Random Processes Problems

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Probability - Self Test

1. Let $x(u)$ be a random variable. Define the cumulative distribution function (cdf), probability density function (pdf), characteristic function, mean and variance of $x(u)$.

2. **Independent vs. Uncorrelated Random Variables -** Let $x(u)$ and $y(u)$ be random variables.

   (a) What is the condition for $x(u)$ and $y(u)$ to be uncorrelated? What is the condition for $x(u)$ and $y(u)$ to be independent?

   (b) Show that $x(u)$ and $y(u)$ are independent *if and only if* $z(u) = g(x(u))$ and $w(u) = h(y(u))$ are uncorrelated random variables for every choice of the functions $g(\cdot)$ and $h(\cdot)$.

   Hint: For the "if" part consider the class of functions $g_\omega(x) = e^{j\omega x}$ for all real ω.

   (c) Part (b) implies that independent random variables are always uncorrelated, but that uncorrelated random variables are not always independent. Give an example where $x(u)$ and $y(u)$ are uncorrelated but not independent.

   (d) What special joint distribution of $x(u)$ and $y(u)$ has the property that independence is the same as being uncorrelated?

3. This problem develops bounds on tail probabilities. In the following a and ϵ are arbitrary positive constants and you may assume that the probability density function exists for all random variables.

   (a) Let $x(u)$ be a non-negative random variable (i.e. $\mathrm{Pr}\{x(u) < 0\} = 0$.) Prove Markov's inequality:

   $$\mathrm{Pr}\{x(u) \geq a\} \leq \frac{\mathbb{E}\{x(u)\}}{a}$$

   Hint: Start by writing down the expression for $\mathbb{E}\{x(u)\}$ and bound this expression from below.

   (b) Now let $y(u)$ be a random variable which may take negative values. Use Markov's inequality to develop the following three bounds:

   $$\text{Chebychev's:} \quad \mathrm{Pr}\{|y(u) - \mathbb{E}\{y(u)\}| \geq \epsilon\} \leq \frac{\mathbb{E}\{|y(u) - \mathbb{E}\{y(u)\}|^2\}}{\epsilon^2}$$

   $$\text{“Non-central Chebychev's:”} \quad \mathrm{Pr}\{|y(u)| \geq \epsilon\} \leq \frac{\mathbb{E}\{|y(u)|^2\}}{\epsilon^2}$$

   $$\text{Chernoff's:} \quad \mathrm{Pr}\{y(u) \geq a\} \leq \min_{\lambda>0} e^{-\lambda a}\mathbb{E}\left\{e^{\lambda y(u)}\right\}$$

   Hint: If $g(\cdot)$ is a strictly monotonically increasing function (i.e. $x > y \iff g(x) > g(y)$), then $\mathrm{Pr}\{x(u) \geq a\} = \mathrm{Pr}\{g(x(u) \geq g(a)\}$.

(c) What type of information about the random variable do you need to apply each of these bounds? Intuitively, which bound do you think is the most powerful? (i.e. which bound do you think is the tightest?)

4. If $\Pr\{x(u) = a\} = 1$ for some constant a, then what is the cdf and pdf of $x(u)$?

Linear Algebra - Self Test

5. Define a linear space (vector space), an inner product and a norm.

6. What is the usual inner product and norm for the linear space $\mathcal{R}^n$; the linear space of $n \times 1$ real vectors?

7. Let
$$\mathbf{K} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$
find the following:
   (a) The determinant of $\mathbf{K}$.
   (b) The transpose of $\mathbf{K}$, $\mathbf{K}^{\mathrm{t}}$.
   (c) The trace of $\mathbf{K}$, $\mathrm{tr}\,(\mathbf{K})$.
   (d) The rank of $\mathbf{K}$.
   (e) The inverse of $\mathbf{K}$, $\mathbf{K}^{-1}$.
   (f) The eigenvaules and eigenvectors of $\mathbf{K}$.

8. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathcal{R}^n$.
   (a) What is the rank of the $(n \times n)$ matrix $\mathbf{b}\mathbf{a}^{\mathrm{t}}$?
   (b) Show that
$$\mathbf{a}^{\mathrm{t}}\mathbf{b} = \mathrm{tr}\,(\mathbf{b}\mathbf{a}^{\mathrm{t}}).$$
   (c) Verify the results of (a) and (b) directly for
$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

9. Let $\mathbf{K}$ be the (2×2) invertable matrix
$$\mathbf{K} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
What is $\mathbf{K}^{-1}$?

10. Let $\mathbf{A}$ be an $(n \times n)$ invertible matrix and $\mathbf{D}$ be an $(m \times m)$ invertible matrix. Let $\mathbf{K}$ be the partitioned matrix

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}.$$

   (a) What are the dimensions of $\mathbf{K}$, $\mathbf{B}$ and $\mathbf{C}$?

   (b) Assuming that $\mathbf{K}^{-1}$ exists, verify (you need not derive this!) that it is given by

$$\mathbf{K}^{-1} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix},$$

   where the partitioning is the same as that for $\mathbf{K}$ and

$$\mathbf{E} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} \qquad \mathbf{F} = -\mathbf{E}\mathbf{B}\mathbf{D}^{-1}$$
$$\mathbf{G} = \mathbf{D}^{-1}\mathbf{C}\mathbf{E} \qquad \mathbf{H} = \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{E}\mathbf{B}\mathbf{D}^{-1} = (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}.$$

   (c) Does this reduce to your answer in problem 9 for $n = m = 1$?

Transform and LTI System Theory - Self Test

11. Define the Dirac delta function and the Kronecker delta function. What is the difference?

12. Define the following transforms and the the associated inverse

   (a) Fourier Transform
   (b) Laplace Transform (two-sided)
   (c) Discrete Time Fourier Transform (DTFT)
   (d) Z Transform (two-sided)
   (e) Fourier Series
   (f) Discrete Fourier Transform

13. Let $h(t)$ be the impulse response of a (continuous time) stable linear time invariant (LTI) system.

   (a) What is the system output when the input is $e^{j2\pi f_0 t}$?

   (b) Describe the region of convergence for $H(s)$, the Laplace transform of $h(t)$, when the system is *stable and causal*.

14. Let $h(n)$ be the impulse response of a (discrete time) stable LTI system.

(a) What is the system output when the input is $e^{j2\pi\nu_0 n}$?

(b) Describe the region of convergence for $H(z)$, the Z transform of $h(n)$, when the system is *stable and causal*.

