, determine numerical values for the following: E_c/N_0, E_c/N_0, ϵ, ρ, L_0, L_E, L_1. State the ML decoding rule for this case in the simplest form.

For this case, decode each of the following observation vectors $\mathbf{y}$ using the rule derived to obtain a decision on the information bit, denoted $\hat{b}$.

$$\mathbf{y}^{\mathrm{t}} = (1\ 1\ E\ 0\ E)$$
$$\mathbf{y}^{\mathrm{t}} = (E\ E\ 0\ E\ E)$$
$$\mathbf{y}^{\mathrm{t}} = (1\ E\ 0\ E\ 0)$$
$$\mathbf{y}^{\mathrm{t}} = (1\ 1\ E\ 0\ 0)$$

4.5. Consider the repetition code over the BPSK-AWGN channel (i.e., with soft-in decoding). Does this provide any coding gain? Discuss this in the context of the results of Problem 1.11.

4.6. Recall the definition of the min* operator is

$$\min{}^*(x_1, x_2 \ldots x_n) \stackrel{\Delta}{=} -\ln(e^{-x_1} + e^{-x_2} + \cdots e^{-x_n})$$

Prove that

(a) $\min^*(x, y) = \min(x, y) - \ln\left(1 + e^{-|x-y|}\right)$

(b) $\min^*(x, y, z) = \min^*(\min^*(x, y), z)$

4.7. Consider the (7,4) Hamming code as described in lecture with i.i.d. input data, each bit equally likely to be a 1 or a 0.

(a) List all the codewords for this code.

(b) Consider using the BPSK-AWGN channel to transmit this information. Specifically, assume that $z_j = (-1)^{c_j} + w_j$ for $j = 0, 1, \ldots 6$, and where w_j is a realization of AWGN. Consider the following realizations of the observation vector ($\mathbf{z}^{\mathrm{t}} = [z_0\ z_1 \ldots z_6]$).

$$\mathbf{z}_A = \begin{bmatrix} -1.77 \\ 1.671 \\ 0.516 \\ 1.445 \\ -1.67 \\ -0.99692 \\ 1.169 \end{bmatrix} \qquad \mathbf{z}_B = \begin{bmatrix} -1.153 \\ 0.827 \\ 0.176 \\ 0.638 \\ -1.047 \\ -0.845 \\ 1.248 \end{bmatrix}$$

where $\mathbf{z}_A$ was generated at $E_b/N_0 = 0$ dB and $\mathbf{z}_B$ was generated at $E_b/N_0 = 6$ dB. For each of these two cases, answer the following:

i. What are the equivalent observations for the binary symmetric channel – i.e., hard decisions on the channel bits?

ii. Specify the properly normalized soft-in metric: $\overline{\mathrm{MI}}[b_i]$ for $i = 0, 1, 2, 3$ and $\overline{\mathrm{MI}}[c_j]$ for $j = 0, 1 \ldots 6$.

iii. Using hard-in information what is the decision that minimizes the probability of block error?

iv. Using soft-in information, what is the decision that minimizes the probability of block error?

v. Using soft-in information what is the decision that minimizes the probability of bit error?

vi. Using min-sum SISO decoding, what is $\overline{\mathrm{MO}}[b_i]$ for $i = 0, 1, 2, 3$?

vii. Using min-sum SISO decoding, what is $\overline{\mathrm{MO}}[c_j]$ for $j = 0, 1, \ldots 6$?

viii. Using min*-sum SISO decoding, what is $\overline{\mathrm{MO}}[b_i]$ for $i = 0, 1, 2, 3$?

ix. Using min*-sum SISO decoding, what is $\overline{\mathrm{MO}}[c_j]$ for $j = 0, 1, \ldots 6$?

x. If the soft-in metrics for c_j are replaced by z_j, how do the output metrics change for the min-sum and min*-sum cases?

4.8. Consider the binary variables $\{d_i\}_{i=0}^{q}$ which are constrained by a repetition code. For example, you may view d_0 as the input to a repetition code and $d_1, d_2, \ldots d_q$ as the outputs. Show that min-sum and min*-sum SISO processing for the repetition code yields

$$\overline{\mathrm{MO}}[d_i] = \sum_{j \neq i} \overline{\mathrm{MI}}[d_j]$$

4.9. Consider real numbers x and y. Prove the following

$$\min(x, y) - \min(0, x + y) = \min(|x|, |y|)\mathrm{sgn}(x)\mathrm{sgn}(y)$$

where $\mathrm{sgn}(z)$ is $+1$ if $z \geq 0$ and -1 if $z < 0$.

4.10. Consider the binary variables $\{d_i\}_{i=0}^{q}$ which are constrained by a single parity check code – i.e., these variables must sum to zero mod 2.

(a) Consider $q = 2$ and show that min-sum SISO processing for the SPC code yields

$$\overline{\mathrm{MO}}[d_i] = \min_{j \neq i} |\overline{\mathrm{MI}}[d_j]| \prod_{j \neq i} \mathrm{sgn}(\overline{\mathrm{MI}}[d_j])$$

Hint: you may consider using the results of Problem 9.

(b) Show that this generalizes to arbitrary q. Hint: One method is to use induction to establish the above recursive relation.

(c) Given the set of incoming metrics, suppose the following has been computed: (i) $M_{min} =$ the minimum of $|\overline{\mathrm{MI}}[d_i]|$ over all i, (ii) $M_{sec} =$ the second smallest value of $|\overline{\mathrm{MI}}[d_i]|$ (possibly the same as M_{min} if two input metrics are the same) and (iii) $S = \prod_{j=0}^{q} \mathrm{sgn}(\overline{\mathrm{MI}}[d_j])$, the product of the signs of all incoming metrics. Show how $\overline{\mathrm{MO}}[d_i]$ can be computed easily given these three quantities.

4.11. We did not consider the case of hard-in/soft-out decoding. Is this a reasonable concept? If so, describe the appropriate method and demonstrate this on the (7,4) Hamming code example.

4.12. Consider a binary digital communication system that transmits no signal to send a '0' and sends a signal to send a '1'. The simplified model for the received signal in this case is

$$z_j(u) = x_j(u) + w_j(u)$$

where $w_j(u)$ is an i.i.d. sequence of Gaussian random variables with mean zero and variance σ^2. The signal $x_j(u)$ is a function of the binary channel input $c_j(u)$. Specifically, when $c_j = 0$, $x_j = 0$ and when $c_j = 1$, $x_j = A$, where A is a positive constant. Note that this is a memoryless channel.

(a) Determine the normalized metric on the channel inputs; simplify your answer as much as possible.

(b) Consider a single parity check code that constrains $\{c_j\}_{j=0}^{4}$. These bits are then sent through the above channel and the following channel observations are made.

$$\begin{aligned} z_0 &= +2.0 \\ z_1 &= +3.0 \\ z_2 &= -1.0 \\ z_3 &= -3.5 \\ z_4 &= +0.5 \end{aligned}$$

Assuming that $A = 2$ and $\sigma^2 = 2$, determine the normalized incoming and outgoing metrics using min-sum processing and specify the best hard decision on the bits.

(c) Assuming the same values for the channel observations, A and σ^2, repeat the SISO decoding using min*-sum processing.

4.13. Consider the BPSK-AWGN channel with channel observation

$$z_j(u) = (-1)^{c_j(u)} + w_j(u)$$

where $w_j(u)$ is an i.i.d. Gaussian sequence with zero mean and variance $N_0/2$.

The real-valued observation z_j is put through a memoryless quantizer to produce q_j. Specifically,

$$q_j = \begin{cases} 0 & z_j < -1 \\ 1 & -1 \leq z_j < 0 \\ 2 & 0 \leq z_j < +1 \\ 3 & z_j \geq +1 \end{cases}$$

This problem deals with decoding based on the observation $\{q_j\}$.

(a) The channel with input c_j and output q_j is a discrete memoryless channel (DMC). Specify this DMC model by determining the conditional probability mass function (pmf) $p_{q_j(u)|c_j(u)}(q|c)$.

(b) Based on the channel observation q_j, what is the incoming, normalized metric $\overline{\mathrm{MI}}[c_j = 1]$?

(c) Consider performing min-sum SISO decoding of the (5,3) code from the toy-SISO spreadsheet for the BSC. The 8 codewords are shown below for reference:

$$\begin{aligned}
\mathbf{c}^{(0)} &= (0\ 0\ 0\ 0\ 0)^{\mathrm{t}} \\
\mathbf{c}^{(1)} &= (0\ 0\ 1\ 0\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(2)} &= (0\ 1\ 0\ 1\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(3)} &= (0\ 1\ 1\ 1\ 0)^{\mathrm{t}} \\
\mathbf{c}^{(4)} &= (1\ 0\ 0\ 1\ 0)^{\mathrm{t}} \\
\mathbf{c}^{(5)} &= (1\ 0\ 1\ 1\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(6)} &= (1\ 1\ 0\ 0\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(7)} &= (1\ 1\ 1\ 0\ 0)^{\mathrm{t}}
\end{aligned}$$

For this code consider the observation vector $\mathbf{q}^{\mathrm{t}} = (1\ 0\ 3\ 2\ 1)$ and $N_0 = 2$. Further assume that the input bits $\{b_i\}$ are i.i.d., and each equally probable to be 0 or 1. Compute the outgoing extrinsic metrics in normalized form. Determine the best hard decisions on these 5 bits.

4.14. A BPSK-AWGN channel has channel observation

$$z_j(u) = \sqrt{E_c}(-1)^{c_j(u)} + w_j(u)$$

where $w_j(u)$ is a mean zero, Gaussian random variable with variance $N_0/2$.

This observation is converted to hard decision information on c_j to produce the observation $y_j \in \{0, 1\}$. Because of front-end hardware imperfections, this does not result in the standard BSC channel. Specifically, the conversion rule is

$$y_j = \begin{cases} 0 & z_j \geq \alpha\sqrt{E_c} \\ 1 & z_j < \alpha\sqrt{E_c} \end{cases}$$

where α is a real-valued constant between 0 and 1.

(a) The resulting channel mapping c_j to y_j is a discrete memoryless channel. Determine the following probabilities associated with this DMC:

- $\epsilon_0 = p(y_j = 1|c_j = 0)$
- $\epsilon_1 = p(y_j = 0|c_j = 1)$

Which is larger, ϵ_0 or ϵ_1?

(b) Draw the DMC transition diagram for this channel and label the edges with the associated conditional probabilities

(c) Determine the Maximum Likelihood Codeword decoding rule for this DMC. Specify this rule in the simplest form possible. This can be in terms of ϵ_0 and ϵ_1. Clearly define any other terms that you use to simplify the expression of the rule.

(d) Consider performing ML-CW decoding of the (5,3) code from the toy-SISO spreadsheet for this DMC. For this code consider the observation vector $\mathbf{y}^{\mathrm{t}} = (1\ 1\ 1\ 0\ 1)$ and, for purposes of numerical evaluation, take $\epsilon_1 = 2.34 \times 10^{-4}$ and $\epsilon_0 = 0.0154$. Determine the best codeword decision

(e) Describe in words the effect that $\epsilon_0 \neq \epsilon_1$ has on the ML-CW decoding rule.

4.15. Consider the standard binary symmetric channel with inputs $c_j(u)$ and outputs $y_j(u)$ and error probability $p(y_j = 1|c_j = 0) = p(y_j = 0|c_j = 1) = \epsilon < 1/2$. In this problem we consider soft-out decoding based on this channel observation. Specifically, assume that any information bits associated with the code are equally likely a-priori to be 0 or 1. The goal is to compute $\overline{\mathrm{MO}}[c_j = 1]$ given the BSC observations and knowledge of the code constraint using min-sum processing.

(a) Based on the channel observation y_j, what is the incoming, normalized metric $\overline{\mathrm{MI}}[c_j = 1]$?

(b) Since min-sum processing is being used and the normalized metrics for the information bits is zero, it is natural to multiply all of the normalized metrics by a constant K so as to remove the dependency on ϵ. Specify this constant.

In the following parts of the problem use these scaled metrics as $\overline{\mathrm{MI}}[c_j = 1]$.

(c) Consider performing min-sum SISO decoding of the (5,3) code from the toy-SISO spreadsheet for the BSC. For this code consider the observation vector $\mathbf{y}^{\mathrm{t}} = (1\ 1\ 1\ 0\ 1)$ and compute the outgoing extrinsic metrics in normalized form.

(d) Consider performing this SISO decoding for an $n = 6$ single parity check code over the BSC. Complete the table for the following two sets of BSC channel observations.

j :	0	1	2	3	4	5
y_j :	0	0	1	0	1	0
$\overline{\text{M0}}[c_j = 1]$:						
$\hat{c}_j$:						

j :	0	1	2	3	4	5
y_j :	1	0	1	0	1	0
$\overline{\text{M0}}[c_j = 1]$:						
$\hat{c}_j$:						

Describe in words what this min-sum SISO rule for the BSC channel does for an SPC.

4.16. Consider a coded binary digital communication system that uses antipodal signaling

$$z_j(u) = (-1)^{c_j(u)}\sqrt{E_c} + w_j(u)$$

where $w_j(u)$ is an i.i.d. sequence of Laplacian random variables with mean zero and variance σ^2. Specifically, the probability density function of $w_j(u)$ is

$$f_{w_j(u)}(w) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}|w|}$$

(a) Sketch the channel likelihoods $f_{z_j(u)|c_j(u)}(z|1)$ and $f_{z_j(u)|c_j(u)}(z|0)$.

(b) Determine the normalized metric on the channel inputs; simplify your answer as much as possible.

(c) Consider a single parity check code that constrains $\{c_j\}_{j=0}^4$. Assuming that $E_c = 4$ and $\sigma^2 = 2$, determine the normalized incoming and outgoing metrics using min-sum processing and specify the best hard decision on the bits given the following observations:

j :	0	1	2	3	4
z_j :	-4	$+3$	-1	-3.5	$+6.6$
$\overline{\text{MI}}[c_j = 1]$:					
$\overline{\text{M0}}[c_j = 1]$:					
$\hat{c}_j$:					

Assuming the same values for E_c and σ^2, repeat the SISO decoding using min*-sum

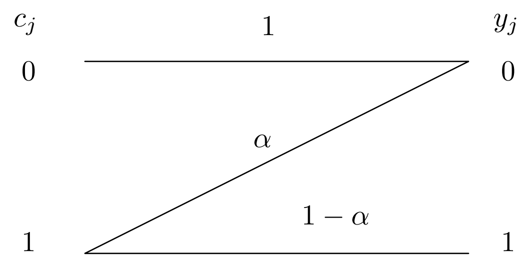


Figure 23: The channel transition diagram for the Z channel.

processing for an $n = 3$ SPC with the given observations:

j :	0	1	2
z_j :	-4	$+3$	-1
$\overline{\mathrm{MI}}[c_j = 1]$:			
$\overline{\mathrm{M0}}[c_j = 1]$:			
$\hat{c}_j$:			

4.17. Consider a binary code comprising all $(n \times 1)$ binary vectors with exactly three 1's.

(a) Determine the number of valid configurations M and specify the valid configurations when $n = 5$.

(b) Is this code linear? What is the minimum distance of this code?

(c) Given a set of incoming normalized metrics for the variables $\{\overline{\mathrm{MI}}[c_j]\}_{j=0}^{n-1}$, determine the outgoing, normalized extrinsic metric for c_i obtained with min-sum processing. Express this rule in the simplest form possible.

(d) Demonstrate the soft-in/soft-out rule derived above by processing the incoming metrics below to obtain the out-going metrics and the best hard decisions on the variables $\{c_j\}$ with $n = 7$.

j :	0	1	2	3	4	5	6
$\overline{\mathrm{MI}}[c_j]$:	0.7	2	-3	4	1	6	-2
$\overline{\mathrm{M0}}[c_j]$:							
$\hat{c}_j$:							

4.18. Consider the DMC called the "Z-channel" as diagrammed in Fig. 23 where $\alpha \in (0, 0.5)$.

(a) Describe the ML codeword decoding rule for this code in the simplest terms possible.

(b) Demonstrate your understanding of this rule by decoding the (5,3) code from the toy-SISO spreadsheet for this channel with the observation $\mathbf{y}^{\mathrm{t}} = (1\ 0\ 0\ 0\ 0)$ and $\alpha = 0.25$. For reference, the codewords in this code are:

$$\begin{aligned}
\mathbf{c}^{(0)} &= (0\ 0\ 0\ 0\ 0)^{\mathrm{t}} \\
\mathbf{c}^{(1)} &= (0\ 0\ 1\ 0\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(2)} &= (0\ 1\ 0\ 1\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(3)} &= (0\ 1\ 1\ 1\ 0)^{\mathrm{t}} \\
\mathbf{c}^{(4)} &= (1\ 0\ 0\ 1\ 0)^{\mathrm{t}} \\
\mathbf{c}^{(5)} &= (1\ 0\ 1\ 1\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(6)} &= (1\ 1\ 0\ 0\ 1)^{\mathrm{t}} \\
\mathbf{c}^{(7)} &= (1\ 1\ 1\ 0\ 0)^{\mathrm{t}}
\end{aligned}$$

Is this decision unique?

