Scholtz Problem 10 Solution – © K. M. Chugg

This is a solution for problem 10 in the Scholtz problem set. As discussed in class, the distinction between the simulation and representation problems is not always clearly made by Scholtz. In this problem assume that we want to simulate the second moment description of the n-dimensional random vector $\mathbf{x}(u)$, where $\mathbf{K}_{\mathbf{x}}$ is *singular*. This problem says that we can design $\mathbf{H}$ and $\mathbf{m}$ so that

$$\mathbf{x}(u) \stackrel{\text{ws}}{=} \mathbf{y}(u) = \mathbf{H}\mathbf{w}(u) + \mathbf{m},$$

where $\mathbf{w}(u)$ has dimension less than n. We know that we should choose

$$\mathbf{m} = \mathbf{m}_{\mathbf{x}}.$$

Since, by design, $\mathbf{K}_{\mathbf{y}} = \mathbf{K}_{\mathbf{x}}$, we can talk about factoring either. Read the following solution with this interpretation.

Solution to Scholtz Problem 10

(a) First let's consider the simulation technique that we know works,

$$\mathbf{y}_0(u) = \mathbf{G}\mathbf{v}(u),$$

where $\mathbf{v}(u)$ is a white $(n \times 1)$ random vector, that is $\mathbf{m}_{\mathbf{v}} = \mathbf{0}$ and $\mathbf{K}_{\mathbf{v}} = \mathbf{I}$, and we use the usual notation $\mathbf{y}_0(u) = \mathbf{y}(u) - \mathbf{m}$. We can choose $\mathbf{G} = \mathbf{E}\mathbf{\Lambda}^{1/2}$. If we denote $r = \text{rank}(\mathbf{K}_{\mathbf{y}})$, then since $\mathbf{K}_{\mathbf{y}}$ is singular we have $r < n$ (strictly less than!). Let's also order the eigenvalues by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0$, where we have noted that all of the eigenvalues are nonnegative and that exactly $(n-r)$ of them are zero. Then we expand our second order representation as

$$\begin{aligned}
\mathbf{y}_0(u) &= \mathbf{G}\mathbf{v}(u) = \mathbf{E}\mathbf{\Lambda}^{1/2}\mathbf{v}(u) \\
&= \sum_{i=1}^{n} \lambda_i^{1/2} v(u,i)\mathbf{e}_i \\
&= \sum_{i=1}^{r} \lambda_i^{1/2} v(u,i)\mathbf{e}_i + \underbrace{\sum_{i=r+1}^{n} \lambda_i^{1/2} v(u,i)\mathbf{e}_i}_{=0} \\
&= \sum_{i=1}^{r} \lambda_i^{1/2} v(u,i)\mathbf{e}_i \\
&= \mathbf{E}_r\mathbf{\Lambda}_r^{1/2}\mathbf{w}(u),
\end{aligned}$$

where

$$\mathbf{E}_r = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_r \end{bmatrix} \qquad (n \times r)$$

$$\mathbf{\Lambda}_r = \mathbf{diag}(\lambda_1, \lambda_2, \ldots \lambda_r) \qquad (r \times r)$$

$$\mathbf{w}(u) = \begin{bmatrix} w(u,1) \\ w(u,2) \\ \vdots \\ w(u,r) \end{bmatrix} = \begin{bmatrix} v(u,1) \\ v(u,2) \\ \vdots \\ v(u,r) \end{bmatrix} \qquad (r \times 1).$$

So we take $\mathbf{H} = \mathbf{E}_r^{1/2}\mathbf{\Lambda}_r$, which is $(n \times r)$ and we have

$$\mathbf{y}_0(u) = \mathbf{H}\mathbf{w}(u).$$

Notice that the question asked for a matrix $\mathbf{H}$ which was $(n \times m)$, with $m < n$. We have found the minimum value for m, namely r.