15. **Amplitude and Phase Modulation:** In this problem you are asked to find the second moment description of two random processes (models for communication signals). You may assume that the *message*, $x(u,t)$, is statistically independent of the carrier phase, $\theta(u)$, for all values of $t \in \mathcal{T} = \mathcal{R}$. Also $\theta(u)$ is uniformly distributed on $[0, 2\pi]$ and $x(u,t)$ is a real random process.

    (a) An Amplitude Modulated (AM) signal has the following form

    $$z(u,t) = x(u,t)\cos(2\pi f_0 t + \theta(u)).$$

    Determine $m_z(t)$, $R_z(t_1,t_2)$ and $K_z(t_1,t_2)$ in terms of the second order description of the message.

    (b) A Phase Modulated (PM) signal has the form

    $$z(u,t) = \cos(2\pi f_0 t + \theta(u) + x(u,t)).$$

    Determine $m_z(t)$, $R_z(t_1,t_2)$ and $K_z(t_1,t_2)$.
    <u>Hint:</u> Express your answer in terms of the joint characteristic function of $x(u,t_1)$ and $x(u,t_2)$

    $$\Phi_{x(u,t_1),x(u,t_2)}(\omega_1,\omega_2) = \mathbb{E}\left\{e^{j(\omega_1 x(u,t_1)+\omega_2 x(u,t_2))}\right\}.$$

    (c) Describe the type of information you need about the message in order to specify the second order description of the AM and PM signals. What is the fundamental property of the modulation formats which caused this difference?
    <u>Hint:</u> Consider what would happen if the message was $x(u,t) + y(u,t)$.

    (d) Would your solution to (a) have been different if you were only given that $x(u,t)$ and $\theta(u)$ were uncorrelated for all values of t? What if you were told only that $x(u,t)$ and $w(u,t) = \cos(2\pi f_0 t + \theta(u))$ were uncorrelated random processes?

16. Let the complex random process, $z(u,t)$, be defined on $\mathcal{T} = \mathcal{R}$ with

    $$z(u,t) = x(u,t) + jy(u,t),$$

    where $x(u,t)$ and $y(u,t)$ are real random processes. Suppose that

    $$\begin{aligned} m_z(t) &= 0 \\ K_z(t_1,t_2) &= e^{\frac{-(t_1-t_2)^2}{4}}\left[1 + e^{\frac{-(t_1-t_2)^2}{4}}\right] \\ \widetilde{K}_z(t_1,t_2) &= e^{\frac{-(t_1-t_2)^2}{4}}\left[1 - e^{\frac{-(t_1-t_2)^2}{4}}\right] + 2je^{\frac{-(t_1-t_2)^2}{6}}. \end{aligned}$$

    Find the second moment description of $x(u,t)$ and $y(u,t)$ (including $K_{xy}(t_1,t_2)$).

17. Let the real random vector $\mathbf{x}(u)$ have the second moment description

$$\mathbf{m}_{\mathbf{x}} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \qquad \mathbf{K}_{\mathbf{x}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

In class we developed the "elliptical region bound" given by

$$\textsc{Pr}\left\{(\mathbf{x}(u) - \mathbf{m}_{\mathbf{x}})^{\mathrm{t}}\mathbf{K}_{\mathbf{x}}^{-1}(\mathbf{x}(u) - \mathbf{m}_{\mathbf{x}}) \geq 20\right\} \leq 0.10,$$

which allows us to plot a "90% confidence region."

(a) Sketch the corresponding 90% confidence region.

This is not the only possible 90% confidence region. Develop each of the bounds below, identifying C in each case, then plot the corresponding 90% confidence region for each.

(b) "Non-central elliptical region bound"

$$\textsc{Pr}\left\{\mathbf{x}^{\mathrm{t}}(u)\mathbf{R}_{\mathbf{x}}^{-1}\mathbf{x}(u) \geq C\right\} \leq 0.10$$

<u>Hint:</u> See the solution to Scholtz 16 (c) and (d).

(c) "Centered circular region bound"

$$\textsc{Pr}\left\{\|\mathbf{x}(u) - \mathbf{m}_{\mathbf{x}}\|^2 \geq C\right\} \leq 0.10$$

(d) "First component bound"

$$\textsc{Pr}\left\{|x(u,1) - m_x(1)| \geq C\right\} \leq 0.10$$

(e) "Principle axis bound"

$$\textsc{Pr}\left\{|\mathbf{b}^{\mathrm{t}}(\mathbf{x}(u) - \mathbf{m}_{\mathbf{x}})| \geq C\right\} \leq 0.10,$$

where the vector $\mathbf{b}$ is

$$\mathbf{b} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(f) What is the area of the 90% confidence region for each of the above bounds? Which appears to be the best?

<u>Hint:</u> The ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has area πab.

18. For this problem, let $\langle \mathbf{a}, \mathbf{b} \rangle$ denote *any valid inner product* on $\mathcal{R}^n$, and $\|\mathbf{b}\| = \sqrt{\langle \mathbf{b}, \mathbf{b} \rangle}$ denote the associated norm (i.e. these are *not* necessarily the standard Euclidean inner product and norm). Consider the standard real binary hypothesis testing problem

$$\mathcal{H}_i : \quad \mathbf{x}(u) = \mathbf{s}_i + \mathbf{n}(u) \quad i = 1, 2,$$

where $\mathbf{m_n} = \mathbf{0}$ and $\mathbf{K_n}$ is invertible.

Start with the *generalized minimum distance criterion* for the binary hypothesis testing problem:

$$\|\mathbf{x}(u) - \mathbf{s}_1\| \underset{\mathcal{H}_1}{\overset{\mathcal{H}_2}{\gtrless}} \|\mathbf{x}(u) - \mathbf{s}_2\|.$$

(a) Show that an equivalent decision is

$$\langle \mathbf{s}_1 - \mathbf{s}_2, \mathbf{x}(u) \rangle \underset{\mathcal{H}_2}{\overset{\mathcal{H}_1}{\gtrless}} \frac{\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2}{2}.$$

(b) Show that the following defines a valid inner product on $\mathcal{R}^n$

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{n}} \triangleq \mathbf{b}^{\mathrm{t}} \mathbf{K}_{\mathbf{n}}^{-1} \mathbf{a}.$$

(c) What is the decision rule corresponding to this choice of inner product? Do you recognize this rule?