4.19. Show that for the BPSK-AWGN channel with equal a-priori probabilities, the random coding bound can be written as

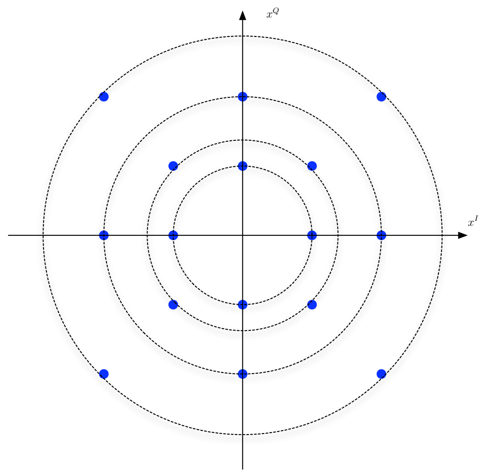
$$\bar{P}_{word} \leq e^{-k(E_b/N_0)} \left\{ \min_{0 \leq \rho \leq 1} 2^{\rho r+1} \int_0^{\infty} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} \cosh^{1+\rho}\left(\frac{y\sqrt{2r(E_b/N_0)}}{1+\rho} \right) dy \right\}^n$$

Note that this is equation (16) in Dolinar, Divsalar and Pollara with a slight correction (i.e., the exponent has a factor of 1/2).

4.20. Consider two potential solutions for a signal design to achieve 2 bps/Hz. System one uses a rate 2/3 binary code with 8PSK modulation. System two uses a rate 1/2 binary code with 16-QAM modulation. For each system the input block length of the code is limited to $k = 2048$ bits. Estimate the minimum value of E_b/N_0 (in dB) required to achieve a block error rate (BLER) of 10^{-4} for each system using any possible code. Which system is preferred from this viewpoint? What is the advantage, in dBs of E_b/N_0, for the preferred system?

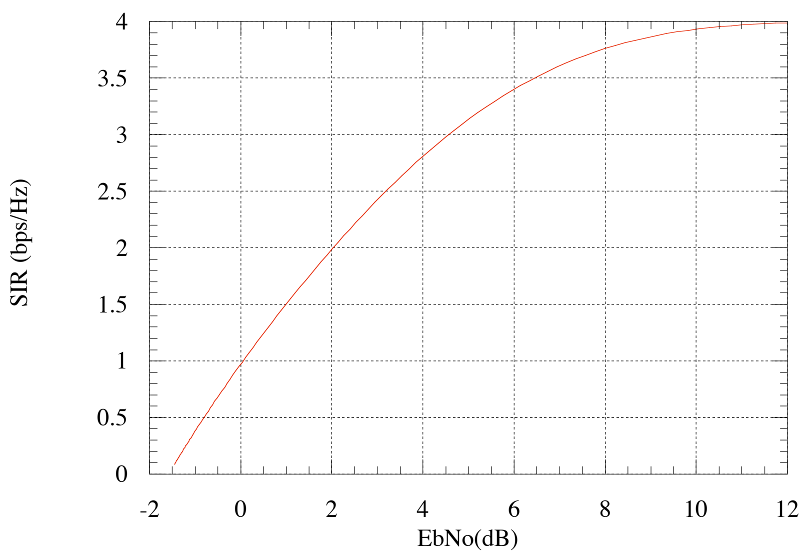
4.21. For a given block size and rate, the symmetric sphere-packing bound approximation and the symmetric random coding bound both predict a value of minimum E_b/N_0 for a given BLER. For BPSK modulation and a target BLER of 10^{-4}, plot the difference between these two values of E_b/N_0 in dB vs. block size. Do this for various code rates. What can you conclude from these plots?

4.22. You have just been hired as a communication systems engineer by FliTunes, a company selling digital music via satellite download. Your first task is to review the current design. The senior engineers have selected a rate $r = 2/3$ binary convolutional code along with a proprietary QASK-type modulation. The constellation for this modulation is shown below:



The system uses a data packet structure with $k = 1024$ input bits per packet. The design goal is to operate with a decoded error probability on these blocks of 10^{-4} or lower. You are told that the current design operates at an E_b/N_0 of 9.0 dB.

You have lost your copy of "`FEC_limits`" but you were able to find the symmetric information rate (SIR) for the above modulation using Matlab. This is shown below and assumes that one symbol per sec per Hz of bandwidth is achieved:



You got this job because during your interview you explained that you knew all about "modern codes" from your USC EE568 class. So now, your bosses are calling on you to determine if using such codes can improve the performance of their modem.

(a) What is the spectral efficiency of the current design – i.e., $\eta_{bps/Hz}$ in bps/Hz assuming one symbol per second per Hz?

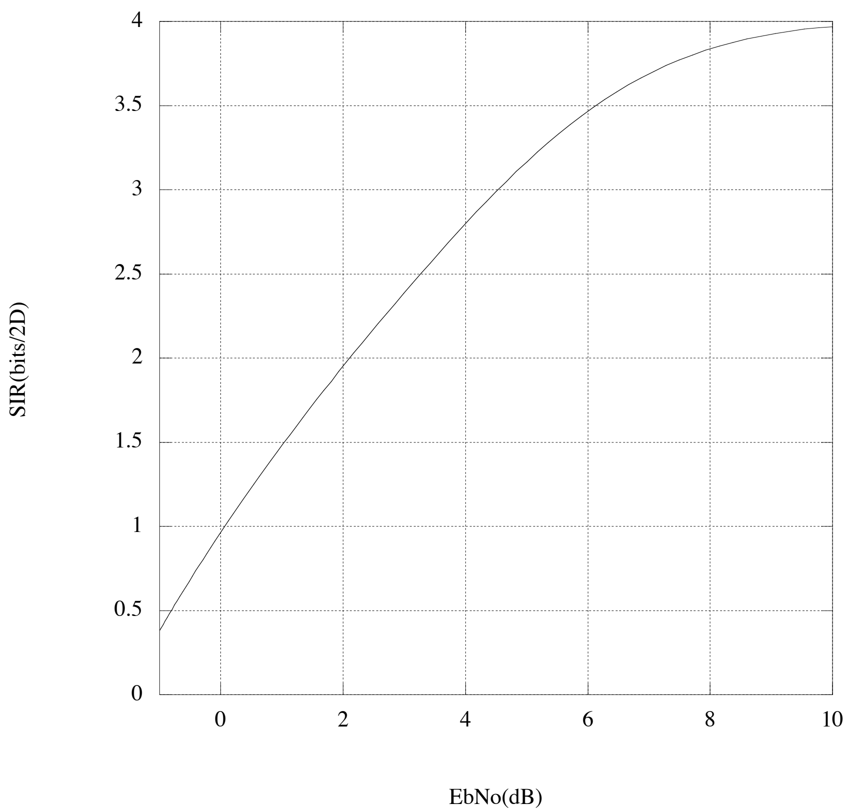
(b) Based on using block of size $k = 1024$ and a required block error rate of 10^{-4}, and maintaining the current spectral efficiency, estimate the required E_b/N_0 for well-designed modern coded system with this modulation and explain your reasoning.

What is the additional coding gain that you predict for such a modern code relative to that provided by the current system?

(c) For the same spectral efficiency and E_b/N_0 found in part b, suppose one desires to achieve a block error probability of 10^{-6} by using a different value of k – i.e., k_{new}. What value of k_{new} would be required to achive this with a well-designed modern code?

(d) For this spectral efficiency, what is the lowest value of E_b/N_0 (in dB) possible for reliable communication using any modulation format?

4.23. Consider designing a modem for the AWGN channel using 4-ary Pulse Amplitude Modulation (4-PAM) modulation. This modulation has constellation points $-3A$, $-A$, $+A$, $+3A$ where A is a positive constant. Note that, like BPSK, this is a one-dimensional signal constellation. The Symmetric Information Rate (SIR) for this modulation is provided below:



(a) A linear binary code with rate $r = 7/10$ is to be used for this link.

- Specify the rate of this coded modulation system – *i.e.*, $\eta_{\text{b/sym}}$ and $\eta_{\text{b/2d}}$.
- Using the SIR as a good approximation for the modulation-constrained capacity, what is the minimum value of E_b/N_0 required to operate at this rate with 4-PAM modulation?

- What is the corresponding value of E_s/N_0 – i.e., the minimum value of the energy per 4-PAM modulation symbol to noise spectral level?

(b) Because of latency constraints, the number of information bits to be coded per block is $k = 768$. A good modern turbo-like code is available with this rate, block size and modulation format. The link will use this modern code and operate at $E_b/N_0 = 6$ dB. Determine the resulting codeword error probability.

4.24. Consider the $(12, 4)$-PSK (also known as $(12, 4)$ amplitude shift keying) constellation given in Fig. 24.

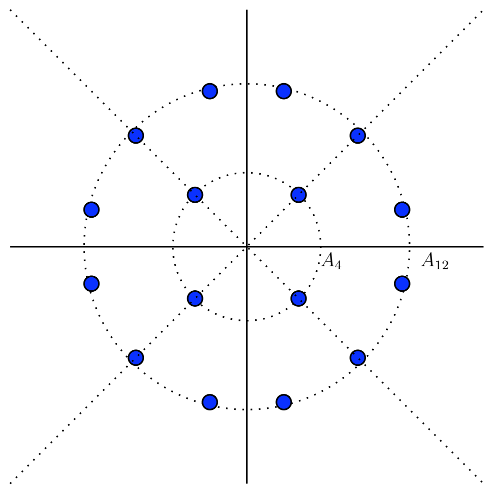


Figure 24: The $(12, 4)$-PSK signal constellation.

The radii of the inner and outer circles satisfy

$$A_4 \sin\left(\frac{\pi}{4}\right) = A_{12} \sin\left(\frac{\pi}{12}\right)$$

so that the distance between closest points on the inner and outer circles is the same. Also, the signals on the inner circle are separated by $\pi/2$ radians and one is located at angle $\theta = \pi/4$ off the x-axis. One of the signals on the outer circle is also at angle $\pi/4$ off of the x-axis. Points on the outer circle are separated by an angle of $\pi/6$.

(a) For equally likely signals ($p_m = 1/(16)$), determine the value of A_4 and A_{12} as a function of E_s.

(b) Consider designing a system that uses this modulation and achieves a throughput of 2.5 bits per symbol, uses an input block length of $k = 512$, and must achieve a codeword error probability of 10^{-3}.

   i. How many output symbols will there be per code block?

   ii. Estimate the minimum value of E_b/N_0 (in dB) for a good modern code to achieve this performance. What value of E_s/N_0 (in dB) does this correspond to?

4.25. A system design is being considered using a binary $r = 1/2$ code with 8PSK modulation over an AWGN channel.

(a) What is the spectral efficiency measured in bits/sec/Hz?

(b) If any modulation could be used, what is the smallest value of E_b/N_0 required to reliably communicate over an AWGN channel at this spectral efficiency?

(c) For a codeword error probability requirement $P_{CW} < 10^{-4}$, what is the minimum value of input block size k required to achieve a $\Delta_{\text{dB}} < 1$ for the 8PSK system?

(d) If a good, practical modern code is used and $P_{CW} < 10^{-4}$ is required, what is the minimum value of k required to operate within 1 dB in E_b/N_0 of the modulation constrained capacity?

(e) For this value of k, how many 8PSK channel symbols are required to be sent?

5 Classical Coding

5.1. Consider a $(8, 4)$ linear code whose parity-check equations are given by

$$v_0 = u_1 + u_2 + u_3 \tag{1}$$
$$v_1 = u_0 + u_1 + u_2 \tag{2}$$
$$v_2 = u_0 + u_1 + u_3 \tag{3}$$
$$v_3 = u_0 + u_2 + u_3 \tag{4}$$

where u_0, u_1, u_2, u_3 are the message symbols and v_0, v_1, v_2, v_3 are the parity-check symbols. Find generator and parity-check matrices for this linear code.

5.2. Write down a parity check matrix for the binary linear code whose generator matrix is given by

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5.3. Determine the minimum distance of the $(7, 3)$ linear block code having parity-check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

5.4. Write down the weight distribution of the $(7, 4)$ Hamming code discussed in lecture.

5.5. Consider a $(5, 2)$ systematic linear code having generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(a) Find the standard array for this code.

(b) What is the minimum distance of the code?

(c) On a certain transmission across the Binary symmetric channel (BSC), the channel introduces the error vector $\mathbf{e} = [01011]^T$, i.e., the received vector $\mathbf{y}$ is given by

$$\mathbf{y} = \mathbf{c} + \mathbf{e}$$

where $\mathbf{c}$ is the transmitted codeword and $\mathbf{e}$ is as given above. If $\hat{\mathbf{c}}$ is the decoded codeword as a result of the maximum likelihood decoding (MLD) using the standard array, what is the Hamming distance $d_H(\mathbf{c}, \hat{\mathbf{c}})$?

(d) What is the probability of the codeword error with minimum Hamming distance decoding? Assume a BSC with channel error rate below 0.5.

(e) What is the probability of message-bit error with minimum Hamming distance decoding? Assume a BSC with channel error rate below 0.5.

5.6. Repeat problem 21 with

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and discuss the results relative to those obtained in problem 19.

5.7. A code with $(n, k, d_{\min}) = (15, 9, 5)$ is desired. Describe a search procedure to find such a code.

5.8. Specify a generator matrix for the RM$(2, 4)$ code. What is the rate and minimum distance of this code?

5.9. Specify $(n, k, d_{\min})$ and a generator matrix for the the $m = 4$ Hamming code.

5.10. Consider the two conditions for a linear code $\mathcal{C}$ and its dual $\mathcal{C}^\perp$ and prove if A, then B:

$$\begin{aligned} A &: \text{ The dual code } \mathcal{C}^\perp \text{ can correct all weight one error patterns.} \\ B &: \text{ The linear code } \mathcal{C} \text{ has no repeated bits} \end{aligned}$$

A code has a repeated bit if two bits of every codeword agree – *i.e.,* $c_j = c_m$ for some $j \neq m$ and this holds for every $\mathbf{c} \in \mathcal{C}$.

5.11. Diagram the parity check trellis for the $(7, 4)$ Hamming code. Using the observation values $\mathbf{z}_A$ from Problem 7, run the FBA to determine the soft-out information and confirm that this is the same as obtained by exhaustive marginalization and combining used in Problem 7.

5.12. Consider the linear block code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(a) *Code Structure:* Determine the following for this code: a parity check matrix $\mathbf{H}$, the rate r, the number of codewords, M.

(b) *Distance Properties:* Determine the following for this code: the weight enumerating function (WEF) $A(D)$, the minimum Hamming distance d_{min}

Determine the number of cosets and list 5 of the coset leaders.

Answer the following questions:

i. Is this a perfect binary code?

ii. For this block size and rate, is it possible a d_{min} larger than that of this code can be achieved?

(c) *Dual Code:* Determine the following for this code: the rate r_D, the number of codewords M_D, the WEF $A_D(D)$, and minimum distance $d_{min,D}$.

(d) Assume that the coded bits are indexed by $\mathbf{c}^{\mathrm{t}} = [\ c_0\ c_1\ c_2\ c_3\ c_4\ c_5\ ]$ and the channel observation model

$$z_j(u) = (-1)^{c_j(u)} + w_j(u) \quad j = 0, 1, \ldots 5$$

where $w_j(u)$ is an i.i.d. sequence of Gaussian random variables with zero mean and variance one. Assume that the information (input) bits are i.i.d. random binary digits, equally likely to be 0 or 1 a-priori.

Consider the realization of the observation

$$\mathbf{z}^{\mathrm{t}} = [\ z_0\ z_1\ z_2\ z_3\ z_4\ z_5\ ] = [\ +0.1\ \ -0.2\ \ -0.8\ \ -0.5\ \ +0.1\ \ -0.4\ ]$$

and determine the normalized ($\overline{\mathrm{MI}}[v = 0] = 0$) input metrics for SISO decoding.

Based on this observation $\mathbf{z}$, find the codeword decision that minimizes the probability of codeword error and specify the associated decision for codeword and information bits: $\hat{\mathbf{c}}_{SID}$ and $\hat{\mathbf{b}}_{SID}$.

(e) *HI Decoding:* Consider the above observation and determine the associated "hard decisions" on the channel bits. Specifically, let $y_j \in \{0, 1\}$ be the decision for c_j based only on z_j and specify y_j for $j = 0, \ldots 5$.