(b) Since $\lambda_n = 0$, we know that $\mathbf{K_y}$ has a nontrivial null space,

$$\mathbf{K_y}\mathbf{e}_n = \mathbf{0},$$
$$\Rightarrow \mathbf{e}_n^{\mathrm{t}}\mathbf{K_y}\mathbf{e}_n = 0.$$

So one (nonunique) choice of $\mathbf{d}$ is $\mathbf{d} = \mathbf{e}_n \neq \mathbf{0}$. Recalling that $\mathbf{d}^{\mathrm{t}}\mathbf{K_y}\mathbf{d} = \mathbb{E}\left\{(\mathbf{d}^{\mathrm{t}}\mathbf{y}_0(u))^2\right\}$, we have

$$\mathbf{K_y} \text{ singular} \iff \mathbb{E}\left\{(\mathbf{d}^{\mathrm{t}}\mathbf{y}_0(u))^2\right\} = 0 \iff \mathbf{d}^{\mathrm{t}}\mathbf{y}_0(u) \stackrel{\mathrm{as}}{=} 0$$
$$\iff \sum_{i=1}^{n} d_i(y(u,i) - m_i) \stackrel{\mathrm{as}}{=} 0$$

Since $\mathbf{d} \neq \mathbf{0}$, at least one component of $\mathbf{d}$ must be nonzero. Assuming that $d_j \neq 0 \Rightarrow$

$$y(u,j) \stackrel{\mathrm{as}}{=} -\sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{d_i}{d_j} y(u,i) + (m_j + \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{d_i}{d_j} m_i),$$

which is of the desired form.

(c) If a random variable, $x(u)$, is equal to a constant with probability one, then the density is a Dirac delta function

$$x(u) \stackrel{\mathrm{as}}{=} a \Rightarrow f_{x(u)}(z) = \delta(z-a).$$

In part (b) we showed that conditioned on knowledge of $\{y(u,i)\}_{i=1, i\neq j}^{n}$ we know the value of $y(u,j)$ with probability one. Let us denote by $\mathbf{y}_j(u)$ the random vector $\mathbf{y}(u)$ without the j^{th} component

$$\mathbf{y}_j(u) = \begin{bmatrix} y(u,1) \\ \vdots \\ y(u,j-1) \\ y(u,j+1) \\ \vdots \\ y(u,n) \end{bmatrix} \qquad ((n-1) \times 1)$$

then

$$f_{y(u,j)|\mathbf{y}_j(u)}(z|y_1,\ldots y_{j-1},y_{j+1}\ldots y_n) = \delta(z-a),$$

where

$$a = -\sum_{\substack{i=1\\i\neq j}}^{n} \frac{d_i}{d_j} y_i + (m_j + \sum_{\substack{i=1\\i\neq j}}^{n} \frac{d_i}{d_j} m_i).$$

(d) In this case $\mathbf{K_x}$ has rank 2, and clearly $\mathbf{d} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^{\mathrm{t}}$ is in the null space. The linear combination is then

$$x(u,2) \overset{\text{as}}{=} x(u,1) + x(u,3).$$

The point of part (c) was to get you thinking about conditional densities, specifically

$$\begin{aligned} f_{\mathbf{x}(u)}(\mathbf{z}) &= f_{x(u,1),x(u,2),x(u,3)}(z_1,z_2,z_3) \\ &= \underbrace{f_{x(u,2)|\,x(u,1),x(u,3)}(z_2|\,z_1,z_3)}_{\text{delta function}} \underbrace{f_{x(u,1),x(u,3)}(z_1,z_3)}_{\text{2-dim. Gaussian}}. \end{aligned}$$

Since the (1,3) and (3,1) elements of $\mathbf{K_x}$ are zero, we also have that $x(u,1)$ and $x(u,3)$ are uncorrelated. Since a Gaussian density only depends on the second moments $\Rightarrow$ $x(u,1)$ and $x(u,3)$ independent. The variances of these two Gaussian random variables can be read directly from the (1,1) and (3,3) elements of $\mathbf{K_x}$ $\Rightarrow$

$$f_{\mathbf{x}(u)}(\mathbf{z}) = \delta(z_2 - (z_1+z_3))(2\pi\sqrt{2})^{-1} \exp\left(\frac{(z_1-1)^2}{2} + \frac{(z_3-2)^2}{4}\right).$$

Notice that we chose to express $x(u,2)$ in terms of the other components (i.e. we took $j=2$). We could have used one of the other components, but because of the uncorrelated nature of $x(u,1)$ and $x(u,3)$ it is easiest to choose $j=2$ in this case.

<u>Remark:</u> Don't get lost in the notation of this problem - and don't feel bad if you didn't get this problem on your own (it's hard the first time through). However, do be sure you understand the significance of this problem. If the covariance matrix is singular, then in some sense we over estimated the dimension of the random vector. We can reduce the dimension of the problem from n to r, the rank of the covariance matrix. The other significant point is that the component of the random vector in the direction of an e-vector corresponding to $\lambda = 0$ is zero with probability one.