(d) Sketch the locus of points which are unit distance from the origin with the distance function implied by the inner product in (18b). Consider the simple case of

$$\mathbf{K_n} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

In other words sketch the curve

$$d_{\mathbf{n}}(\mathbf{b}, \mathbf{0}) = 1,$$

where

$$d_{\mathbf{n}}(\mathbf{b}, \mathbf{0}) = \|\mathbf{b} - \mathbf{0}\|_{\mathbf{n}} = \left(\langle \mathbf{b}, \mathbf{b} \rangle_{\mathbf{n}}\right)^{1/2}.$$

19. Consider the following real binary hypothesis testing problem

$$\mathcal{H}_i : \quad \mathbf{x}(u) = \mathbf{s}_i + \mathbf{n}(u) \quad i = 1, 2,$$

where

$$\mathbf{m_n} = \mathbf{0} \qquad \mathbf{K_n} = \begin{bmatrix} 1 & (1-\epsilon) \\ (1-\epsilon) & 1 \end{bmatrix},$$

with $0 < \epsilon < 2$. The signals are given by

$$\mathbf{s}_1 = -\mathbf{s}_2 = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + 3\sqrt{2}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

<u>HINT:</u> I've given you the eigenvector expansion.

(a) Determine a good decision rule.

(b) Describe the behavior of your decision rule as $\epsilon \to 0$. What happens as $\epsilon \to 2$?

20. In the binary hypothesis testing problem considered in class, the signals are given. This problem considers the case in which we can *design* the signals to optimize performance. Consider the standard (real) hypothesis testing problem:

$$\mathcal{H}_i : \quad \mathbf{x}(u) = \mathbf{s}_i + \mathbf{n}(u) \quad i = 1, 2,$$

where $\mathbf{m_n} = \mathbf{0}$ and $\mathbf{K_n}$ is invertible, but we are allowed to choose the signal vectors. We impose an *energy constraint* on the signals; i.e. we require

$$\|\mathbf{s}_i\|^2 = \mathbf{s}_i^{\mathrm{t}}\mathbf{s}_i = E, \quad i = 1, 2.$$

(a) Determine the choice(s) of $\mathbf{s}_1$ and $\mathbf{s}_2$ which minimize the error probability bound given in class.
<u>HINT:</u> Write $\mathbf{s}_1 - \mathbf{s}_2 = M\mathbf{b}$, where

$$M = \|\mathbf{s}_1 - \mathbf{s}_2\| \qquad \mathbf{b} = \frac{\mathbf{s}_1 - \mathbf{s}_2}{M},$$

then perform the minimization with respect to M and $\mathbf{b}$ separately.

(b) Does your solution to (a) insure that the probability of error is minimized? Explain.

(c) What is the resulting minimum value of the bound?

(d) Consider the minimum value of the bound found in the previous part under the condition of a fixed signal to noise ratio (SNR):

$$\mathrm{SNR} = \frac{E}{\mathbb{E}\left\{\|\mathbf{n}(u)\|^2\right\}} = \text{constant}.$$

Under this assumption, describe what appears to be the "worst case" $\mathbf{K_n}$.
<u>HINT:</u> As a systems engineer, if you design the signals according to part (a), which noise covariance matrix would you prefer:

$$\mathbf{K_n} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{or} \quad \mathbf{K_n} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}?$$

21. Describe *in words* why the inner product and associated distance function defined in problem 18 are appropriate for colored noise hypothesis testing.

22. **Review Problem:** There seems to be some confusion regarding the difference between the simulation and representation problems; this simple problem is intended to clarify the difference.

    Let $\mathbf{x}(u)$ be a mean zero random vector, with covariance matrix $\mathbf{K_x} = \mathbf{H}\mathbf{H}^\dagger = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^\dagger$. We can simulate (up to the second moment description) $\mathbf{x}(u)$ by

    $$\mathbf{y}(u) = \mathbf{H}\mathbf{w}(u),$$

    where $\mathbf{w}(u)$ is a white random vector, i.e. $\mathbf{m_w} = \mathbf{0}$ and $\mathbf{K_w} = \mathbf{I}$. Assume that $\mathbf{x}(u)$ and $\mathbf{w}(u)$ are uncorrelated (i.e. $\mathbf{K_{xw}} = \mathbf{O}$).

    We can represent $\mathbf{x}(u)$ by the following random vector (i.e. the K-L expansion)

    $$\mathbf{z}(u) = \sum_{k=1}^{n} a_k(u)\mathbf{e}_k \qquad a_k(u) = \mathbf{e}_k^\dagger \mathbf{x}(u) \qquad k = 1, 2 \dots n.$$

    $$\mathbf{z}(u) = \mathbf{E}\mathbf{a}(u) \qquad\qquad \mathbf{a}(u) = \begin{bmatrix} a_1(u) \\ a_2(u) \\ \vdots \\ a_n(u) \end{bmatrix} = \mathbf{E}^\dagger \mathbf{x}(u),$$

    where we have written the n scalar equations in vector form.

    (a) Determine $\mathbb{E}\left\{\|\mathbf{x}(u) - \mathbf{y}(u)\|^2\right\}$ and $\mathbb{E}\left\{\|\mathbf{x}(u) - \mathbf{z}(u)\|^2\right\}$ in terms of $\mathbf{K_x}$.

    (b) Answer the following:
    - Are $\mathbf{x}(u)$ and $\mathbf{y}(u)$ equivalent in the wide sense? i.e. Is $\mathbf{x}(u) \overset{\text{ws}}{=} \mathbf{y}(u)$?
    - Are $\mathbf{x}(u)$ and $\mathbf{y}(u)$ equal with probability 1? i.e. Is $\mathbf{x}(u) \overset{\text{as}}{=} \mathbf{y}(u)$?
    - Are $\mathbf{x}(u)$ and $\mathbf{z}(u)$ equal with probability 1? i.e. Is $\mathbf{x}(u) \overset{\text{as}}{=} \mathbf{z}(u)$?
    - Are $\mathbf{x}(u)$ and $\mathbf{z}(u)$ equivalent in the wide sense? i.e. Is $\mathbf{x}(u) \overset{\text{ws}}{=} \mathbf{z}(u)$?

    (c) Does it make sense to simulate $\mathbf{x}(u)$ from $\mathbf{z}(u)$? Explain.

    (d) What is the LMMSE estimate of $\mathbf{x}(u)$ based on the observation $\mathbf{y}(u)$? What is the associated minimum MSE?

    (e) What is the LMMSE estimate of $\mathbf{x}(u)$ based on the observation $\mathbf{z}(u)$? What is the associated minimum MSE?