Based on this observation $\mathbf{y}$, find the codeword decision that minimizes the probability of codeword error – i.e., specify $\hat{\mathbf{c}}_{HID}$ and $\hat{\mathbf{b}}_{HID}$.

i. Are the decoded decisions the same under soft-in and hard-in decoding?

ii. What can be said about the actual value of the transmitted codeword? Explain.

(f) *SISO Decoding:* Based on the channel observation $\mathbf{z}$ given above, determine the normalized ($\overline{\mathrm{MO}}[v = 0] = 0$) output extrinsic metrics based on min-sum processing for each of the information and coded bits.

5.13. A rate 1/2 code with 3 input bits b_0, b_1, b_2 is generated using

$$\begin{aligned}
c_0 &= b_0 \\
c_1 &= b_1 \\
c_2 &= b_2 \\
c_3 &= b_0 + \overline{b}_1 \\
c_4 &= b_1 + \overline{b}_2 \\
c_5 &= \overline{b}_0 + b_1 + \overline{b}_2
\end{aligned}$$

where all addition is modulo two and $\overline{b}$ represents the logical complement of the binary variable – i.e., if $b = 0$, $\overline{b} = 1$ and if $b = 1$, $\overline{b} = 0$.

(a) For each of the following properties, answer 'yes', 'no', or 'not enough information' according to whether the code exhibits the property. Properties: Systematic, Linear, Perfect, MDS, Self-Dual. What is the minimum distance of this code?

(b) Suppose that this code is used in conjunction with BPSK modulation over the AWGN channel. After proper scaling and normalization, the incoming channel metrics are

$$\begin{aligned}
\overline{\mathrm{MI}}[c_0] &= +2 \\
\overline{\mathrm{MI}}[c_1] &= +4 \\
\overline{\mathrm{MI}}[c_2] &= +1 \\
\overline{\mathrm{MI}}[c_3] &= +3 \\
\overline{\mathrm{MI}}[c_4] &= +1 \\
\overline{\mathrm{MI}}[c_5] &= +5
\end{aligned}$$

Assuming that there is no a-priori information on the input bits, find the codeword decision that minimizes the probablity of codeword error: $\hat{\mathbf{c}}$.

For standard min-sum SISO processing, determine the following normalized extrinsic output metrics: $\overline{\mathrm{MO}}[b_0]$, $\overline{\mathrm{MO}}[c_0]$, $\overline{\mathrm{MO}}[b_2]$, $\overline{\mathrm{MO}}[c_2]$, $\overline{\mathrm{MO}}[c_5]$.

5.14. Consider a linear binary code with parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(a) Specify the input (k) and output (n) block sizes and the code rate r.

(b) Determine the minimum distance of this code. What does the Hamming bound on d_{min} say for these code parameters?

(c) List the possible syndromes and for each, list a coset leader. Suppose that this code is used over a binary symmetric channel with $\epsilon = 0.2$ and the received vector is

$$\mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Using standard array decoding based on the coset leaders specified above, provide the codeword decision $\hat{\mathbf{c}}$. Repeat this for

$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

5.15. Consider a linear binary code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This problem considers this code with an input bits b_i that are i.i.d., taking on 0 and 1 with equal probability.

(a) Determine the following regarding this code: k, n, r, $d_{\min}$, and $\mathbf{H}$.

(b) Consider the following observation from a BPSK-AWGN channel:

$$\mathbf{z} = \begin{pmatrix} +1 \\ +2 \\ -3 \\ -5 \\ +4 \\ +4 \\ -2 \end{pmatrix}$$

If this is processed by the receiver to convert to hard decisions on the coded bits without using the code structure, what will be the vector of corresponding binary observations ($\mathbf{y}$)?

Using the observation $\mathbf{y}$ perform hard-in decoding to minimize the codeword error probability and provide the best codeword.

Using the observation $\mathbf{z}$ perform soft-in decoding to minimize the codeword error probability and provide the best codeword:

(c) Using the same observation $\mathbf{z}$ from above, and taking $\overline{\mathrm{MI}}[c_j] = z_j$, compute the following normalized extrinsic messages using min-sum processing: $\overline{\mathrm{MO}}[c_0]$, $\overline{\mathrm{MO}}[b_0]$, and $\overline{\mathrm{MO}}[c_6]$.

5.16. Consider a set of n binary variables $\{d_j\}_{j=1}^{n}$ with $d_j \in \{0, 1\}$ that are constrained by a local switching circuit. This switching circuit allows the configuration with all of the d_j variables equal to 0 and it also allows all configurations with exactly one of the d_j variables equal to 1 and the rest set to 0.

(a) Determine the number of valid configurations, M. Specify the valid configurations when $n = 5$. This constraint defines a code; is this code linear? What is the minimum distance of this code?

(b) Given a set of incoming normalized metrics for the variables $\{\overline{\mathrm{MI}}[d_j]\}_{j=1}^{n}$, determine the outgoing, normalized extrinsic metric for d_i obtained with min-sum processing.

(c) Demonstrate the soft-in/soft-out rule derived above by processing the incoming metrics below to obtain the out-going metrics and the best hard decisions on the variables $\{d_j\}$ with $n = 5$. Fill in the tables below for

j :	1	2	3	4	5
$\overline{\mathrm{MI}}[d_j]$:	0.7	2	-3	4	1
$\overline{\mathrm{M0}}[d_j]$:					
$\hat{d}_j$:					

5.17. Consider the toy (5,3) code from the spreadsheet used to first explain SISO decoding. This code has generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(a) Determine a parity check matrix for this code.

(b) Give the weight distribution for this code $\{A_d\}$ for all values of d such that $A_d \neq 0$.

(c) Give the input-output weight distribution for this code $\{B_{w,d}\}$ for all values of input weight w and output weight d for which $B_{w,d}$ is nonzero.

(d) Give upper bounds on the average bit error probability and codeword error probability for ML-CW decoding over the BPSK-AWGN channel.

5.18. Consider a set of n binary variables $\{d_j\}_{j=1}^{n}$ with $d_j \in \{0, 1\}$ that are constrained to odd parity. Specifically, $\sum_{j=1}^{n} d_j = 1 \bmod 2$.

(a) Determine the number of valid configurations, M. Specify the valid configurations when $n = 4$:

(b) This constraint defines a code; determine the rate and the minimum distance this code. Is this code linear?

(c) Given a set of incoming normalized metrics for the variables $\{\overline{\mathrm{MI}}[d_j = 1]\}_{j=1}^{n}$, determine the outgoing, normalized extrinsic metric for d_i obtained with min-sum processing (simplify as much as possible).

(d) Demonstrate the soft-in/soft-out rule derived above by processing the incoming metrics below to obtain the out-going metrics and the best hard decisions on the variables $\{d_j\}$

with $n = 5$. Fill in the tables below for

j :	1	2	3	4	5
$\overline{\mathrm{MI}}[d_j = 1]$:	0.7	2	-3	-4	1
$\overline{\mathrm{MO}}[d_j = 1]$:					
$\hat{d}_j$:					

j :	1	2	3	4	5
$\overline{\mathrm{MI}}[d_j = 1]$:	1	3	2	0.1	0.1
$\overline{\mathrm{MO}}[d_j = 1]$:					
$\hat{d}_j$:					

5.19. This problem is based on the map of the continental US given in lecture.

(a) Run the Viterbi algorithm from the east coast to the west coast. Show your results in the same format shown in lecture.

(b) Identify the shortest path from the east coast to Newport, Oregon.

(c) Using the results of the forward and backward Viterbi algorithm runs, determine the milage of the shortest route that passes through each of the cities on the map (i.e., this is an MSM calculation).

(d) You want to visit friends in Salt Lake and Nashville during your trip. What is the best route? Can you describe a general algorithm approach for obtaining $\mathrm{MSM}[s_i, s_{i+D}]$ where D is a given integer?

5.20. The third generation digital cellular standard uses a parallel concatenated convolutional ("turbo") code based on an eight-state, rate 1/2, recursive systematic convoutional code which has generators

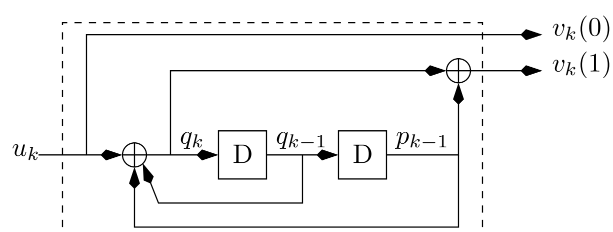
$$G(D) = \left[1, \frac{1 + D + D^3}{1 + D^2 + D^3}\right]$$

Specify an FSM next-state and output table for this code. Draw the state transition diagram and a trellis section for this code.

5.21. Consider the sample four-state RSC discussed in lecture – i.e., with

$$G(D) = \left[1, \frac{1 + D^2}{1 + D + D^2}\right]$$

and shown below.



where $v_k(0)$ and $v_k(1)$ are the systematic and parity channel bit sequences. Assume that the channel is a BPSK-AWGN channel with

$$z_k(0) = (-1)^{v_k(0)} + w_k(0) \quad (5)$$
$$z_k(1) = (-1)^{v_k(1)} + w_k(1) \quad (6)$$

where the two noise sequences are mutually independent and both are AWGN with variance 0.5 – i.e., $E_c/N_0 = 0$ dB. Consider the case where the encoder is started in $s_0 = (00)$ and 8 input bits are to be encoded – i.e., $\{b_k\}_{k=0}^{7}$. An additional two bits b_8 and b_9 are used by the encoder to force termination into the zero state. The following information is available from the source statistics and the channel:

$k =$	0	1	2	3	4	5	6	7	8	9
$z_k(0) =$	0.1	-0.2	$+0.4$	-1.1	-0.5	$+0.1$	$+0.1$	$+0.8$	-0.6	-0.3
$z_k(1) =$	-0.6	-0.8	$+1.4$	0	$+0.3$	$+0.6$	-0.5	$+0.8$	$+0.9$	$+0.6$
$p(b_k = 1) =$	0.7	0.4	0.5	0.2	0.7	0.1	0.4	0.6	–	–

Note that the values of b_8 b_9 are determined with probability given the state s_8.

(a) Convert the observations into hard decisions on the coded bits. Ignoring the a-priori probabilities (i.e., assume that input bits are equally likely), run the Viterbi algorithm. Document your results by showing the state metric values and the survivor sequences at each stage as well as identifying the decoded bit decisions.

(b) Convert the observation and a-priori probabilities into the proper normalized input metrics. Run the Viterbi algorithm and document it as above.

(c) Repeat, part b running the backwards Viterbi algorithm.

(d) Use the results of the previous two parts to determine $\mathrm{MSM}_0^9[b_k]$ $0 \le k \le 7$ and $\mathrm{MSM}_0^9[s_k]$ for $0 \le k \le 10$.

(e) Determine the second best path for this soft-in min-sum decoding. How much larger is the metric of this path and the best path?

(f) What is the metric of the shortest path consistent with $v_5(1) = 0$? What about $v_5(1) = 1$? Determine the normalized extrinsic output metric for $v_5(1)$ for the min-sum SISO decoder.

(g) Repeat the above SISO operations using min*-sum processing.

Note/Hint: You probably want to create a spreadsheet or program to compute these results. Future problems will require similar calculations to perform iterative decoding of a PCCC.

5.22. The goal of this problem is to develop some insight into survivor merging or the decay of message effects in min-sum processing. Consider a four-state trellis that is fully connected – i.e., each state at time k can be reached from each state at time $k-1$. For example, this can be a model double steps through the four state trellises considered in class. Consider a Viterbi-like algorithm that randomly selects survivors for each state at each time. Plot the probability that all survivors have merged in d stages vs. d. Do you expect merging in the VA to occur sooner or later than this algorithm?

Hint: Define $\mathbf{p}_d$ as the vector of probabilities that there are $1, 2, 3$, or 4 survivors d steps back – i.e., the component corresponding to 1 survivor is the probability of merging. Obtain a recursion on $\mathbf{p}_d$ using the combinatorics of the random selection algorithm.

5.23. Consider the 8-state recursive systematic convolutional code considered in a previous HW. Specifically, the code with generators

$$G(D) = \left[1, \frac{1 + D + D^3}{1 + D^2 + D^3}\right]$$

Determine the free distance of this code.

5.24. Draw the state diagram for the rate 1/2, 4-state feedforward convolutional code with generators (110) and (101). This code is catastrophic. Show that there are input sequences with an arbitrarily large number of ones that are mapped into an output sequence with a fixed number of ones.

5.25. In this problem you will consider decoding of a four-state convolutional code over the binary symmetric channel. The code has generator polynomials $G_1(D) = 1$ and $G_2(D) = (1 + D)/(1 + D + D^2)$ and maps a sequence of input bits b_i into two sequences of coded bits $c_i^{(1)}$ and $c_i^{(2)}$. The input bits are independent and each has equal probability of being 0 or 1.

The input information bit sequence is

$$(b_0\ b_1\ b_2\ b_3) = (1010)$$

and the encoder is started in the zero state.

(a) Draw one section of the trellis diagram for this convolutional code, being sure to clearly indicate all the values of the input and output bits associated with each transition.

(b) Determine the encoded sequences and the state of the encoder after b_3 has been encoded: Two tail-bits b_4 and b_5 are to be used to terminate into the zero state – i.e., $s_6 = 0$. Determine the required value of these two tail-bits and the associated coded bits.

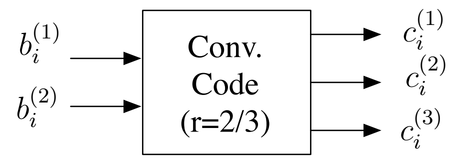
(c) Consider the case where the coded bits are sent through a binary symmetric channel. The observations from this channel are $\{z_i^{(1)}\}_{i=0}^5$ and $\{z_i^{(2)}\}_{i=0}^5$, which are either 0 or 1. Consider the case where there are three channel errors:

$$z_2^{(2)} \neq c_2^{(2)}, \quad z_3^{(1)} \neq c_3^{(1)}, \quad z_3^{(2)} \neq c_3^{(2)}$$

Except for these three, $z_i^{(j)} = c_i^{(j)}$ for $j = 1, 2$ and $i \in \{0, \dots 5\}$.
Run the Viterbi algorithm to decode this code with this set of observations. Document this by showing the survivor sequences and forward state metrics for each state at each time.

(d) If instead of the above errors, the channel caused two errors – i.e., $z_i^{(j)} \neq c_i^{(j)}$ for exactly two variables – would this Viterbi decoder correct them? Prove your assertion.

5.26. Consider a rate $r = 2/3$ convolutional code that takes in two bits and outputs 3 bits at every time as shown below



The input output relations are

$$
\begin{aligned}
c_i^{(1)} &= b_i^{(1)} \oplus b_{i-1}^{(2)} \\
c_i^{(2)} &= b_i^{(2)} \oplus b_{i-1}^{(2)} \\
c_i^{(3)} &= b_i^{(1)} \oplus b_i^{(2)} \oplus b_{i-1}^{(2)}
\end{aligned}
$$

(a) Show that this convolutional code can be represented by a two-state FSM. Show this by defining the state variable s_i and showing all allowable state transitions. Show each transition and label each with the values of the variables as indicated below.

Label with: $(b_i^{(1)}, b_i^{(2)}, c_i^{(1)}, c_i^{(2)}, c_i^{(3)})$

(b) Determine the free distance of this code.

(c) Consider the case when 6 input bits $b_0^{(1)}, b_0^{(2)}, b_1^{(1)}, b_1^{(2)}, b_2^{(1)}, b_2^{(2)}$ are encoded, starting from $s_0 = 0$, without any termination of the endocer. Let $z_i^{(m)}$ be the output of a BPSK-AWGN channel when the input is $c_i^{(m)}$ and consider the following set of observations:

i	0	1	2
$z_i^{(1)}$	-3	$+1$	-2
$z_i^{(2)}$	-4	-2	-2
$z_i^{(3)}$	-1	$+5$	$+2$

Consider min-sum decoding, taking these channel measurements as the incoming channel metrics and assuming that the inputs are independent, and have uniform a-priori probability distribution over $\{0,1\}$. Determine the forward and backward state metrics $\{\mathrm{F}_i[s_1 = j]\}$ for $i = 0,1$ and $j = 0,1$ and $\{\mathrm{B}_i[s_1 = j]\}$ for $i = 1,2$ and $j = 0,1$. Determine the normalized, extrinsic output metrics for each of the variables at time $i = 1$.

5.27. Consider a two state convolutional encoder with two input bits b_0, b_1, output bits $d_i^{(1)}$ and $d_i^{(2)}$ generated by generator polynomials $G_1(D) = 1$ and $G_2(D) = 1 + D$, respectively. After the two input bits are encoded, the encoder is driven to the zero state with a single tail bit. The bits $d_0^{(2)}$ and $d_2^{(1)}$ are punctured.

