Mean-Square Calculus Summary

For a continuous time process $x(u,t)$, we may consider limiting operations such as differentiation, integration, and continuity tests. As stochastic limits, these concepts must be tied to a particular mode of stochastic convergence (i.e., a process may be continuous in the mean square sense (mss), but not almost surely). This handout contains a brief summary of the results of "mean-square calculus" without the development. The book by Stark and Woods is a good engineering-level reference for this topic. Assume throughout that the process is real valued.

1 Mean-Square Continuity

Definition: Let a_n be any sequence such that $\lim_{n\to\infty} a_n = 0$, then $x(u,t)$ is ms-continuous at $t = t_0 \iff$

$$\lim_{n\to\infty} \mathbb{E}\left\{[x(u,t_0) - x(u,t_0+a_n)]^2\right\} = 0. \tag{1}$$

An equivalent statement is that $y_n(u,t_0) = x(u,t_0+a_n)$ converges in the mss. Since a_n is an arbitrary sequence going to zero, this is equivalent to the limit as a continuous parameter (i.e., Δ) goes to zero

$$\lim_{\Delta\to\infty} \mathbb{E}\left\{[x(u,t_0) - x(u,t_0+\Delta)]^2\right\} = 0. \tag{2}$$

Necessary and Sufficient Condition: Checking for convergence of $y_n(u,t_0)$ yields

$$x(u,t) \text{ is ms-cont. at } t = t_0 \iff R_x(t,t) \text{ is cont. at } t = t_0. \tag{3}$$

Note that a deterministic function $g(t)$ is continuous at $t = t_0$ if $\lim_{n\to\infty} g(t_0+a_n) = g(t_0)$ for all choices of a_n going to zero.

Wide-Sense Stationary Simplification: If $x(u,t)$ is WSS, then the condition for ms-continuity is

$$x(u,t) \text{ is ms-cont. at } \forall\, t \in \mathcal{R} \iff R_x(\tau) \text{ is cont. at } \tau = 0. \tag{4}$$

Note that a WSS process is either continuous at all times or discontinuous at all times.

2 Mean-Square Differentiability

Definition: Let a_n be any sequence such that $\lim_{n\to\infty} a_n = 0$ and define

$$y_n(t_0) = \frac{x(u,t_0) - x(u,t_0+a_n)}{a_n}, \tag{5}$$

then $x(u,t)$ is ms-differentiable at $t = t_0 \iff y_n(t_0)$ converges in the mss. The mss-limit of this sequence, when it exists, will be referred to as the mss-derivative of $x(u,t)$ and denoted by $\dot{x}(u,t_0) = \text{mss} - \lim_{n\to\infty} y_n(u,t_0)$.

Necessary and Sufficient Condition: Checking for convergence of $y_n(u, t_0)$ yields

$$x(u,t) \text{ is ms-diff'able at } t = t_0 \iff \frac{\partial^2}{\partial t_1 \partial t_2} R_x(t_1, t_2) \text{ exists at } t_1 = t_2 = t_0. \quad (6)$$

If $x(u,t)$ is ms-differentiable at two times t_1 and t_2, then the correlation between the two random variables $\dot{x}(u, t_1)$ and $\dot{x}(u, t_2)$ is

$$R_{\dot{x}}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_x(t_1, t_2). \quad (7)$$

Wide-Sense Stationary Simplification: If $x(u,t)$ is WSS, then the condition for ms-differentiability is

$$x(u,t) \text{ is mss-diff'able } \forall\, t \in \mathcal{R} \iff \frac{-d^2}{d\tau^2} R_x(\tau) \text{ exists at } \tau = 0. \quad (8)$$

If $x(u,t)$ is differentiable in the mss and WSS, then

$$R_{\dot{x}}(\tau) = \frac{-d^2}{d\tau^2} R_x(\tau). \quad (9)$$

Note that this may be thought of as putting $x(u,t)$ through an LTI filter with $H(f) = j2\pi f$, so that $S_{\dot{x}}(f) = -(j2\pi f)^2 S_x(f)$, which is equivalent to the above expression for the correlation function.

3 Mean-Square Integrability

A Riemann sum can be defined for which ms-convergence can be tested. Thus, the integral

$$z(u) = \int_a^b x(u,t)\, dt \quad \text{(mss-limit)} \quad (10)$$

exists if the Riemann sum converges in the mss. A necessary and sufficient condition for this convergence is

$$x(u,t) \text{ is ms-integrable on } [a,b] \iff \int_a^b \int_a^b R_x(t_1, t_2)\, dt_1 dt_2 < \infty. \quad (11)$$

It is clear that any finite power process is ms-integrable over a finite interval.

4 Example

A WSS process with $R_x(\tau) = e^{-a|\tau|}$ is ms-continuous and integrable over any finite length interval. However, since this correlation function is not differentiable at $\tau = 0$, the process is not ms-differentiable at any t.