23. Let $\mathbf{w}(u)$ be a real Gaussian random vector, with second moment description $\mathbf{m_w} = \mathbf{0}$ and $\mathbf{K_w} = \mathbf{I}$. Let $\mathbf{x}(u)$ be generated by

    $$\mathbf{x}(u) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{w}(u) + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Independence

i. Are $x(u,1)$ and $x(u,2)$ independent?

ii. Are $x(u,1)$ and $x(u,3)$ independent?

iii. Are $x(u,2)$ and $x(u,3)$ independent?

iv. Are $x(u,1)$, $x(u,2)$ and $x(u,3)$ mutually independent?

(b) What is the pdf of $\mathbf{x}(u)$?

(c) What is the pdf of $x(u,k)$ as a function of $k \in \{1,2,3\}$?

(d) What is the $\mathbb{E}\{x(u,1)|x(u,2)\}$?

(e) What is the $\mathbb{E}\{x(u,1)|x(u,3)\}$?

24. **Recursive Estimation - A Simple Kalman Filter**
What if we have a sequence of observations, $\{x(u,i)\}_{i=1}^{\infty}$, and we would like to estimate an n-dimensional random vector, $\mathbf{v}(u)$? Suppose that we know the best estimate of $\mathbf{v}(u)$ based on the observations $\{x(u,i)\}_{i=1}^{k}$ and we now observe $x(u,k+1)$: Do we need start over and solve the new (larger dimensional) estimation problem, or can we somehow update the estimate to account for the new information provided by $x(u,k+1)$? This is the subject of this problem.

Let $\mathbf{v}(u)$ be an n-dimensional mean zero, Gaussian random vector. Let the i^{th} observation be the zero mean, Gaussian random variable $x(u,i)$ and consider the estimation problem described above. You may assume that $\mathbf{v}(u)$ and $\{x(u,i)\}_{i=1}^{\infty}$ are jointly Gaussian. Denote the $(k \times 1)$ vector of observations by

$$\mathbf{x}_k(u) \triangleq \begin{bmatrix} x(u,k) \\ x(u,k-1) \\ \vdots \\ x(u,1) \end{bmatrix},$$

and denote the unconstrained MMSE estimate of $\mathbf{v}(u)$ based on the k observations by

$$\hat{\mathbf{v}}_k(u) \triangleq \mathbb{E}\{\mathbf{v}(u)|x(u,k), x(u,k-1)\ldots x(u,1)\} = \mathbb{E}\{\mathbf{v}(u)|\mathbf{x}_k(u)\}.$$

The following shorthand notation is also useful:

$$\begin{aligned}
\mathbf{r}_{x\mathbf{v}}(i) &\triangleq \mathbb{E}\{x(u,i)\mathbf{v}(u)\} && (n \times 1) \\
\mathbf{R}_{\mathbf{vx}}(i) &\triangleq \mathbb{E}\{\mathbf{v}(u)\mathbf{x}_i^{\mathrm{t}}(u)\} && (n \times i) \\
\sigma_x^2(i) &\triangleq R_x(i,i) = \mathbb{E}\{x(u,i)^2\} && (1 \times 1) \\
\mathbf{r}_{\mathbf{x}}(i+1) &\triangleq \mathbb{E}\{x(u,i+1)\mathbf{x}_i(u)\} && (i \times 1) \\
\mathbf{R}_{\mathbf{x}}(i) &\triangleq \mathbb{E}\{\mathbf{x}_i(u)\mathbf{x}_i^{\mathrm{t}}(u)\} && (i \times i).
\end{aligned}$$

You may assume that $\mathbf{R}_{\mathbf{x}}(i)$ is invertible for all i.

(a) Show that we can update the estimate as follows:

$$\hat{\mathbf{v}}_{k+1}(u) = \hat{\mathbf{v}}_k(u) + \mathbf{g}(k+1)\left[x(u,k+1) - \mathbf{r}_{\mathbf{x}}^{\mathrm{t}}(k+1)\mathbf{R}_{\mathbf{x}}^{-1}(k)\mathbf{x}_k(u)\right],$$

and identify the $(n \times 1)$ *Kalman Gain Vector*, $\mathbf{g}(k+1)$, in terms of the above quantities.

<u>HINT:</u>

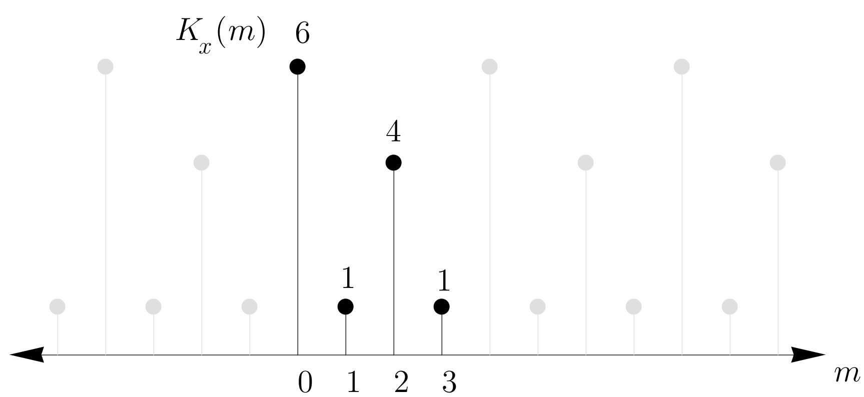
- What does the fact that the observations and $\mathbf{v}(u)$ are jointly Gaussian imply?
- Use partitioned matrices to express $\hat{\mathbf{v}}_{k+1}(u)$, then apply the results of Problem 10.

(b) The term

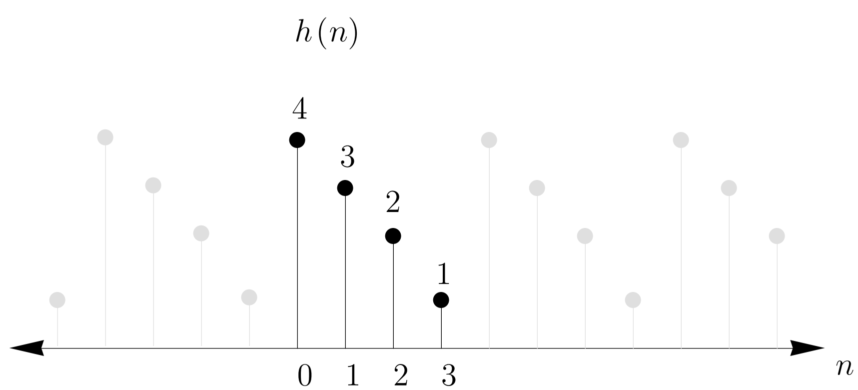
$$\left[x(u,k+1) - \mathbf{r}_{\mathbf{x}}^{\mathrm{t}}(k+1)\mathbf{R}_{\mathbf{x}}^{-1}(k)\mathbf{x}_k(u)\right]$$

is called the "innovation" provided by the $(k+1)^{th}$ observable; describe the meaning of this term.