(a) Draw the full trellis diagram for this terminated convolutional code, labeling each state transition with input bit and output bit(s).

Specify the k, n, r, and $d_{\min}$ for this code.

(b) Specify a parity check matrix for this code and specify a corresponding standard decoding array, including the syndromes.

(c) Assume that independent, random input bits are encoded and the codeword is transmitted through a binary symmetric channel with $\epsilon < 1/2$ and that the received binary vector is

$$\mathbf{y}^{\mathrm{t}} = (1\ 1\ 1\ 1)$$

Use the standard array to perform ML codeword decoder. Specify your result by giving the syndrome, the corresponding coset leader and the codeword decision.

Repeat this hard-in decoding using the Viterbi algorithm running on the trellis from part (a). Report your results by drawing the trellis from part (a) with survivors and state metrics. Illustrate the best path and report the decoded bit decisions

(d) Using the parity check matrix from (b), specify the parity check trellis for this code. Is this trellis related to the trellis from part (a)? Explain.

5.28. Consider using a two-state convolutional code with generators $G_1(D) = G_2(D) = 1 + D$ to encode 100 information bits. What is the minimum distance of this code if the encoder is started and terminated into the zero state ($d_{\min,0}$)? What is the minimum distance of this code when tail-biting termination is used ($d_{\min,\mathrm{tb}}$)?

5.29. Consider a rate one, convolutional code with generator $g(D) = 1/(1 + D)$.

(a) Determine the associated difference equation and sketch an encoder block diagram.

(b) Draw and label the trellis diagram.

(c) Describe how one can terminate the encoder into the zero state.

(d) Let b_i be the input sequence and c_i be the output sequence. Let $\overline{\mathrm{MI}}[b_i]$ and $\overline{\mathrm{MI}}[c_i]$ be the normalized input metric on these variables. Also, define $\overline{\mathrm{F}}_{i-1}[s_i]$ and $\overline{\mathrm{B}}_{i+1}[s_{i+1}]$ be the forward and backward state metrics in normalized form (*e.g.*, so that $\overline{\mathrm{F}}_{i-1}[s_i = 0] = 0$). Show that the forward state recursion, backward state recursion and completion operations can each be carried out with one call of the function

$$g(x, y) = \min(x, y) - \min(0, x + y) = \min(|x|, |y|)\mathrm{sgn}(x)\mathrm{sgn}(y)$$

(e) How many additions are required to process one trellis stage as above?

(f) This code is called an accumulator and may be viewed as outputting the running parity check over all previous inputs. Consider this code is terminated into the zero state and discuss the relationship between this code and the SPC code and SISO considered in problem 11.

(g) Consider the convolutional code with generator $g(D) = 1 + D$. Show that the SISO processor for the above $1/(1 + D)$ code can used to carry out the SISO processing for this $(1 + D)$ code with a change in the input message ports used.

5.30. This problem concerns codes that contain the following three codewords

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

(a) Consider the *smallest linear code* that contains these codewords, $\mathcal{C}_L$. Specify all the codewords in this code excluding the three given above.

(b) Specify the following standard parameters of this code: k, n, r, $d_{\min}$, and t_c.

(c) Provide valid generator and parity check matrices for $\mathcal{C}_L$ from the linear code defined above.

(d) Consider the *smallest code* (*i.e.,* not necessarily linear) that contains the three codewords, $\mathcal{C}$. Specify all the codewords in this code excluding the three given above.

(e) Specify the following standard parameters of this code: k, n, r, $d_{\min}$, and t_c.

5.31. Consider a set of n binary variables $\{d_j\}_{j=1}^n$ with $d_j \in \{0, 1\}$ that are constrained by a local switching circuit. This switching circuit allows only configurations with exactly two of the variables set to 1 (others must be 0). Assume that $n \geq 3$.

(a) Determine the number of valid configurations M.

(b) Specify the valid configurations when $n = 5$:

(c) This constraint defines a code. Is this code linear? What is $d_{\min}$ for this code?

(d) Given a set of incoming normalized metrics for the variables $\{\overline{\mathrm{MI}}[d_j]\}_{j=1}^n$, determine the outgoing, normalized extrinsic metric for d_i obtained with min-sum processing (simplify as much as possible).

(e) Demonstrate the soft-in/soft-out rule derived above by processing the incoming metrics below to obtain the out-going metrics and the best hard decisions on the variables $\{d_j\}$ with $n = 5$. Fill in the tables below for

j :	1	2	3	4	5
$\overline{\mathrm{MI}}[d_j]$:	0.7	2	-3	-4	1
$\overline{\mathrm{M0}}[d_j]$:					
$\hat{d}_j$:					

j :	1	2	3	4	5
$\overline{\mathrm{MI}}[d_j]$:	1	3	2	0.1	0.1
$\overline{\mathrm{M0}}[d_j]$:					
$\hat{d}_j$:					

5.32. Consider an additive noise channel with real-valued observation

$$z_j(u) = c_j(u) + w_j(u)$$

where $c_j(u) \in \{0,1\}$ is the j^{th} coded bit, and $w_j(u)$ is an i.i.d. sequence of exponentially distributed random variables with probability density function (pdf) given by

$$f_{w_j(u)}(w) = \lambda e^{-\lambda w}\mathrm{u}(w) = \begin{cases} \lambda e^{-\lambda w} & w \geq 0 \\ 0 & w < 0 \end{cases}$$

where $\lambda > 0$ is a constant characterizing the noise power and $\mathrm{u}(w)$ is the unit step function. This problem considers both soft-in/hard-out and hard-in/hard-out decoding on this channel.

(a) Determine and provide a labeled sketch of the conditional pdf of $z_j(u)$ given $c_j(u)$.

(b) It can be shown that if one is to make hard decisions on the bits c_j without taking into consideration the structure of the code (*i.e.,* convert soft-decision information into hard decisions information), the best rule is

$$y_j = \begin{cases} 1 & z_j \geq 1 \\ 0 & z_j < 1 \end{cases}$$

Given this rule for conversion from $\{z_j\}$ to $\{y_j\}$, the channel mapping $\{c_j\}$ to $\{y_j\}$ is a discrete memoryless channel (DMC). Determine the transition probabilities $p_{y_j(u)|c_j(u)}(y|c)$ for all values of y and c (you may choose to present this in tabular format).

Provide a labeled sketch of this DMC (*i.e.,* the channel transition diagram used for DMCs):

(c) Consider performing maximum likelihood codeword decoding (assuming a purely random source of information bits) on this channel. Determine the normalized channel metrics for each of the two values of y_j:

(d) Give the simplified ML-CW decoding rule for this channel based on the observations $\{y_j\}$. You may describe this using words and mathematical formulas as you see fit to make your answer most clear.

(e) Demonstrate your understanding of the above decoding rule by decoding the (5,3) code from the toy-SISO spreadsheet. The 8 codewords are shown below for reference:

$$\mathbf{c}^{(0)} = (0\ 0\ 0\ 0\ 0)$$
$$\mathbf{c}^{(1)} = (0\ 0\ 1\ 0\ 1)$$
$$\mathbf{c}^{(2)} = (0\ 1\ 0\ 1\ 1)$$
$$\mathbf{c}^{(3)} = (0\ 1\ 1\ 1\ 0)$$
$$\mathbf{c}^{(4)} = (1\ 0\ 0\ 1\ 0)$$
$$\mathbf{c}^{(5)} = (1\ 0\ 1\ 1\ 1)$$
$$\mathbf{c}^{(6)} = (1\ 1\ 0\ 0\ 1)$$
$$\mathbf{c}^{(7)} = (1\ 1\ 1\ 0\ 0)$$

For this code, decode each of the following observation vectors $\mathbf{y}$ using the rule derived (you may specify the decision as $\mathbf{c}^{(m)}$ with m numerical).

$$\mathbf{y}^{\mathrm{t}} = (0\ 0\ 1\ 1\ 1)$$
$$\mathbf{y}^{\mathrm{t}} = (1\ 1\ 1\ 1\ 1)$$

(f) Now consider soft-in decoding on this channel – *i.e.,* using $\{z_j\}$ directly to decode. Determine the normalized channel metrics as a function of z_j

(g) Give the simplified ML-CW decoding rule for this channel based on the observations $\{z_j\}$. You may describe this using words and mathematical formulas as you see fit to make your answer most clear.

(h) For the (5,3) code given in (c), decode each of the following observation vectors $\mathbf{z}$ using the rule derived (you may specify the decision as $\mathbf{c}^{(m)}$ with m numerical).

$$\mathbf{z}^{\mathrm{t}} = (0.7\ \ 0.9\ \ 1.2\ \ 12\ \ 4)$$
$$\mathbf{z}^{\mathrm{t}} = (1.5\ \ 2.1\ \ 1.04\ \ 2.1\ \ 3.3)$$

(i) For the BPSK-AWGN channel, we saw that soft-in decoding provides approximately 0.2 dB of additional coding gain relative to hard-in decoding. How much additional coding gain would you expect on this channel when soft-in decoding is used over hard-in decoding? EXPLAIN.

5.33. In his notes, Prof. Kumar uses a $(n, k, d_{\min}) = (6, 3, 3)$ code with the following parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

In this problem you will use a parity check trellis for this code in order to perform ML codeword decoding on the BPSK-AWGN channel.

(a) Draw and label a trellis diagram for this code. You may use a line style convention to indicate data values (*e.g.,* dashed and solid).

(b) Let the BPSK-AWGN channel observation for bit c_j be z_j. Consider the set of observations

j	0	1	2	3	4	5
z_j	-4	$+2$	-6	-4	$+1$	-5

Show the results of running the Viterbi algorithm using these observations by redrawing the trellis and showing the survivor sequences and the forward state metrics at each stage. Illustrate the best path by boxing the corresponding transitions.

Summarize your results by giving the decisions on c_j and the information bits b_i by filling out the table below with a 0 or 1 in each position.

j	0	1	2	3	4	5
$\hat{c}_j$						
$\hat{b}_j$				•	•	•

5.34. Consider the binary linear code with parity check matrix

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Determine the following parameters of this code: n, k, r and give a valid generator matrix for this code.

(b) Draw the *unterminated* parity check trellis for the code, labeling the syndromes corresponding to states.

(c) Draw the *terminated* parity check trellis for the code (*i.e.,* all paths are valid codewords).

(d) Using the parity check trellis from part (c), determine the ML codeword decision for the BSC channel by running the Viterbi algorithm when the channel observation is:

$$\mathbf{y}^{\mathrm{t}} = (\ 0\ 0\ 0\ 0\ 1\ 1)$$

Show the results on the trellis – *i.e.,* show the survivor state metric for each state at each time – and give the resulting codeword decision.

(e) Using the unterminated parity check trellis from part (b), determine the coset leaders for this code by running the Viterbi algorithm. Show the results on the trellis – *i.e.,* show the survivor state metric for each each state at each time. Give the coset leaders in the order of the states on the trellis diagram:

(f) What is $d_{\min}$ for this code?

(g) For the channel observation from part (d), specifically,

$$\mathbf{y}^{\mathrm{t}} = (\ 0\ 0\ 0\ 0\ 1\ 1)$$

what is the corresponding syndrome and coset leader? Is this consistent with result obtained with the Viterbi decoder from part (d)? Explain.

5.35. Consider the rate 1/2 convolutional code with generator polynomials given by $G^{(1)}(D) = 1 + D + D^2 + D^3$ and $G^{(2)}(D) = 1 + D + D^3$. Diagram an encoder for this code using delay elements and mod-2 adders. Diagram one stage of the trellis diagram for this code. Use dashed lines to indicate a 0 input and solid lines for a 1 input. Label each transition with two output bits.

5.36. Consider the rate 1/2 convolutional code with generator polynomials $G^{(1)}(D) = 1 + D^2$ and $G^{(2)}(D) = 1 + D + D^2$. This code is punctured in the following way: for odd times i, only $c_i^{(2)}$ is output – *i.e.*, , $c_i^{(1)}$ is punctured for odd i. Both $c_i^{(1)}$ and $c_i^{(2)}$ are sent for even i. Determine the rate of this code and find the free distance.

5.37. Consider the binary linear block code obtained using a terminated convolutional code with generator polynomials $G^{(1)}(D) = 1 + D^2$ and $G^{(2)}(D) = 1 + D + D^2$. Specifically, let b_0, b_1 and b_2 be input bits to this encoder and terminate the encoder into the zero state using tail bits. The three input bits are independent and each is equally likely to be 0 or 1.

(a) Determine the following parameters of this code: n, k, r, and the generator matrix.

(b) Draw a full trellis diagram for this code. Label the transitions with the output bits and use dashed lines for zero input and solid for one input.

(c) Determine the minimum distance of this code.

(d) The coded bits are sent through a BPSK-AWGN channel and the channel observations are

j	0	1	2	3	4
$z_j^{(1)}$	2	-3	4	2	0
$z_j^{(2)}$	-5	-1	1	-1	1

Use these channel measurements to run the min-sum version of the forward-backward algorithm. Document your results by drawing the code trellis with the forward state metrics and backward state metrics for each time and each state clearly labeled.

(e) Using the state metrics from the previous part, determine the normalized MSM for each of the information bits and the corresponding best hard decisions.

5.38. Consider the linear block code defined by the following generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Specify the n, k, and r for this code.

(b) Specify a valid parity check matrix for this code.

(c) What is the weight enumerating function of this code, $A(D)$?

(d) What is the minimum distance of this code?

(e) Now consider the dual of this code and specify the following parameters of the dual code: $n^{\perp}$, $k^{\perp}$, $r^{\perp}$, and $A^{\perp}(D)$.

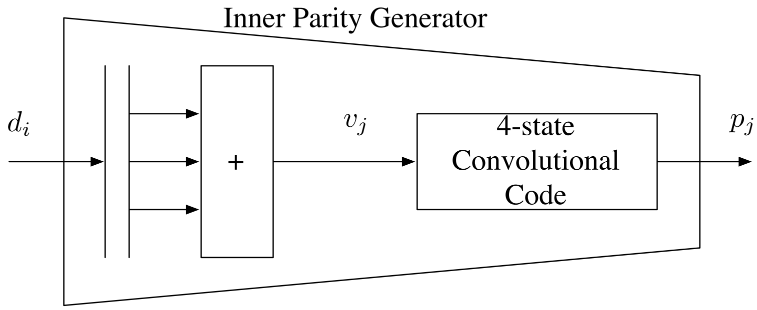


Figure 25: A parity generator used in a modern code.

5.39. The encoder in Fig. 25 is used as a parity generator (PG) in a modern code.

(a) The generator for the convolutional code is $G(D) = 1/(1 + D + D^2)$. Provide a block diagram of an encoder for this convolutional code.

(b) There are 9 bits $\{d_0, d_1, \ldots d_8\}$ at the input to the $J = 3$ mod-2 block summer resulting in 3 input bits to the convolutional code $\{v_0, v_1, v_2\}$. The convolutional code is terminated into the zero state using 2 tail bits. This results in 5 output bits $\{p_0, p_1, p_2, p_3, p_4\}$. The channel metrics on p_j are

$$\overline{\mathrm{MI}}[p_0 = 1] = +2$$
$$\overline{\mathrm{MI}}[p_1 = 1] = +1$$
$$\overline{\mathrm{MI}}[p_2 = 1] = +4$$
$$\overline{\mathrm{MI}}[p_3 = 1] = -1$$
$$\overline{\mathrm{MI}}[p_4 = 1] = -3$$

In addition to these channel metrics, an outer code SISO provides normalized input metrics on d_i. Using the above channel metrics and the values of $\overline{\mathrm{MI}}[d_i = 1]$ shown below, perform the locally optimal SISO processing for the PG and determine the normalized extrinsic output metrics for d_j.

i :	0	1	2	3	4	5	6	7	8
$\overline{\mathrm{MI}}[d_i = 1]$:	$+1$	-2	-3	$+4$	$+1$	$+6$	-2	0	$+4$
$\overline{\mathrm{M0}}[d_i = 1]$:									

5.40. A rate 1/2 convolutional code has generators

$$G^{(1)}(D) = 1 + D + D^2 + D^3$$
$$G^{(2)}(D) = 1 + D + D^3$$

(a) Provide a block diagram of an encoder for this code and diagram one stage of the trellis for this code; use dashed lines to indicate a 0 input and solid for a 1 input and label each transition with $c_i^{(1)}, c_i^{(2)}$.

(b) Find the free distance of this code and provide and input/output sequence pair realizing this free distance (*i.e.,* give an output sequence with weigh d_{free} and its corresponding input sequence) – include your reasoning.

