K-L Expansion Summary – © Keith M. Chugg

"Properties"	Vector Case	Continuous Time $t \in [0,T]$
Assumptions: $\boldsymbol{x}(u)$ is random element of $\mathcal{S}_{\mathcal{T}}$	$\mathcal{S}_{\mathcal{T}} = \mathcal{C}^n$ $\mathbf{x}(u)$ is a second moment $(n \times 1)$ random vector $\mathbf{m}_{\mathbf{x}} = \mathbf{0}$	$\mathcal{S}_{\mathcal{T}} = \mathcal{L}_2([0,T]) = \left\{\boldsymbol{f} : \int_0^T \lvert f(t)\rvert^2 < \infty\right\}$ $x(u,t)$ is a second moment random waveform with: $x(u,t)$ continuous in the mss $\forall\ t \in [0,T]$ $\int_0^T \int_0^T \lvert K_x(t_1,t_2)\rvert^2 dt_1 dt_2 < \infty$ $m_x(t) = 0 \quad t \in [0,T]$
Inner product for $\mathcal{S}_{\mathcal{T}}$	$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{b}^\dagger \mathbf{a}$	$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_0^T f(t) g^*(t) dt$
Eigen-equation Real, non-negative e-values Orthonormal e-vectors form full basis for $\mathcal{S}_{\mathcal{T}}$	$\mathbf{K}_{\mathbf{x}} \mathbf{e}_k = \lambda_k \mathbf{e}_k \quad k = 1, 2, \ldots n$ $\mathbf{e}_k^\dagger \mathbf{e}_l = \delta_K(k-l) \quad \lambda_k \geq 0$ $\mathbf{a} \in \mathcal{C}^n \implies \mathbf{a} = \sum_{k=1}^n \alpha_k \mathbf{e}_k$ $\alpha_k = \mathbf{e}_k^\dagger \mathbf{a}$	$\int_0^T K_x(t_1,t_2) e_k(t_2) dt_2 = \lambda_k e_k(t_1) \quad t_1 \in [0,T],\ k = 1, 2, \ldots$ $\int_0^T e_k(t) e_l^*(t) dt = \delta_K(k-l) \quad \lambda_k \geq 0$ $\boldsymbol{f} \in \mathcal{L}_2([0,T]) \implies f(t) = \sum_{k=1}^\infty F_k e_k(t) \quad t \in [0,T]$ $F_k = \int_0^T f(t) e_k^*(t) dt$
KL-Expansion: Eigen expansion with mean zero, <u>uncorrelated</u> coefficients	$\mathbf{x}(u) = \sum_{k=1}^n X_k(u) \mathbf{e}_k$ $X_k(u) = \mathbf{e}_k^\dagger \mathbf{x}(u)$ $\mathbb{E}\{X_k(u)\} = 0$ $\mathbb{E}\{X_k(u) X_l^*(u)\} = \lambda_k \delta_K(k-l)$	$x(u,t) = \sum_{k=1}^\infty X_k(u) e_k(t) \quad t \in [0,T]$ (mss) $X_k(u) = \int_0^T x(u,t) e_k^*(t) dt$ (mss) $\mathbb{E}\{X_k(u)\} = 0$ $\mathbb{E}\{X_k(u) X_l^*(u)\} = \lambda_k \delta_K(k-l)$
Mercer's Theorem	$\mathbf{K}_{\mathbf{x}} = \sum_{k=1}^n \lambda_k \mathbf{e}_k \mathbf{e}_k^\dagger$	$K_x(t_1,t_2) = \sum_{k=1}^\infty \lambda_k e_k(t_1) e_k^*(t_2) \quad t_1, t_2 \in [0,T],$
Total Energy	$\mathbb{E}\left\{\lVert \mathbf{x}(u) \rVert^2\right\} = \mathrm{tr}\left(\mathbf{K}_{\mathbf{x}}\right) = \sum_{k=1}^n \lambda_k$	$\mathbb{E}\left\{\int_0^T \lvert x(u,t)\rvert^2 dt\right\} = \int_0^T K_x(t,t) dt = \sum_{k=1}^\infty \lambda_k$
Singularity and null directions	$\mathbf{K}_{\mathbf{x}}$ singular $\iff \mathbf{K}_{\mathbf{x}} \mathbf{a} = \mathbf{0} \quad \mathbf{a} \neq \mathbf{0}$ $\iff \lambda_m = 0$ for some $m \implies \mathbf{e}_m^\dagger \mathbf{x}(u) \stackrel{\text{as}}{=} 0$ $\mathbf{e}_m$ is the "null-direction"	$K_x(t_1,t_2)$ singular $\iff \int_0^T K_x(t_1,t_2) f(t_2) dt_2 = 0 \quad \boldsymbol{f} \neq \mathbf{0}$ $\iff \lambda_m = 0$ for some $m \implies \int_0^T x(u,t) e_k^*(t) = 0$ (mss) $e_m(t)$ is the "null-direction"