25. Let $x(u,n)$ be a wide-sense-stationary random process defined on the index set $\mathcal{Z}_4$. The covariance function of $x(u,n)$ is $K_x(m)$, which is sketched below:



The mean of $x(u,n)$ is $m_x = 2$. $x(u,n)$ is passed through an LTI system (on $\mathcal{Z}_4$) with impulse response $h(n)$, defined by:



Denote the output process by $y(u, n)$.

(a) What are the eigenvalues of the covariance function of $y(u, n)$, corresponding to the standard e-vectors for circulant matrices.

(b) What is $K_y(n_1, n_2)$ for $n_1, n_2 \in \mathcal{Z}_4$?

(c) What is m_y?

(d) Verify your solution to parts (b) and (c) directly by solving for $\mathbf{K_y}$ and $\mathbf{m_y}$, the second moment description of the corresponding random vector, using the methods developed in the first half of the course (i.e. $\mathbf{K_y} = \mathbf{H K_x H}^{\mathrm{t}}$ and $\mathbf{m_y} = \mathbf{H m_x}$).

26. Prove that convergence in the mean-square sense implies convergence in probability.

Hint: Use the "Non-central Chebychev Bound."

27. State each of the following results from your probability theory class as $y_n(u) \to y(u)$ as $n \to \infty$; identifying in each case, $\{y_n(u)\}$, $y(u)$, and the mode of stochastic convergence (i.e. the sense of the limit).

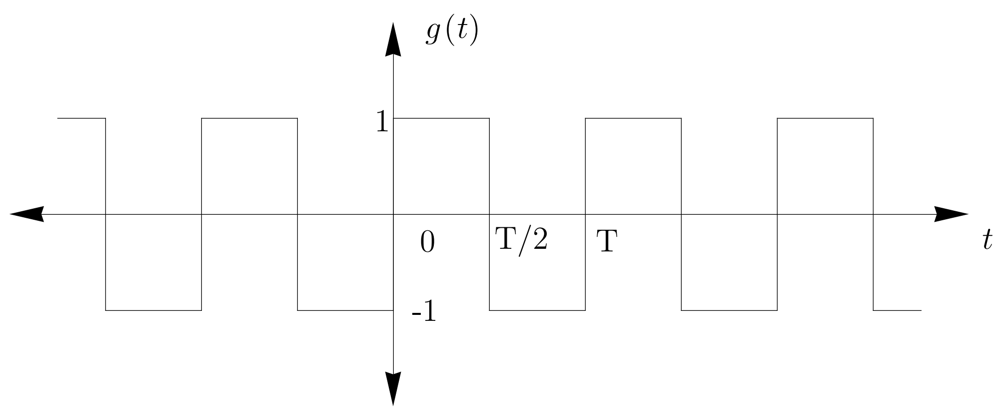
(a) The Weak Law of Large Numbers

(b) The Strong Law of Large Numbers

(c) The Central Limit Theorem

(d) In what sense is the Strong Law of Large Numbers "stronger" than the Weak Law of Large Numbers?

28. Let $g(t)$ be the deterministic, periodic square-wave, sketched below:



Define the random process $y(u, t) = g(t + \theta(u))$, where $\theta(u)$ is uniformly distributed over $[0, T]$. Find the second moment description of $y(u, t)$.

HINT: You should find that $y(u, t)$ is wide-sense-stationary on the index set $\mathcal{R}$.

29. The Central Limit Theorem (CLT) deals with the sequence $\{x_n(u)\}$ defined by

$$x_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{w_i(u) - m}{\sigma} \right),$$

where $\{w_i(u)\}$ is a sequence of independent, identically distributed random variables, each with mean m and variance σ^2. The CLT states that this sequence converges in distribution to a mean zero, unit variance Gaussian random variable. Does this sequence converge in the mean square sense? Do you think that this sequence converges in probability (use "non-mathematical" reasoning).

30. Suppose that $x(u,t)$ is a Gaussian random process input into a linear system. Assume that we know that the output of this system, $y(u,t)$, exists in the mean-square-sense for all values of t. Explain (prove) that $y(u,t)$, the mss limit process, is also Gaussian.

HINT: $y(u,t)$ is the mss limit of $y_n(u,t)$ as $n \to \infty$, where $y_n(u,t)$ is a linear combination of a *finite* number of Gaussian random variables.

31. **Hypothesis testing in discrete time:** (Modified Final Exam Problem - Fall-92 - Hinedi/Chugg).

Consider the task of deciding between two hypotheses regarding signals defined on $\mathcal{T} = \mathcal{Z}$:

$$\begin{aligned} \mathcal{H}_0 : \quad & x(u,n) = s_0(n) + w(u,n) \\ \mathcal{H}_1 : \quad & x(u,n) = s_1(n) + w(u,n), \end{aligned}$$

where the deterministic signals are defined by

$$s_i(n) = \begin{cases} \sqrt{P}\cos(\pi(n+i)) & n \in \{0,1\} \\ 0 & n = \ldots -2, -1, 2, 3 \ldots \end{cases} \qquad i = 0, 1,$$

and the (real) Gaussian noise process, $w(u,n)$, has PSD given by

$$S_w(\nu) = \frac{\sigma^2(1-\rho^2)}{1 - 2\rho\cos(2\pi\nu) + \rho^2},$$

where $|\rho| < 1$.

(a) Based on observing only $x(u,n)$ for $n = 0, 1$ design a good rule for deciding which Hypothesis is true.

(b) What is the bound on the probability of error?

(c) Determine an exact expression for the probability of error. Your answer should involve the Q function:

$$Q(z) = \int_z^\infty \mathcal{N}_1(x;0;1)dx = \int_z^\infty \frac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi}} dx.$$

(d) Using the fact that $Q(3) \approx 0.001$, and assuming that the noise is white (i.e. $\rho = 0$), what is the minimum signal to noise ratio (SNR$= \frac{P}{\sigma^2}$) required to ensure that the error probability is at most 1/1000?

(e) Discuss the performance of this system as ρ varies. Do you see the relation to Problem 20?