(c) Consider terminating this convolutional code by driving the encoder to the zero state. The resulting code can be viewed as a linear block code with k input bits, n output bits, and minimum Hamming distance $d_{\min}$. Determine n and $d_{\min}$ as a function of k for $k \geq 1$.

Consider the specific case of $k = 2$. Specify the following:

- n, $d_{\min}$, t_c
- Number of Coset Leaders defined by t_c value.
- Does a linear code exists with this same n, k, with larger $d_{\min}$? Explain.

6 Modern Coding

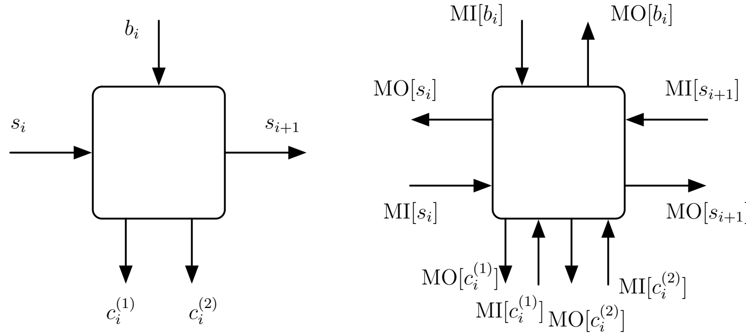
6.1. Consider the random variable $Y(u) = (-1)^{c_i(u)} \frac{4\sqrt{E_c}}{N_0} z_i(u)$. This random variable may be viewed as a measure of confidence for the correct bit value when the observation model

$$z_i(u) = (-1)^{c_i(u)} \sqrt{E_c} + w_i(u)$$

is assumed with $w_i(u)$ being a Gaussian random variable with mean zero and variance $N_0/2$. Show that $Y(u)$ is a Gaussian random variable and find the mean and variance of $Y(u)$. Express the variance as a function of the mean. This simple relation is called the *symmetry condition* for Gaussian channel negative log-likelihood ratios.

6.2. Consider one stage of an FSM trellis as a simple code constraint. Namely, the inputs are the input information bit, b_i, the current state, s_i, and the outputs are the coded bits $\{c_i^{(l)}\}_l$, and the next state, s_{i+1}. For a specific example, consider the rate 1/2, four state feed-forward code considered in lecture (i.e., with generators (101) and (111)). The objective of this problem is to show that the forward-backward algorithm may be viewed as activating locally optimal SISO processing nodes corresponding to these simple code constraints. This notion is shown in the figure below.

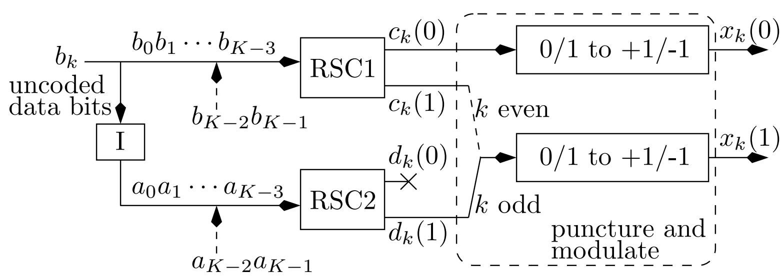
How many valid configurations does this simple code constraint have? List this valid configurations, with the associated values of all input and output variables.



Describe the standard (min-sum) SISO processing for this simple code constraint. Show that the following correspondences hold:

- MI[s_i] $\longrightarrow$ MO[s_{i+1}] is the forward state metric recursion.
- MI[s_{i+1}] $\longrightarrow$ MO[s_i] is the backward state metric recursion.
- MO[b_i] computation is completion on the input bit.
- MO[$c_i^{(l)}$] computation is completion on the coded bit.

6.3. Consider a PCCC as shown in the figure below where the recursive systematic encoders are each the 4-state convolutional code used in HW4, problem 3 – i.e., $c_k(0)$ and $d_k(0)$ are systematic bits and $c_k(1)$ and $d_k(1)$ are generated by $G(D) = (1 + D^2)/(1 + D + D^2)$.



Consider an input block size of 8 bits and consider encoding the bit sequence

$$(b_0\ b_1 \cdots b_7) = (1\ 0\ 0\ 1\ 1\ 1\ 0\ 0)$$

with the interleaver given by

k	0	1	2	3	4	5	6	7
$I(k)$	2	6	3	0	4	1	7	5

Specifically, $a_{I(k)} = b_k$. For example, $a_6 = b_1$.

(a) Determine the input sequence to the bottom RSC, $\mathbf{a}$.

(b) Determine the two tail bits required to terminate each encoder into the zero state.

(c) Determine the two parity bit sequences $\{c_k(1)\}_{k=0}^{9}$ and $\{d_k(1)\}_{k=0}^{9}$.

(d) Determine the sequences $\{x_k(0)\}_{k=0}^{9}$ and $\{x_k(1)\}_{k=0}^{9}$.

(e) What is the rate of this code, including the effects of the tail? What would the rate be if the input block size was 2048?

(f) Perform 3 iterations of the min-sum iterative decoder for this encoded sequence realization. The noise sequence realization is:

k	0	1	2	3	4	5	6	7	8	9
$w_k(0)$	-0.2	$+0.3$	-1.2	-0.5	-0.1	$+1.2$	$+0.4$	$+0.5$	-0.3	$+0.1$
$w_k(1)$	$+0.3$	$+1.3$	$+0.5$	-1.4	$+1.2$	$+0.8$	-0.4	-0.6	$+1.6$	-0.5

for the observation model is $z_k(i) = \pm 1 + w_k(i)$, with $E_c/N_0 = 1$ (0 dB).

Define one iteration as an activation of SISO1, the interleaver, SISO2, and the deinterleaver. At the end of each iteration, give the best hard decisions for $\{b_k\}_{k=0}^{7}$ available, as well as the associated soft decision information. How many errors are there in the final decisions?

6.4. Using the results of Problems 1.8 and 1.**??**, consider iterative decoding of the (7,4) Hamming code. More specifically,

(a) Draw the parity check graph representing this code.

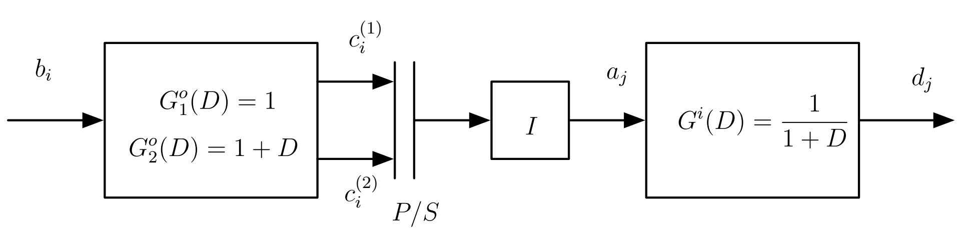
(b) Using the observation values $\mathbf{z}_A$ from Problem 7, perform 4 iterations of the iterative decoder based on this parity check graph. At each iteration, compute the best soft decision information for the coded bits – i.e., $\overline{\mathrm{MI}}[c_j] + \overline{\mathrm{MO}}[c_j]$ – and the corresponding implied hard decisions. How close are these hard and soft decisions to the optimal values computed in Problem 7?

6.5. Repeat Problem 4 using the graphical model associated with the systematic generator matrix. Why is it necessary to use the systematic generator form?

6.6. Verify equations (5)-(7) in PiHu01 for the for the forward, backward, and completion operations of the min-sum SISO for the recursive single parity check code (RSPC)

6.7. Consider a repetition code constraint with 4 associated binary variables $\{d_i\}_{i=0}^{3}$. Consider the incoming normalized metrics $\overline{\mathrm{MI}}[d_i]$ and assume that these are well-modeled by negative log-likelihood ratios for an AWGN channel (i.e., assume they are Gaussian and satisfy the symmetry condition). If the mean of $(-1)^{d_i}\overline{\mathrm{MI}}[d_i]$ is m_i, and these are assumed to be mutually independent, determine the distribution of $(-1)^{d_i}\overline{\mathrm{MO}}[d_i]$.

6.8. Consider the concatenated convolutional code shown below:



The input bits are independent and each has equal probability of being 0 or 1.

(a) Draw one section of the trellis diagram for the outer and inner codes. Be sure to clearly indicate all the values of the input and output bits associated with each transition.

(b) What is the minimum distance of the outer code?

Does uniform interleaver analysis suggest that this code will provide interleaver gain for bit error probability performance? Specify the maximum exponent of n for this case.

(c) In this part, you will execute the min-sum version of the soft-in/soft-out decoder for the outer code. Specifically, assume that the input block size is 4 bits, and that the outer encoder is started in state 0 and is not terminated. At the particular iteration under consideration, the incoming metrics provided to the outer SISO are

i :	0	1	2	3
$\overline{\mathrm{MI}}[c_i^{(1)}]$	$+1$	-4	$+2$	$+3$
$\overline{\mathrm{MI}}[c_i^{(2)}]$	$+1$	$+1$	-2	$+1$

These incoming metrics are in normalized extrinsic form, so that the metric associated with the zero conditional value is 0 and the numbers above are for the one conditional values.

Run the forward-backward algorithm based on your trellis description above to obtain the normalized extrinsic output metrics for $\{b_i\}$, $\{c_i^{(1)}\}$ and $\{c_i^{(2)}\}$ for $i = 0, 1, 2, 3$ and the best decisions for the information bits after this SISO activation. Fill in the table below with these values:

i :	0	1	2	3
$\overline{\mathrm{MO}}[c_i^{(1)}]$				
$\overline{\mathrm{MO}}[c_i^{(2)}]$				
$\overline{\mathrm{MO}}[b_i]$				
$\hat{b}_i$				

6.9. During this class, we repeatedly used the fact that the min-sum SISO for an SPC constraint can be computed using the function

$$g(x, y) = \min\{|x|, |y|\}\mathrm{sgn}(x)\mathrm{sgn}(y)$$

Specifically, for $c_0, c_1, \ldots c_{n-1}$, constrained to satisfy an even SPC, we found that

$$\overline{\mathrm{MO}}[c_i] = g\left(\overline{\mathrm{MI}}[c_0], \ldots \overline{\mathrm{MI}}[c_{i-1}], \overline{\mathrm{MI}}[c_{i+1}], \ldots \overline{\mathrm{MI}}[c_{n-1}]\right)$$

with $g(\cdot)$ defined for multiple arguments using the recursion

$$g(x, y, z) = g(g(x, y), z)$$

and $\overline{\mathrm{MI}}[c_j]$ being the incoming normalized, extrinsic metrics.

The goal of this problem is to express the min*-sum SISO for this SPC constraint in a related form.

(a) The min*-sum SISO for this SPC constraint can be computed as above where $g(x, y)$ is replaced by

$$g^*(x, y) = g(x, y) + \Delta(x, y)$$

Determine the "correction factor" function $\Delta(x, y)$ and show your derivation of this result.

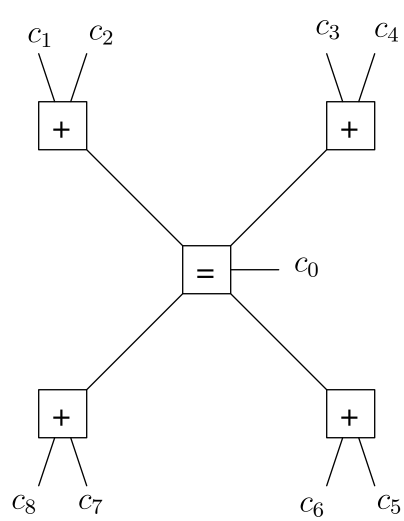
(b) Demonstrate the operation of this min*-sum SISO calculation by considering 4 bits, $\{c_j\}_{j=0}^{3}$ constrained by an SPC with

$$\overline{\mathrm{MI}}[c_0 = 1] = +3, \quad \overline{\mathrm{MI}}[c_1 = 1] = -2, \quad \overline{\mathrm{MI}}[c_2 = 1] = -0.5, \quad \overline{\mathrm{MI}}[c_3 = 1] = -1.5$$

Specifically, determine the normalized ($\overline{\mathrm{MO}}[c_j = 0] = 0$) output metrics for min*-sum SISO decoding.

(c) Compare this result to that obtained using min-sum processing.

6.10. Consider a linear binary code defined by the ("star") graphical model shown below:



Here, lines represent variables and boxes represent constraints. The two types of boxes are an even parity constraint and an equality constraint. The graph shows the constraints for the codeword bits. The input bits are independent and each has equal a-priori probability of being 0 or 1.

(a) Determine the rate of this code, a parity check matrix $\mathbf{H}$, the number of states in the associated parity check trellis S, and the minimum distance of this code.

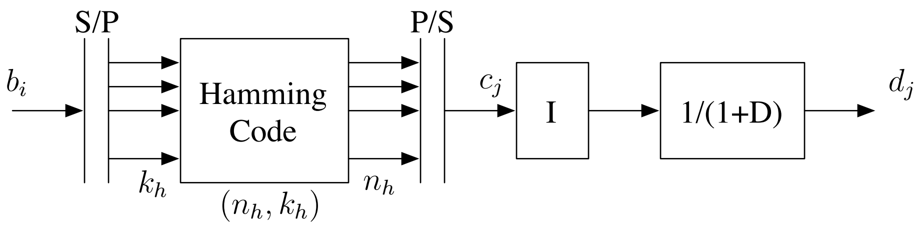
(b) A codeword from this code has been transmitted over the BPSK-AWGN channel using the standard convention and the corresponding observations are:

z_0	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
-3	$+5$	-1	$+1$	-2	-5	-4	$+2$	-2

It is desired to compute the ML codeword decision. Use message-passing decoding on the graph provided to obtain this result. Document your work by showing the stable set of messages at convergence and the final hard decisions implied – i.e., show each of these messages near the corresponding arrow.

(c) Consider performing hard-in decoding of this code. First, convert the set of observations $\{z_i\}$ to the corresponding hard channel decisions $\{y_i\}$, where each $y_i \in \{0, 1\}$. Is $\mathbf{y}$ a valid codeword? Explain. Perform minimum Hamming distance decoding of this code using message-passing on the graph provided. Document your results by labeling the graph with the stable set of messages and providing the final hard decisions.

6.11. Consider the concatenated code shown below:

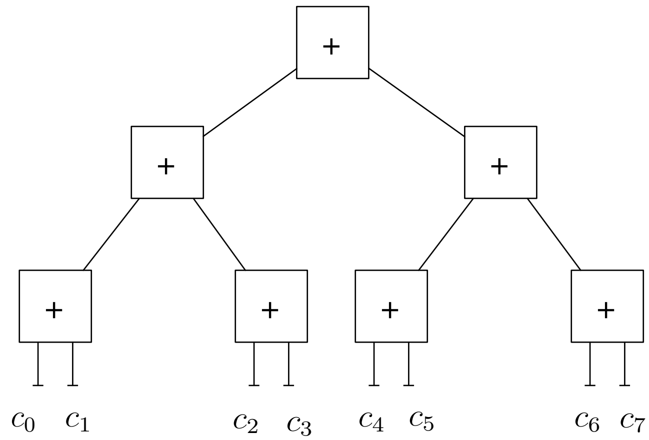


The k input bits are encoded first, in block by block manner, by a Hamming code. These coded bits are then interleaved and put through an accumulator, that is not terminated, to produce the n coded bits d_j. The Hamming code takes in k_h bits and outputs n_h output bits.

(a) Answerthe following:
- Will interleaver gain in P_{CW} be achieved? Explain.
- What is the size of interleaver (function of k, k_h, n_h)
- What is the overall code rate (function of k, k_h, n_h)

(b) If an overall code rate of $r \geq 0.93$ is sought, then what is the smallest value of k_h, $k_{h,\min}$, possible? Consider the code with this minimum value of k_h. The soft-in/soft processing for the outer code is implemented using the forward-backward algorithm on the parity check trellis of each Hamming block. Determine the following for the Hamming code parity check trellis:
- State complexity of the trellis.
- Number of trellis sections per Hamming block.

6.12. Consider the code defined by the graph below.



Describe the global code structure by specifying the parity check matrix. Consider the following incoming metrics for $\{c_j\}$

$\overline{\text{MI}}[c_0]$	$\overline{\text{MI}}[c_1]$	$\overline{\text{MI}}[c_2]$	$\overline{\text{MI}}[c_3]$	$\overline{\text{MI}}[c_4]$	$\overline{\text{MI}}[c_5]$	$\overline{\text{MI}}[c_6]$	$\overline{\text{MI}}[c_7]$
-4	-2	-9	$+2$	-3	$+1$	-6	$+2$

Label the diagram below with the stable set of messages associated with min-sum processing.

6.13. Consider a code constructed by adding a single parity check bit to each row and each column of an $(m \times m)$ array of input bits. Specifically, the code structure is diagrammed in the table below

b_0	b_1	$\cdots$	b_{m-1}	p_0
b_m	b_{m+1}	$\cdots$	b_{2m-1}	p_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$b_{m(m-1)}$	$b_{m(m-1)+1}$	$\cdots$	b_{m^2-1}	p_{m-1}
q_0	q_1	$\cdots$	q_{m-1}	$\cdot$

where p_i is the parity over row i and q_j is the parity over column j

$$p_i = b_{im} + b_{im+1} + \cdots + b_{(i+1)m-1}$$
$$q_j = b_j + b_{j+m} + \cdots + b_{j+m(m-1)}$$

where all addition is modulo 2.

All information bits, row parity bits and column parity bits are transmitted as the codeword.

(a) Find k, n, r, and $d_{\min}$ as a function of m.

(b) Specify a cyclic graphical model for this code that comprises a set of equality constraints and single parity check constraints. Describe a soft-in decoder associated with graphical model. Specify an activation schedule, define one iteration, and state whether the decoder is optimal or not. Your activation schedule will need to start with the equality constraints. Do not count the initial activation of these equality constraints as part of the first iteration.

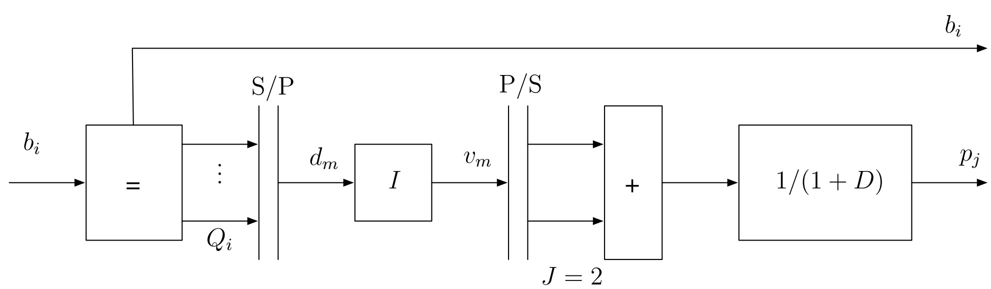
(c) Consider the specific case of $m = 3$. The values in the table below are the channel likelihood values for each codeword bit.

$+2$	$+2$	-1	$+3$
-4	-1	$+3$	$+3$
-1	-1	$+2$	-3
-2	-6	$+5$	$\cdot$

Run your min-sum decoder for two iterations and specify the results for each iteration by filling in tables with the best soft decision value for each of the 9 information bits after that iteration.

(d) Show that this code can be viewed as an LDPC code. Specify the parity check matrix for a general value of m. Clearly label the matrix so that the dimensions and the locations of nonzero elements is clear. Specify the degree distribution for the check nodes and variables nodes. Specifically, give all values for the check degree ($d_c(i)$) along with the fraction of check nodes with each of these values. Report the variable degree distribution similarly.

6.14. Consider a systematic repeat accumulate code (as shown below) with 4 input bits b_0, b_1, b_2, and b_3. There are Q_i repetitions of bit b_i fed into the interleaver where $Q_i = 2$ for even i and $Q_i = 4$ for odd i. The inner accumulator is unterminated.



The interleaver is listed below:

$m =$	0	1	2	3	4	5	6	7	8	9	10	11
$I(m) =$	6	1	3	8	10	4	2	11	9	0	5	7

Note the convention is $v_m = d_{I(m)}$.

(a) Determine the output blocksize of this code, the rate, and the parity check matrix.

(b) Draw a graphical model for this code corresponding to the encoder shown above. Label all variables.

Draw the parity check graph for this code corresponding to the parity check matrix given above. Label all variables.

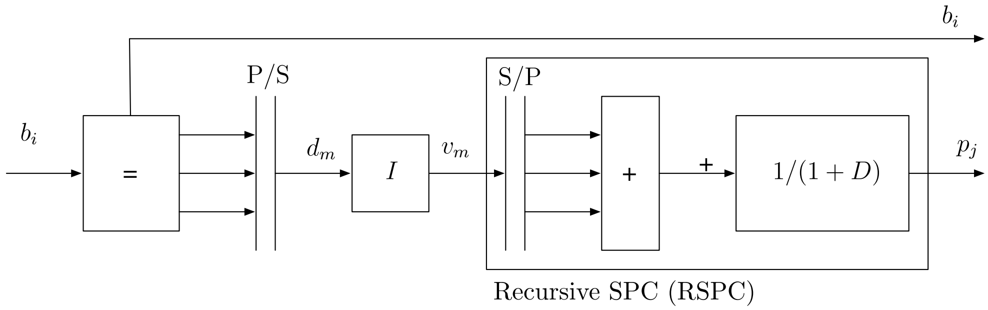


Figure 26: The encoder for the systematic RA code considered.

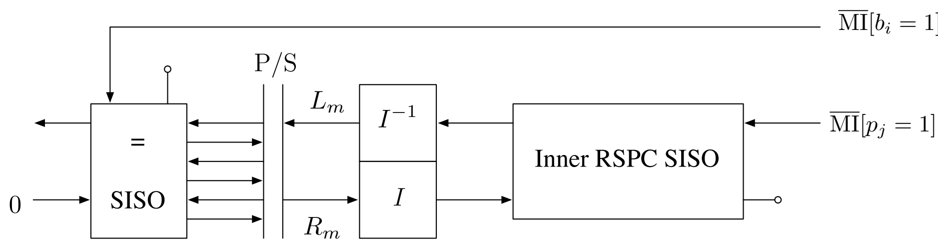


Figure 27: The iterative decoder considered for the RA code.

6.15. Consider a systematic repeat accumulate code (as shown in Fig. 26) with 3 input bits b_0, b_1, and b_2. There are $Q = 3$ repetitions of each bit b_i fed into the interleaver. The size of the inner SPC is $J = 3$ and the inner accumulator is unterminated.

The interleaver is listed below:

$m =$	0	1	2	3	4	5	6	7	8
$I(m) =$	0	3	7	6	2	5	4	1	8

Note the convention is $v_m = d_{I(m)}$.

(a) Determine the output blocksize of this code, the rate, and the parity check matrix.

(b) Consider the iterative decoder for this code as illustrated in Fig. 27.

Consider the systematic channel metrics (in normalized form):

$$\overline{\mathrm{MI}}[b_0 = 1] = -2, \qquad \overline{\mathrm{MI}}[b_1 = 1] = +5, \qquad \overline{\mathrm{MI}}[b_2 = 1] = -1$$

and parity channel metrics

$$\overline{\mathrm{MI}}[p_0 = 1] = +2, \qquad \overline{\mathrm{MI}}[p_1 = 1] = -4, \qquad \overline{\mathrm{MI}}[p_2 = 1] = +6$$

You will perform a few iterations of this decoder and list the messages for variables d_m. These are listed as L_m and R_m for the left-going and right-going normalized metrics on d_m.

Starting the decoding by activating the outer equality SISO, specify R_m.

$m =$	0	1	2	3	4	5	6	7	8
$R_m =$									

After activating the interleaver, the inner RSPC SISO and the deinterleaver, what are L_m?

$m =$	0	1	2	3	4	5	6	7	8
$L_m =$									

After activating the outer equality SISO again, specify R_m.

$m =$	0	1	2	3	4	5	6	7	8
$R_m =$									

After activating the interleaver, the inner RSPC SISO and the deinterleaver again, what are the new values of L_m?

$m =$	0	1	2	3	4	5	6	7	8
$L_m =$									

(c) The outer equality SISO is activated one last time to compute decision metrics for the information bits; determine the resulting hard decisions on the information bits.

6.16. Consider a rate 1/2 convolutional code with generators $G_1(D) = G_2(D) = 1 + D$. This problem considers the case where this encoder is started and terminated (using tail bits as necessary) in the zero state. The BPSK-AWGN channel is considered.

(a) Denote the input to the convolutional encoder by b_i and the outputs $c_i^{(1)}$ and $c_i^{(2)}$. Write the difference equation for generating each output. Sketch an encoder diagram for this code. Sketch and label one trellis section for this code.

(b) If a large number of input bits are encoded, what is the minimum distance of the resulting code?

(c) If 1000 input bits are encoded with this convolutional code, the probability of bit error with soft-in Viterbi decoding will be approximately

$$P_b = K\mathrm{Q}(A)$$

where K is a constant. What is A?

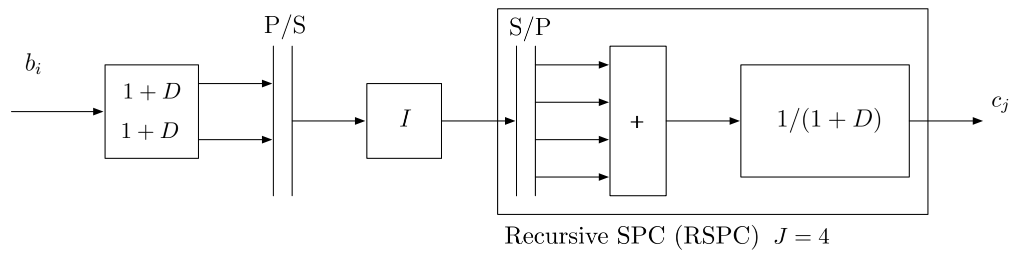
(d) Consider the case where 5 input bits $b_0, b_1, \ldots, b_4$ are encoded along with a single tail bit to produce 12 output bits. Draw the graphical model corresponding to the trellis model for this code. Label all variables (hidden and visible). Consider the case when the information bits are a-priori equally likely to be 1 or 0 and the coded bits are transmitted across a BPSK-AWGN channel with corresponding observations:

i	0	1	2	3	4	5
$z_i^{(1)}$	$+3$	-2	$+5$	-4	-1	$+2$
$z_i^{(2)}$	$+1$	-3	0	$+3$	$+3$	$+4$

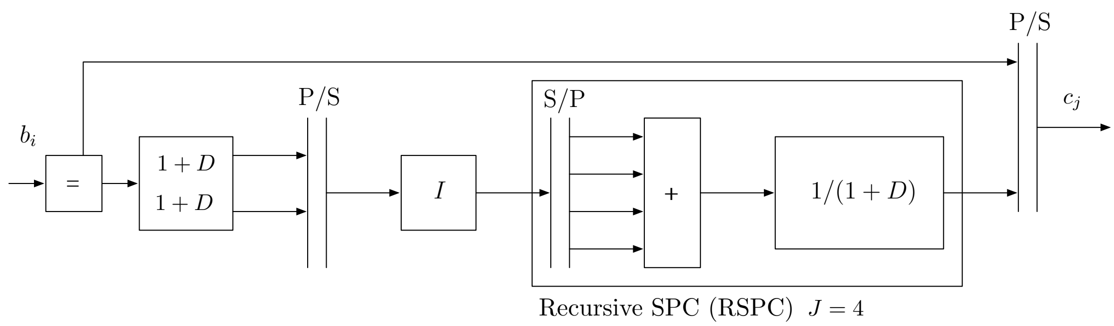
Consider min-sum decoding, so that the above may be taken as normalized incoming metrics on the coded bits. Sketch the graphical model from above without the variable labels and label the edges with a convergent set of messages. State the associated ML-CW hard decisions for the information bits.

(e) Consider using the convolutional code above in a modern code. Specifically, consider the two codes below:

CODE A



CODE B



Let r_A and r_B be the rate of code A and B, respectively (neglecting the effect of tail bits). Determine these rates. Let $\alpha_{\mathrm{A,max}}$ and $\alpha_{\mathrm{B,max}}$ be the maximum exponent of N in the uniform interleaver analysis for code A nd B, respectvely. What are these values?

6.17. Consider the code defined by the graph shown in Fig. 28. Given the incoming messages (normalized metrics), perform min-sum decoding. Show your results by labeling the arrows with the set of stable messages. Also note the final hard decision for each of the visible variables.

6.18. Consider the linear block code defined by the following parity check matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

This problem deals with this code used over the erasure channel. The erasure channel is a discrete memoryless channel with transition diagram shown in Fig. 29 where p is the eraure probability and$0 < p < 0.5$. The channel output E corresponds to an erasure – i.e., the decoder either receives the bit without error or the bit is erased on the channel. Assume each of the codewords is sent with the same probability.

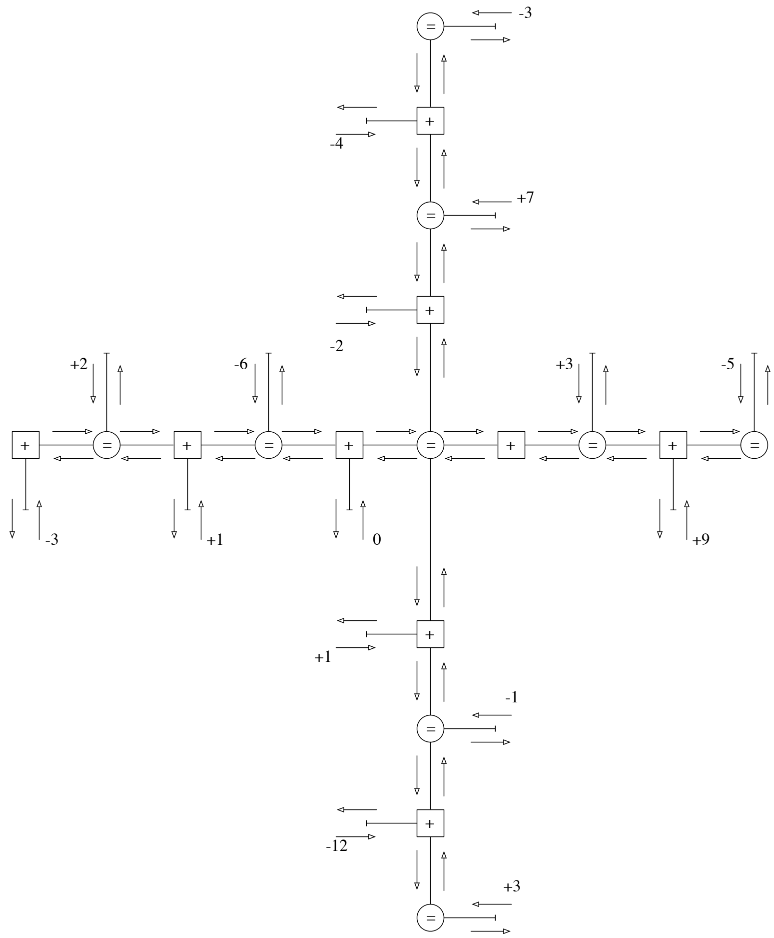


Figure 28: A cycle free graph defining a code.

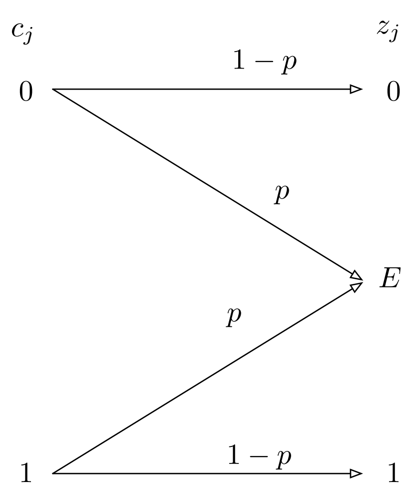


Figure 29: The DMC diagram for the erasure channel.

(a) Draw the Tanner graph for this parity check matrix.

(b) Determine the normalized channel metrics for this channel. Specifically, determine $\overline{\text{MI}}[c_j = 1]$ for each of the possible values of the channel output (z_j). Summarize in words what the decoding rule is for this erasure channel.

(c) Consider the set of observations:

j	0	1	2	3	4	5
z_j	E	E	E	1	1	1

Use these observations to perform min-sum iterative decoding on the Tanner graph from part (a). This iterative decoder will converge, but fail to decode to a codeword. Show this by drawing the Tanner graph and labeling the edges with convergent messages in both directions.

Can an optimal decoder decode this observation? If so, show how and provide the codeword decision. If not, explain why it cannot.

(d) Is it possible to specify a different parity check matrix for this code for which iterative decoding on the associated Tanner graph will successfully decode the above channel observations? If yes, then provide such a parity check matrix and show the convergent set of messages on the corresponding Tanner graph. If not, explain why this is not possible.

6.19. Consider iterative decoding of the $(6, 3, 3)$ code from Kumar's notes using its parity check graph. This graph is shown in Fig. 30 – the degree 1 variables nodes are shown above the check nodes to simplify the diagram. Also shown on this diagram are the incoming channel metrics in normalized form.

(a) Show the messages after the following processing: (i) activation of all variable nodes, (ii) activation of all check nodes, (iii) activation of all variable nodes.