(f) What is the conditional pdf of $w(u,0), w(u,1)$ given $w(u,k)$, where k is an integer other than 0 or 1? In other words, determine

$$f_{w(u,0),w(u,1)|w(u,k)}(z_0, z_1|z_k).$$

With this result, intuitively would you expect better performance if the observation interval was longer (i.e. observe $x(u,n)$ for $n = 0, 1, 2 \dots k$)? Discuss this for white and colored noise.

32. Let $x(u,n)$ be a wide-sense-stationary (real) Gaussian random sequence with zero mean. Determine which of the following functions are valid correlation functions for $x(u,n)$:

$$R_a(m) = \begin{cases} 1 & m = -1, 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$R_b(m) = \begin{cases} |m| + 1 & |m| \leq 3 \\ 3 - ||m| - 3| & |m| = 4, 5, 6, 7 \\ 0 & |m| > 7. \end{cases}$$

$$R_c(m) = \left(\frac{1}{2}\right)^{|m|} \cos\left(\frac{\pi}{4} m\right).$$

$$R_d(m) = \left(\frac{1}{3}\right)^{|m|} \sin\left(\frac{\pi}{8} m\right).$$

For each function which is a valid correlation function, determine the pdf of $x(u,n)$ and the conditional pdf of $x(u,n+1)$ given $x(u,n)$.

33. **Gain Control! on $\mathcal{T} = \mathcal{Z}$:** Your task is to obtain an estimate of $s(u,n)$ from the noisy observation $x(u,n)$ given by

$$x(u,n) = s(u,n) + w(u,n),$$

where $s(u,n)$ and $w(u,n)$ are independent processes. Let the correlation function of $s(u,n)$ be $R_s(m) = \frac{4}{3}\left(\frac{1}{2}\right)^{|m|}$. Assume that the noise, $w(u,n)$, is unit variance white noise.

(a) Find the optimal LMMSE (Wiener) filter (possibly non-causal) - $G_{\text{opt}}(\nu)$.

(b) Find the best Causal filter - $G_{\text{opt,C}}(\nu)$.

HINT: The following factorization is easy to obtain:

$$\frac{9}{2} - (z + z^{-1}) = \frac{1}{q}(1 - qz^{-1})(1 - qz),$$

where $q = \frac{1}{4}(9 + \sqrt{65})$.

(c) Suppose that due to cost and complexity considerations, you decided to try a simple estimator with a *fixed gain structure*. In other words you constrain your estimate to be of the form $ax(u, n)$, where a is a real constant. What is the best choice of a? What is the corresponding estimation filter impulse response?

(d) Without performing any calculations, rank the performance of these three estimators.

34. A *hard-limiter* is a system (operating on continuous time signals) with the following input/output characteristic

$$\tilde{\mathbf{y}} = \mathbb{H}\tilde{\mathbf{x}} \iff y(t) = \text{sgn}(x(t)) = \begin{cases} +1 & \text{if } x(t) > 0 \\ 0 & \text{if } x(t) = 0 \\ -1 & \text{if } x(t) < 0. \end{cases}$$

(a) Is this system
- Linear?
- Time-Invariant?
- Causal?

(b) Define the wss random process $x(u, t) = A\sin(2\pi f_0 t + \phi(u))$, where $\phi(u)$ is uniform on $[-\pi, \pi)$ and A is a positive real constant. Determine the PSD of $x(u, t)$.

(c) Let $y(u, t)$ be the output of the hard-limiter when $x(u, t)$ is the input. Find the PSD of $y(u, t)$.

HINT: Determine how Problem 28 is related to this problem.

(d) Describe how your answers to (b) and (c) are related to (a) - i.e., can you tell if this system is LTI based on the input/output PSD's?

35. **PSD of AM Signal** Recall the definitions of Amplitude (AM) and Phase Modulated Signals from problem 3 of HW # 1:

$$\text{AM: } z(u, t) = x(u, t)\cos(2\pi f_0 t + \theta(u))$$
$$\text{PM: } z(u, t) = \cos(2\pi f_0 t + x(u, t) + \theta(u)),$$

where $\theta(u)$ is uniform on $[0, 2\pi)$ and independent of the message process $x(u, t)$. Assume that the message is wide-sense stationary with correlation function

$$R_x(\tau) = \text{trian}\left(\frac{f_0\tau}{1000}\right).$$

(a) What is the mean of $x(u,t)$?

(b) The AM signal is wide-sense-stationary, is the PM signal?

(c) Find and sketch the PSD of the AM signal.

36. **PSD of a Digital (PAM) Signal:** Let $x(u,t)$ be the continuous time random process defined by

$$x(u,t) = \sum_{k=-\infty}^{\infty} a_k(u) p(t - kT - \theta(u)),$$

where the limit is in the mss and the following conditions hold

$\{a_k(u)\}_{k=-\infty}^{\infty}$ is a sequence of iid random variables with mean 0, and variance σ_a^2,
$\theta(u)$ is a uniformly distributed on $[0,T)$ and is independent of $a_k(u)$ for all k,
$p(t)$ is a deterministic pulse shape.

(a) Determine the mean and correlation function of $x(u,t)$ in terms of the given parameters.

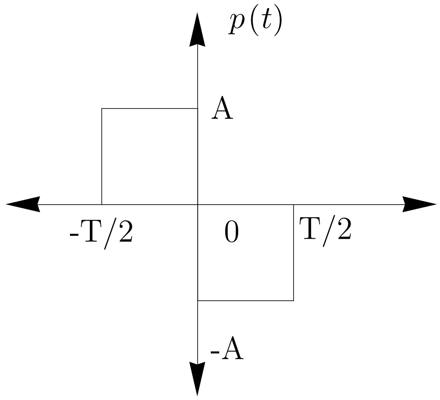
<u>HINT:</u> Your answer should involve the "auto-correlation of the pulse" defined as

$$E_p(\tau) \triangleq p(\tau) * p^*(-\tau) = \int_{-\infty}^{\infty} p(\tau - w) p^*(-w) dw = \int_{-\infty}^{\infty} p(w) p^*(w - \tau) dw.$$

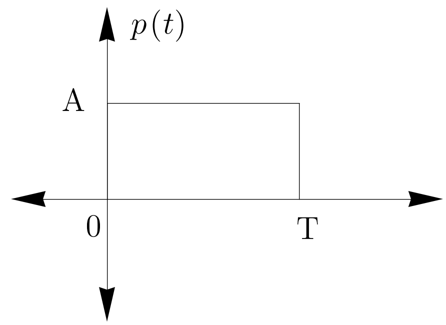
(b) Determine the PSD of $x(u,t)$ in term of the given parameters and $P(f) = \mathbb{FT}\{p(t)\}$.