(b) Continue the iterative decoding and show the messages after additional activation of (i) all check nodes and (ii) all variable nodes. Also indicate the best hard decision for each of the 6 visible variables.

6.20. Consider the linear block code defined by the graphical model shown in Fig. 31

(a) Given the channel and a-priori metrics

$$\overline{\text{MI}}[b_0 = 1] = -4 \qquad \overline{\text{MI}}[c_0 = 1] = +2$$
$$\overline{\text{MI}}[b_1 = 1] = -3 \qquad \overline{\text{MI}}[c_1 = 1] = +1$$
$$\overline{\text{MI}}[b_2 = 1] = +1 \qquad \overline{\text{MI}}[c_2 = 1] = +4$$
$$\overline{\text{MI}}[c_3 = 1] = -1$$
$$\overline{\text{MI}}[c_4 = 1] = -3$$

perform min-sum decoding and provide a convergent set of messages on a labeled graphical model.

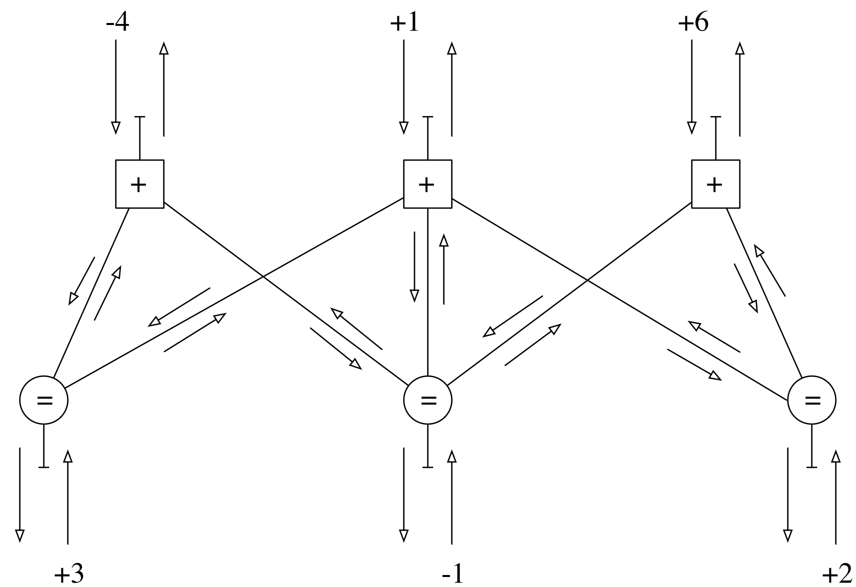


Figure 30: The Tanner graph for the $(6, 3, 3)$ code.

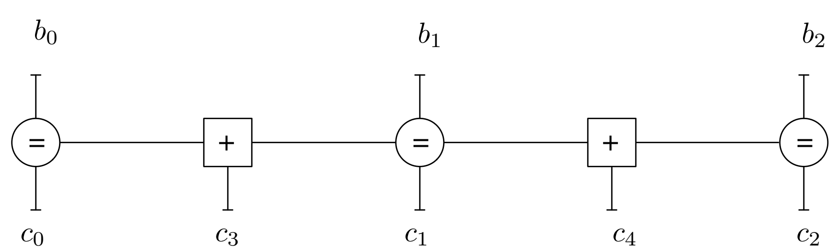


Figure 31: A graphical model defining a code.

What are the decisions on the information bits based on minimum codeword error probability decoding?

(b) Provide a generator and parity check matrix for the code defined by this graph.

(c) Consider syndrome-based decoding of this code. Suppose the information bits are a-priori uniform on $\{0,1\}$ and that BSC channel observations $\{y_j\}$ are obtained by thresholding the channel negative log-likelihoods $\{\overline{\mathrm{MI}}[c_0=1]\}$. Specify this observation vector $\mathbf{y}$ and perform syndrome-based decoding. Is this decision unique? Is it the same as the decision obtained in part (b)?

7 Short Problems on Signals, Modulation, and Detection

7.1. (a) TRUE
(b) FALSE

Reliable communication over an AWGN channel is not possible if the energy per bit E_b in Joules is less than the one-sided noise PSD level N_0 in Watts/Hz.

7.2. For a QASK modulation scheme, the Nyquist condition on the pulse assures that

(a) the sampling is robust to synchronization error
(b) the samples of the matched filter are independent
(c) the pulse bandwidth is sufficiently limited

7.3. (a) TRUE
(b) FALSE

For a given E_b/N_0, the bit error performance of QPSK with Gray mapping is the same as the bit error probability of BPSK

7.4. What is the minimum required value of E_b/N_0 (in dB) required to communicate with a spectral efficiency of 10 bits/sec/Hz?

7.5. (a) TRUE
(b) FALSE

The minimum distance characterizes the low SNR performance of a signaling scheme on an AWGN channel

7.6. (a) TRUE
(b) FALSE

For an M-ary system, the minimum bit error probability decision rule is the same as the minimum symbol error probability rule as long as the noise is AWGN and all hypotheses are equally likely

7.7. A receiver for a BPSK assumes square pulse shaping, but the actual pulse shape used, while limited to $[0,T]$ and having the same energy, is not square. What is the degradation in SNR due to this mismatch?

7.8. The performance of simplex signaling is the same as that of orthogonal; what is the advantage of simplex signaling? Why would orthogonal signaling ever be used over simplex signaling?

7.9. (a) TRUE

(b) FALSE

Accurate phase synchronization becomes less critical for QAM as M is increased

7.10. Name one modulation technique that is appropriate for each of the following cases:

(a) Bandwidth constrained, high throughput desired

(b) Power constrained, low error rates required

7.11. (a) TRUE

(b) FALSE

There is always an implementation of the MAP detector for (colored) Gaussian noise channels with exactly k correlators (k being the dimension of the signal space).

7.12. (a) TRUE

(b) FALSE

MAP detection is equivalent to minimum distance detection only if the a-priori probabilities are equal and the noise is AWGN.

7.13. (a) TRUE

(b) FALSE

16-QAM is approximately 10 dB in E_b/N_0 better than 16-PSK.

7.14. (a) TRUE

(b) FALSE

64-ary Walsh function modulation was used in the IS-95 cellular standard because it slightly outperforms 64-ary orthogonal FSK.

7.15. (a) TRUE

(b) FALSE

$P(\mathcal{E}) \leq (M-1)\mathrm{Q}\left(\sqrt{\frac{d^2_{min}}{2N_0}}\right)$ is a valid upper bound for any M-ary signaling technique with $\pi_m = 1/M$ and an AWGN channel (here $d^2_{min} = \min_{i \neq j} d^2(\mathbf{s}_i, \mathbf{s}_j)$).

7.16. (a) TRUE

(b) FALSE

Rayleigh fading results in a degradation in E_b/N_0 of about 3 dB relative to no fading for BPSK.

7.17. (a) TRUE

(b) FALSE

The robustness to sampling time error for a matched filter decreases with the pulse bandwidth.

7.18. (a) TRUE

(b) FALSE

For an SNR= $\frac{P_{ave}}{WN_0} = 10^{-10}$, reliable communication is not possible at any data rate.

7.19. (a) TRUE

(b) FALSE

Two user waveforms with a correlation coefficient $\rho > 0.5$ cannot be reliably separated once added together on a common AWGN channel.

7.20. (a) TRUE

(b) FALSE

A matched filter provides a small advantage over the corresponding correlator because the output SNR is maximized.

7.21. (a) TRUE

(b) FALSE

Any data pulse with duration less than or equal to a symbol time satisfies the Nyquist condition for no ISI.

7.22. The topic of Claude Shannon's M.S. thesis was (no penalty for guessing):

(a) Boolean mathematics applied to digital circuit design

(b) The maximum achievable compression ratio for English text

(c) IQ testing

(d) Shannon never received a Masters or Ph.D. degree.

7.23. (a) TRUE

(b) FALSE

For 8PSK with natural bit labeling and uniform a-priori probabilities, the MAP bit and symbol detection rules are equivalent.

7.24. (a) TRUE

(b) FALSE

The I and Q channel of a QPSK signal may be staggered to provide better symbol error probability for an AWGN channel

7.25. (a) TRUE

(b) FALSE

The only pulses satisfying the Nyquist criteria are those that are zero outside $[0, T]$ and the family of raised-cosine pulses.

7.26. (a) TRUE
(b) FALSE

MPSK is preferred over MQAM because it yields better performance for a given E_b/N_0.

7.27. (a) TRUE
(b) FALSE

If two binary signal sets have the same distance, then the performance over an AWGN channel will be the same.

7.28. (a) TRUE
(b) FALSE

Phase noncoherent methods are used because they provide a small performance improvement at the cost of additional receiver complexity.

7.29. (a) TRUE
(b) FALSE

For the colored-Guassian noise channel, the signals can be rotated without affecting the performance of the optimal receiver.

7.30. (a) TRUE
(b) FALSE

It is theoretically possible to communicate without error using finite energy waveforms over some colored-Gaussian noise channels with non-singular covariance functions (no zero eigenvalues).

7.31. (a) TRUE
(b) FALSE

Upper and lower bounds should be used for M-ary PAM signaling since a closed form expression for $P(\mathcal{E})$ is not available.

7.32. (a) TRUE
(b) FALSE

The Maximum Likelihood decision criterion is equivalent to the minimum distance rule.

7.33. The narrowband assumption requires

(a) That f_cT is an integer
(b) That $f_c \gg B$ where B is the bandwidth of the equivalent baseband modulation.
(c) That the carrier phase θ_c is known precisely at the receiver.

7.34. (a) TRUE
(b) FALSE

The a-priori probabilities yielding largest $P(\mathcal{E})$ for binary MAP detection are $\pi_0 = \pi_1 = 1/2$?

7.35. (a) TRUE
(b) FALSE

Fading typically leads to an effective SNR degradation.

7.36. (a) TRUE
(b) FALSE

A good QASK constellation will be symmetric around the origin.

7.37. (a) TRUE
(b) FALSE

A matched-filter can be used to replace many correlators by sampling at different time instants.

7.38. (a) TRUE
(b) FALSE

FSK with a tone separation of $1/(2T)$ provides a noncoherent orthogonal signal set.

7.39. (a) TRUE
(b) FALSE

The minimum number of correlators needed for the MAP symbol-detector is always determined by M, the number of possible signals.

7.40. (a) TRUE
(b) FALSE

If $s_0(t)$ and $s_1(t)$ are passband signals and $\int_0^T s_0(t)s_1(t)dt = 0$, then these two signals can be detected (phase) noncoherently.

7.41. Staggering of the I and Q channels is used in some cases to

(a) Relax the requirements for synchronization accuracy
(b) Add robustness against filtered nonlinearities
(c) Simplify the integrate and dump circuits

7.42. (a) TRUE
(b) FALSE

A BPSK system will perform better in an AWGN channel if the bandwidth is spread using high bandwidth pulse shapes.

7.43. (a) TRUE
(b) FALSE

A deep-space satellite system is using QPSK modulation and only a portion of the available bandwidth. To improve the link BER, one reasonable option is to use 8PSK.

(a) TRUE
(b) FALSE

7.44. (a) TRUE
(b) FALSE

For a standard QAM system, the pulse shape affects on the spectrum, not the performance.

7.45. (a) TRUE
(b) FALSE

In theory, it is possible to communication effectively over an AWGN channel with E_b/N_0 below 0 dB.

7.46. (a) TRUE
(b) FALSE

The Nyquist condition for no ISI ensures that the noise samples at the output of the matched filter are independent.

7.47. (a) TRUE
(b) FALSE

The Craig form for the Q-function is used to get tighter upper bounds on the performance of MPSK in AWGN.

7.48. (a) TRUE
(b) FALSE

Mini-max detection requires that $\pi_0 = \pi_1$.

7.49. (a) TRUE
(b) FALSE

Adding a constant signal; to each of a possible set of message signals will typically improve the performance.

7.50. (a) TRUE
(b) FALSE

The sensitivity of MPSK to synchronization errors typically increases with M

7.51. (a) TRUE
(b) FALSE

There is a standard degradation of 3 dB to be expected between coherent detection and noncoherent detection.

7.52. (a) TRUE
(b) FALSE

Fast Hadamard transforms can be used for 64 orthogonal FSK demodulation

7.53. (a) TRUE

(b) FALSE

As few as one correlator is sometimes required for M-ary detection.

7.54. The set of 64-ary Walsh functions provides an orthogonal signal set in the noncoherent sense.

(a) TRUE

(b) FALSE

7.55. A satellite link uses OQPSK with a standard rectangular pulse shape. It is desired to use shaped OQPSK (precoded MSK) by replacing the T-second rectangular pulse with the following T-second pulse:

$$p(t) = \sqrt{\frac{2}{T}} \sin(\pi t/T) \quad t \in [0, T]$$

The satellite receiver cannot be reprogrammed, however, so it will still correlate to the rectangular pulse – *e.g.,* integrate and dump. This will result in a degradation in performance of X dB of E/N_0. Determine X:

7.56. Consider the unknown phase model

$$\mathcal{H}_m : \quad \mathbf{r}(u) = \mathbf{s}_m e^{j\theta_c} + \mathbf{w}(u) \qquad m = 0, 1, \ldots (M-1)$$

Using the fact that θ_c is a constant in $[0, 2\pi)$, determine the generalized likelihood given $\mathcal{H}_m$, $G(\mathbf{r}|\mathcal{H}_m)$.

If $M = 2$ and orthogonal, equal energy (E) signals are used, what is the maximum generalized likelihood rule and the associated probability of error?

7.57. For binary AWGN communications, coherent detection of orthogonal signals performs better than differentially coherent BPSK.

(a) TRUE

(b) FALSE

7.58. A company licenses 10 MHz of bandwidth. Assuming a standard AWGN channel, what is the minimum E_b/N_0 value required to achieve reliable communications at a rate of 60 Mbps in this band?

7.59. Due to imperfect receiver processing in a standard QPSK system, the signal constellation is shifted to the right by $\epsilon E > 0$ before entering the slicer as shown in Fig. 32.

Specifically, the slicer decision boundaries remain the axes, but the signals are

$$\mathbf{s}_m = \sqrt{E} \begin{bmatrix} \cos(m\pi/4) + \epsilon \\ \sin(m\pi/4) \end{bmatrix} \quad m = 0, 1, 2, 3$$

and this is observed in AWGN with zero mean and variance $N_0/2$ per dimension. Find the probability of symbol error in this case.

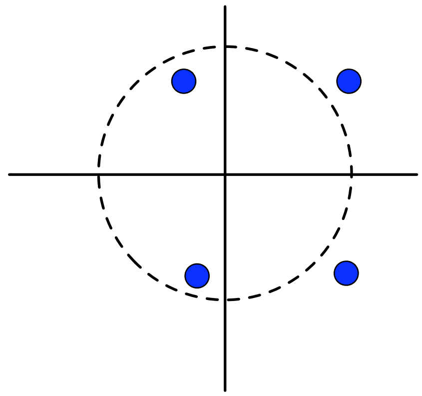


Figure 32: The effect of imperfect receiver processing is to shift teh constellation.

7.60. It is desired to achieve 2.5 bps/Hz over an AWGN channel. What is the minimum value of the E_b/N_0 required to achieve this spectral efficiency using any signaling approach? Can this spectral efficiency be approached at this E_b/N_0 with very low error probability using the following 8PSK? 64-QAM? If yes, then briefly explain how; if no, the explain why not.

7.61. For EE564 we model the information sequence as an independent, identically distributed Bernoulli sequence with $p = 1/2$ because this a good model for most multimedia courses such as voice, images, and text files.

7.62. For phase noncoherent reception, antipodal binary signaling is preferred since it maximizes the signal distance and therefore minimizes the error probability.

(a) TRUE
(b) FALSE

7.63. For equally likely signals in any additive noise channel, the MAP decision rule is a minimum distance rule.

(a) TRUE
(b) FALSE

7.64. For equally likely information bits and ML symbol detection, the bit error probability of QPSK and BPSK are the same regardless of the bit labeling.

(a) TRUE
(b) FALSE

7.65. Differentially coherent detection of differentially encoded MPSK performs within 1 dB of coherent MPSK for all values of M.

(a) TRUE

(b) FALSE

7.66. The separation theorem implies that, with sufficient latency, source coding can be done separately from channel coding in practice without any loss of optimality

(a) TRUE
(b) FALSE

8 Short Problems on Coding

8.1. Consider 4 bits, $\{c_j\}_{j=0}^{3}$ that are inputs/outputs associated with an even single parity check constraint and the normalized ($\overline{\text{MI}}[c_j = 0] = 0$) input metrics given by

$$\overline{\text{MI}}[c_0 = 1] = +3, \quad \overline{\text{MI}}[c_1 = 1] = -2, \quad \overline{\text{MI}}[c_2 = 1] = -0.5, \quad \overline{\text{MI}}[c_3 = 1] = -1.5$$

determine the normalized ($\overline{\text{MO}}[c_j = 0] = 0$) output metrics for min-sum SISO decoding.