<u>HINT:</u> Use the "autocorrelation" property of the Fourier Transform.

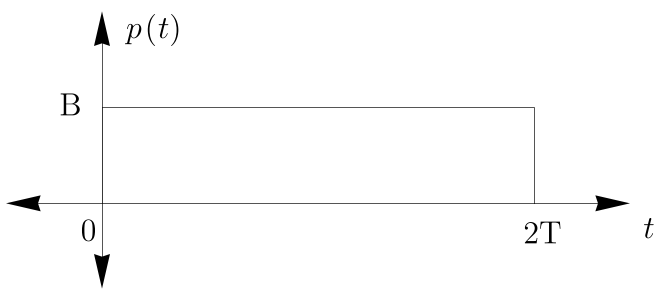
(c) Find and sketch the PSD of $x(u,t)$ for the three pulses shown below:



(i)



(ii)



(iii)

Here $A = \frac{1}{\sqrt{T}}$ and $B = \frac{1}{\sqrt{2T}}$.

37. Consider an LTI system, $\mathbb{M}$, characterized by the following differential equation:

$$\tilde{\mathbf{y}} = \mathbb{M}\tilde{\mathbf{x}} \iff \ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 3x(t) - \dot{x}(t). \tag{1}$$

A continuous time, wss random process, $x(u,t)$, with correlation function

$$R_x(\tau) = \frac{1}{8}e^{-4|\tau|}$$

is passed through the above LTI system, with the output denoted by $y(u,t)$.

(a) What is the frequency response of the system - i.e. $M(f)$?

(b) Determine $S_y(f)$ and $S_{xy}(f)$.

(c) Determine the optimal (Wiener) causal filter for estimating $x(u,t)$ from $y(u,t)$; specify the frequency response of this filter.

(d) What is the PSD of the best estimate, $\hat{x}(u,t)$, in terms of $S_x(f)$?
<u>Hint:</u> Determine the frequency response of the cascade of $M(f)$ and the Wiener filter found in the previous part.

(e) What is the associated MMSE of this estimator?

(f) Explain how your solution would change if the right-hand side of (1) were $3x(t) + \dot{x}(t)$. What is the system characteristic which is changed by this sign change.

38. **A "`randumb`" Simulation** (Midterm, Summer 1993). Mr. Plug N. Chugg has designed a new signal processing algorithm which operates on the real Gaussian random vector, $\mathbf{x}(u)$; defined on the index set $\mathcal{T} = \{1, 2, 3\}$. He has analyzed his algorithm, but must verify his results through computer simulation. From the analysis, Plug knows that

$$\mathbf{K_x} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 30 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ 1 & 0 & 0 \\ 0 & \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 12 & 6 \\ 0 & 6 & 28 \end{bmatrix}$$

$$\mathbf{m_x} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Plug doesn't remember much from EE562a, but he does have a copy of the "Supplemental Notes." From the notes he figures that he can simulate the second moment statistics of $\mathbf{x}(u)$ by producing

$$\mathbf{y}(u) = \mathbf{H}\mathbf{w}(u) + \mathbf{c},$$

from any real random vector $\mathbf{w}(u)$ with

$$\mathbf{K_w} = \mathbf{I} \qquad \mathbf{m_w} = \mathbf{0}.$$

He decides that, if he can generate $\mathbf{w}(u)$ on the computer, he can design $\mathbf{H}$ and $\mathbf{c}$ so that $\mathbf{x}(u) \overset{\text{ws}}{=} \mathbf{y}(u)$.

Plug's computer has a subroutine, `randumb(a,b)`, which returns a random variable which is uniformly distributed on $[a, b]$. Successive calls to `randumb(a,b)` produce uncorrelated random variables.

(a) To review his EE562a material, Plug first decides to obtain the second moment description of

$$\mathbf{z}(u) = \begin{bmatrix} 2x(u,1) \\ x(u,3) \end{bmatrix}.$$

Determine the second moment description of $\mathbf{z}(u)$; i.e., $\mathbf{m_z}$ and $\mathbf{K_z}$.

Since $\mathbf{K_x}$ is non-singular, Plug decides to produce the components of $\mathbf{w}(u)$ by 3 successive calls to `random(a,b)`. Determine numerical values for the determinant of $\mathbf{K_x}$ and the values of a and b which should be used so that $\mathbf{m_w} = \mathbf{0}$ and $\mathbf{K_w} = \mathbf{I}$.

(b) Since it is possible to design $\mathbf{H}$ to be causal, Plug decides to do so. Give numerical values for $\mathbf{c}$ and lower triangular $\mathbf{H}$, so that $\mathbf{m_y} = \mathbf{m_x}$ and $\mathbf{K_y} = \mathbf{K_x}$.

(c) Plug reasons that he can also express $\mathbf{x}(u)$ exactly using

$$\mathbf{x}(u) = a_1(u)\begin{bmatrix} 0 \\ \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} + a_2(u)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_3(u)\begin{bmatrix} 0 \\ \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

Determine the following:

- $\mathbb{E}\left\{(a_2(u))^2\right\}$
- $\mathbb{E}\left\{a_1(u)a_3(u)\right\}$
- $f_{a_1(u)|a_2(u)}(z_1|z_2)$

(d) Give a set of orthonormal eigenvectors and the corresponding eigenvalues of $\mathbf{R_x}$, the correlation matrix of $\mathbf{x}(u)$.

(e) Answer the following questions:

- Is $\mathbf{y}(u)$ Gaussian? – YES NO MAYBE
- Are $\mathbf{y}(u)$ and $\mathbf{x}(u)$ equal *almost surely*? – YES NO MAYBE
- Are the components of $\mathbf{w}(u)$ independent? – YES NO MAYBE
- Are $a_1(u)$ and $a_3(u)$ independent? – YES NO MAYBE

(f) After performing the computer simulation described above, Plug finds that the results do not agree with his analysis. He asks you for help; explain why his results disagree and suggest a method for improving his simulation.

39. **Unexpected Results?** (Midterm, Summer 1993) A certain (real valued) communication channel is represented by

$$\mathbf{x}(u) = \mathbf{s}(u) + \mathbf{n}(u),$$

where the desired signal, $\mathbf{s}(u)$, takes on only two possible values

$$\Pr\{\mathbf{s}(u) = \mathbf{a}\} = \Pr\{\mathbf{s}(u) = -\mathbf{a}\} = \frac{1}{2}.$$

Numerical values for this system model are

$$\mathbf{m_n} = \mathbf{0} \qquad \mathbf{K_n} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

You may assume that the signal and noise vectors are statistically independent.