8.2. Repeat the above for the case in which the 4 bits are constrained to be equal.

8.3. Consider a "non-perfect" linear block code with minimum Hamming distance 5 transmitted over the binary symmetric channel. With ML codeword decoding, this code can

(a) correct all error patterns of weight 0, 1, 2, 3
(b) correct only error patterns of weight 0, 1, 2
(c) correct all error patterns of weight 0, 1, 2, and at least one pattern with weight ≥ 3

8.4. **True or False?:** Computed exactly, the block error probability predicted by the sphere packing bound must always be lower than that of the random coding bound.

8.5. **True or False?:** For the BPSK-AWGN channel, one needs to use an 8-bit analog to digital converter to achieve most of the soft-in decoding advantage.

8.6. Consider an $(n, k, d_{min}) = (6, 2, 3)$ linear block code used over the binary symmetric channel (BSC). Assume that the BSC is an abstraction of the BPSK-AWGN channel. Give a good upper bound on the codeword error probability as a function of E_b/N_0 on the underlying BPSK-AWGN channel for the ML codeword decoder.

8.7. The repetition code can provide coding gain on the BPSK-AWGN channel if optimal decoding is performed

(a) TRUE
(b) FALSE

8.8. Find the minimum possible value for E_b/N_0 to achieve a spectral efficiency of 1 bps/Hz using any modulation:

8.9. Based only on information theory results, the expected performance gain associated with soft-in decision decoding on the BPSK-AWGN channel is approximately how many dB in E_b/N_0 over hard-in decoding.

8.10. The most important characteristic for determining the performance of a modern turbo-like code is the minimum Hamming distance.

(a) TRUE
(b) FALSE

8.11. Convolutional codes were used extensively in the 1980s because they mimic the properties of the random codes used in proving the Shannon limit.

(a) TRUE
(b) FALSE

8.12. Let $d_0, d_1 \dots d_{n-1}$ be binary variables constrained to take even parity. Suppose that the incoming normalized metrics for these bits are $\overline{\mathrm{MI}}[d_0] = \overline{\mathrm{MI}}[d_1] = 0$ and $\overline{\mathrm{MI}}[d_i] = i^2$ for $i = 2, 3, \dots n$. Determine the output normalized metrics (in extrinsic form) for standard min-sum SISO processing.

8.13. For the BPSK/AWGN channel, the performance gain associated with soft-in decoding relative to hard-in decoding for most code rates is approximately

(a) 1 dB in E_b/N_0
(b) 0.5 dB in E_b/N_0
(c) 2 dB in E_b/N_0
(d) No gain, if optimal decoding is used for both hard-in and soft-in

8.14. Trellis-Coded Modulation and similar coded modulation methods achieve coding gain without bandwidth expansion via

(a) The use of a modulation constellation with more points and redundancy through error correction coding
(b) A special modulation constellation formed by superposition of other constellations
(c) Binary channel signaling with non-binary error correction codes

8.15. Consider the interleaver that swaps even and odd indices (assume that there are an even number of bits in the block). Specifically

$$I(k) = \begin{cases} k+1 & k \text{ even} \\ k-1 & k \text{ odd} \end{cases}$$

Consider this interleaver used in constructing a parallel concatenated convolutional code with input block size 4096 bits. Briefly describe how you expect this to perform and describe why.

8.16. As described in lecture, the RSPC (recursive single parity check or zig-zag) 'code' takes in J bits and outputs one bit. What is the minimum distance of this 'code'?

8.17. A tail-biting convolutional code is one in which the final state of the encoder must be equal to the initial state. This may be used to avoid inserting tail bits to terminate the encoder. Specifically, if 8 bits, $\{b_i\}_{i=0}^{7}$, are encoded $s_0 = s_8$, but the value of this common state can be any state of the convolutional encoder. For example, considering an 8-state encoder, instead of just starting and ending in state 0, it could also start/end in state $1, 2 \ldots 7$. Draw a graphical model and briefly describe the associated min-sum decoder processing based on AWGN channel observations for times $i = 0, \ldots 7$.

8.18. The minimum distance is the characteristic of a code that determines the performance in nearly all practical situations

8.19. The number of states in the constituent codes used in the third generation cellular telephone system turbo code is

(a) 2
(b) 4
(c) 8
(d) 16

8.20. **True or False?:** The limits established by the random coding bound are far from what is practically achievable, so we rely on the channel cut-off rate to guide our practical designs

8.21. **True or False?:** There is a min*-sum version of the Viterbi algorithm that provides hard-decisions that minimize the probability of bit error.

8.22. A rate 1/2 convolutional code has an output weight enumerator function for simple sequences that is

$$2D^9 + 6D^{12} + 12D^{15} + \cdots$$

What is the asymptotic coding gain for this code in dB?

8.23. The standard decoding algorithm for turbo codes performs MAP decoding

(a) TRUE
(b) FALSE

8.24. Consider a code with parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Draw the associated parity check trellis below. Indicate transistions associated with $c_j = 0$ by a dashed line and $c_j = 1$ by a solid line.

8.25. **True or False?:** Since Gaussian signaling achieves capacity on the AWGN channel, it is the modulation most often used in modern digital communication systems.

8.26. If a link operates at $E_b/N_0 = 40$ dB, what is the maximum spectral efficiency one could achieve using any modulation (you may round to the nearest integer value)?

8.27. You are system engineer in need of a $r = 2/3$ code with an input block size of $k = 2048$ using QPSK modulation. What is your target spectral efficiency?

A company offers a decoding chip that achieves this spectral efficiency with a codeword error probability of 10^{-4} at an E_s/N_0 value of 5.0 dB. Is this good performance or should you keep searching for a better product?

8.28. **True or False?:** The standard decoding algorithm for turbo codes performs MAP decoding

8.29. **True or False?:** The parallel concatenated convolutional codes (turbo codes) studied in this class are linear block codes

8.30. **True or False?:** Uniform interleaver analysis for parallel concatenated convolutional codes imply that these codes cannot have a block error probability of below 10^{-2} regardless of block size.

8.31. A new modern code has been suggested and uniform interleaver analysis determines that $\alpha_{\max} = -2$.

(a) This code will exhibit interleaver gain in P_b, but not P_{cw}

(b) This code will exhibit interleaver gain in P_b and P_{cw}

(c) One needs to known what the specific form of the new modern code is before determining the interleaver gain properties.

(d) This code will not exhibit interleaver gain.

8.32. **True or False?:** EXIT charts can be used to determine which constituent codes should be used in modern codes to optimize the SNR threshold.

8.33. **True or False?:** Minimum Euclidean Distance decoding over the BPSK-AWGN channel is the same as selecting the m that minimizes $\sum_{j=0}^{n-1} \overline{\mathrm{MI}}[c_j^{(m)}] = \sum_{j=0,c_j^{(m)}\neq 0}^{n-1} \frac{4\sqrt{E_c}}{N_0} z_j$.

8.34. You are consulting for a venture capital firm. A potential client is proposing to market a point to point communication system operating in AWGN. It is claimed that the system will achieve 3 bps/Hz at an E_b/N_0 of 4 dB. Is this possible? Is it practical? Explain.

8.35. A current system uses a good modern coding scheme comprising a binary $r = 2/3$ code mapped onto 8PSK. The input block length k is 868 bits and $P_{cw} = 10^{-3}$. Holding everything else fixed, it is desired to increase k to achieve an additional 0.5 dB of coding gain.

(a) Estimate the smallest value of k that can achieve this.

(b) If $k = 868$ is maintained and it is desired to achieve $P_{cw} = 10^{-5}$, how much larger would E_b/N_0 have to be (in dB) relative to the current system's required E_b/N_0?

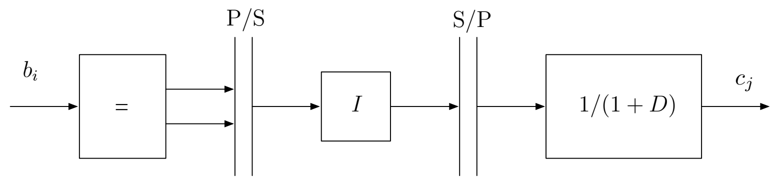


Figure 33: Concatenated code structure considered.

8.36. The binary variables $\{d_j\}_{j=0}^{39}$ are constrained by a single parity check (even). How many allowable configurations of these variables M are there?

The following is given regarding the incoming messages on these variables

$$\prod_{j=0}^{39} \operatorname{sgn}(\overline{\mathrm{MI}}[d_j]) = -1$$
$$\min_j |\overline{\mathrm{MI}}[d_j]| = 0.2$$

If $\overline{\mathrm{MI}}[d_6] = 1.2$ determine $\overline{\mathrm{MO}}[d_6]$.

8.37. Consider the concatenated code shown in Fig. 33.

Select the best answer:

(a) This code will exhibit interleaver gain in P_b, but not P_{cw}

(b) This code will exhibit interleaver gain in P_b and P_{cw}

(c) This code will not exhibit interleaver gain.

(d) The block size needs to be specified to determine if this code will exhibit interleaver gain.

8.38. Consider the linear code on the variables $\{c_j\}_{j=0}^{7}$ defined by the cyclic graphical model shown in Fig. 34.

(a) Determine the rate and minimum distance for this code.

(b) Give a cycle-free graphical model for this code (label completely).

(c) Let the normalized input metrics for the 8 code bits be $\overline{\mathrm{MI}}[c_j = 1]$ be as given below:

i	0	1	2	3	4	5	6	7
$\overline{\mathrm{MI}}[c_j = 1]$	-3	$+2$	-6	-4	$+1$	-5	-10	-4

Redraw your acyclic graphical model and label the edges with the stable set of messages after convergence of the min-sum algorithm. Note that each edge should have a message going in each direction.

8.39. Consider a linear block code with $n = 16$ and $d_{\min} = 8$. What values of $k \in \{0, 1, 2, \ldots 16\}$ are not achievable? Explain your answer.

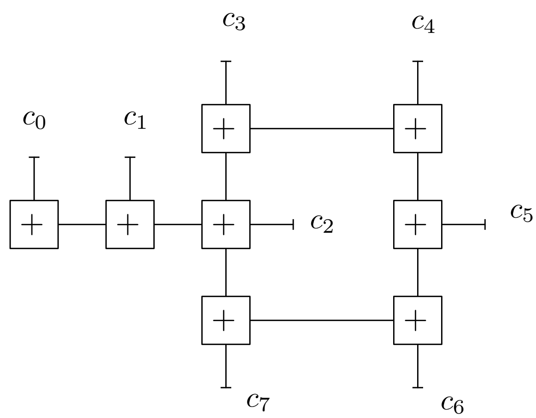


Figure 34: Cyclic graphical model constraining the 8 binary variables $\{c_j\}_{j=0}^{7}$.

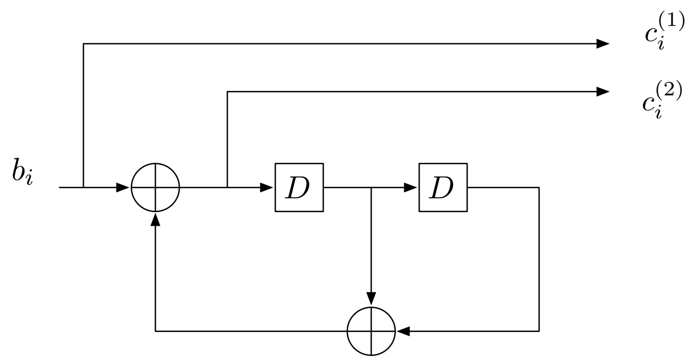


Figure 35: Encoder diagram for $r = 1/2$ convolutional code.

8.40. The Golay code is a perfect code with $n = 23$, $k = 12$.

(a) What is the error correcting capability of this code t_c?

(b) Describe all of the coset leaders for this code.

8.41. Consider performing min-sum iterative decoding of an LDPC code based on observations from the binary symmetric channel. The same message update rules used for the BPSK-AWGN channel can be used if the channel metrics are defined properly. Suppose the BSC has error probability $\epsilon = 0.1$. The BSC channel observation y_j takes values 0 and 1; determine the normalized metric value $\overline{\mathrm{MI}}[c_j = 1]$ for each of these observation values.

8.42. An encoder for a rate 1/2 convolutional code is shown in Fig. 35. Determine the two generator polynomials and draw one section of the trellis.

8.43. **True or False?:** *Shannon codes* are capacity achieving codes introduced in the proof of channel capacity and remain widely used in practical systems.

8.44. For cellular voice communications a typical decoder bit error probability is

(a) 10^{-1}
(b) 10^{-3}
(c) 10^{-6}
(d) 10^{-9}

8.45. **True or False?:** *Turbo codes* were discovered and published in 1983.

8.46. Iterative decoding of modern codes is optimal in the minimum codeword error probability sense.

(a) TRUE
(b) FALSE
(c) It depends on whether min-sum or min*-sum iterative decoding is used.

8.47. **True or False?:** *LDPC codes* are linear codes.

(a) TRUE
(b) FALSE

8.48. Determine a generator matrix for the (15,11) Hamming code.

8.49. A short single-error correcting code with rate $r \geq 1/2$ is sought. Based on the Hamming bound, values of n can you rule out as too short for this to be possible?

8.50. A block code has parity check matrix given by

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Draw the Tanner graph for this code

8.51. **True or False?:** A convolutional code with generators $G_1(D) = 1 + D^2$ and $G_2(D) = 1 + D + D^2$ is not a linear code if the encoder is started in the state (11)

8.52. **True or False?:** A coded modulation satellite data system operating at 1 bps/Hz requires an E_b/N_0 of 0.6 dB. This design cannot be significantly improved in terms of E_b/N_0 sensitivity.

8.53. Consider the linear block code defined by the parity check matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(a) Specify the number of codewords in this code and the rate of this code.
(b) Draw the Tanner Graph corresponding to this parity check matrix.

8.54. A communication channel has been modeled as a binary symmetric channel with error probability $\epsilon = 0.0179$. Determine the maximum rate of reliable (*i.e.,* arbitrarily small error probability) communication across this channel in bits/channel-use.

8.55. The binary variables d_0, d_1, d_2, and d_3 are constrained by a repetition code. Consider performing min-sum SISO decoding for this code with following incoming normalized metrics:

$$\begin{aligned}\overline{\mathrm{MI}}[d_0 = 1] &= +1.5\\ \overline{\mathrm{MI}}[d_1 = 1] &= -2\\ \overline{\mathrm{MI}}[d_2 = 1] &= +2.5\\ \overline{\mathrm{MI}}[d_3 = 1] &= -1\end{aligned}$$

(a) Determine the corresponding outgoing normalized metrics and teh associated hard decisions.

(b) Repeat the above for min*-sum SISO processing using the same incoming metrics.

8.56. Specify the minimum Hamming distance for the following linear block codes:

(a) Hamming (7,4)

(b) Repetition (n,1)

(c) Single Parity Check (n,$n-1$)

(d) Toy-SISO (5,3)

8.57. The relation $\min^*(x, y) \leq \min(x, y)$ holds for all real x, y.

(a) TRUE

(b) FALSE

8.58. A design engineer tells you that a good rule of thumb for good modern code performance is the following: *if you quadruple the block size (*i.e., *make 4 times larger), you obtain approximately 0.5 dB of additional coding gain.* Is this statement true or false? Explain.

(a) TRUE

(b) FALSE

8.59. List the coset leaders for the $n = 5$ repetition code.

8.60. Assuming that the $(7, 4)$ Hamming code is decoded with soft decisions on the BPSK-AWGN channel, what is the expected coding gain at high E_b/N_0?

8.61. **(5 points)** The relation $\min^*(x, y) - \min^*(x + y, 0) = \min^*(|x|, |y|)\mathrm{sgn}(x)\mathrm{sgn}(y)$ holds for all real x, y.

(a) TRUE

(b) FALSE

8.62. Historically, soft-in decoding methods were slow to be adopted because they require approximately 8-bit quantization of the channel observations to realize the additional coding gain. So, soft-decoding did not become commonplace until 8-bit A/D converters were readily available.

(a) TRUE
(b) FALSE

8.63. Low Density Parity Check codes were discovered shortly after Turbo codes were introduced in 1993.

(a) TRUE
(b) FALSE

8.64. The reason one is interested in performing "soft-out" decoding is

(a) Debugging decoder hardware;
(b) Iterative decoding/processing;
(c) It is usually most computationally efficient to produce soft-decisions on variables and then threshold these to obtain hard decisions;
(d) All of the above.