The goal is to design a good estimator of $\mathbf{s}(u)$ based on $\mathbf{x}(u)$. There are two approaches: (1) design a minimum mean-squared error estimator or (2) frame this as a binary hypothesis testing problem.

(a) Give numerical answer for $\mathbf{m_s}$, $\mathbf{K_s}$, $\mathbf{K_{sx}}$, and $\mathbf{K_x}$.

(b) Determine the best Linear Minimum Mean Squared Error (LMMSE) Estimator of $\mathbf{s}(u)$ based on $\mathbf{x}(u)$ and the associated value of the MSE – denote this estimate by $\hat{\mathbf{s}}(u)$

(c) Since $\mathbf{s}(u)$ takes on only two vector values, the problem can be formulated as binary hypothesis testing problem. Design a good rule for deciding between the following two hypotheses

$$\begin{aligned} \mathcal{H}_1 : \quad & \mathbf{s}(u) = \mathbf{a} \\ \mathcal{H}_2 : \quad & \mathbf{s}(u) = -\mathbf{a}. \end{aligned}$$

This decision rule yields an estimate of $\mathbf{s}(u)$ (denote this by $\hat{\mathbf{s}}_{HT}(u)$) in the following sense

$$\hat{\mathbf{s}}_{HT}(u) = \begin{cases} \mathbf{a} & \text{if we decide } \mathcal{H}_1 \text{ is true,} \\ -\mathbf{a} & \text{if we decide } \mathcal{H}_2 \text{ is true.} \end{cases}$$

What is the associated mean square error of this estimator? – i.e., $\mathbb{E}\left\{\|\mathbf{s}(u) - \hat{\mathbf{s}}_{HT}(u)\|^2\right\}$

(d) Answer the following and provide a brief explanation (or work) for each.

- Are $\hat{\mathbf{s}}(u)$ and $\hat{\mathbf{s}}_{HT}(u)$ the same? – YES NO MAYBE
- $\mathbb{E}\left\{\mathbf{s}(u)|\mathbf{x}(u)\right\} =$
- Are $\mathbf{x}(u)$ and $\mathbf{s}(u)$ jointly Gaussian? – YES NO MAYBE

40. **Gain Control!** (Midterm, Summer 1993) You are faced with the following design task: Design an estimator of $\mathbf{s}(u)$ from the observation $\mathbf{x}(u)$, given by:

$$\mathbf{x}(u) = \mathbf{H}\mathbf{s}(u) + \mathbf{n}(u),$$

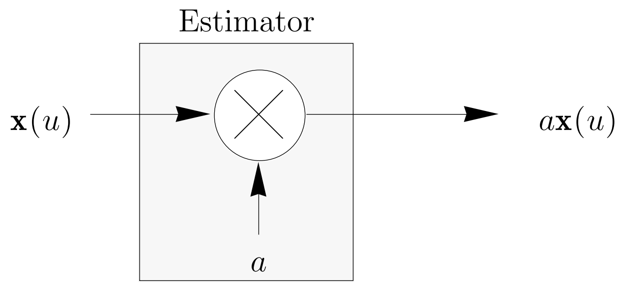
where $\mathbf{s}(u)$ and $\mathbf{n}(u)$ are uncorrelated real Gaussian random vectors defined on the index set $\mathcal{T} = \{1, 2\}$. The numerical values associated with this model are

$$\mathbf{m_n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{K_n} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\mathbf{m_s} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{K_s} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Due to cost and complexity constraints, you must design your estimator to have the following *fixed gain structure*:



where a is a real number.

(a) Determine the second moment description of $\mathbf{x}(u)$ and $\mathbf{s}(u)$; i.e., $\mathbf{m_x}$, $\mathbf{R_s}$, $\mathbf{R_x}$, and $\mathbf{R_{sx}}$.

(b) Determine a_{opt}, the value of a which optimizes the estimator illustrated above (i.e. the value of a which minimizes the mean squared error).

- What is the associated minimum MSE estimate with this constrained structure?
- What is the associated minimum value of the MSE for this constrained structure?
- What is the probability density function of the associated error vector, $\mathbf{e}(u) = \mathbf{s}(u) - \hat{\mathbf{s}}(u)$?

(c) Answer the following questions regarding the estimator derived in part (b).

- Is this a causal estimator of $\mathbf{s}(u)$? – YES NO MAYBE
- Is this a linear estimator of $\mathbf{s}(u)$? – YES NO MAYBE
- Is $\hat{\mathbf{s}}(u)$ a biased estimate of $\mathbf{s}(u)$? – YES NO MAYBE

Rank this estimator by filling each blank below with "=", "≥", or "≤".

MSE(Estimator in (b)) $\Bigg(\quad \Bigg)$ MSE(Linear MMSE Estimator)

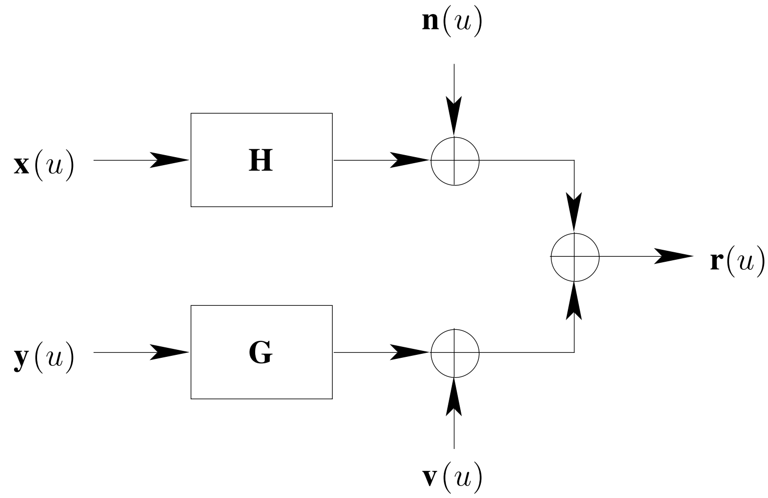
MSE(Estimator in (b)) $\Bigg(\quad \Bigg)$ MSE(Linear, Causal MMSE Estimator)

MSE(Estimator in (b)) $\Bigg(\quad \Bigg)$ MSE(Affine MMSE Estimator)

MSE(Estimator in (b)) $\Bigg(\quad \Bigg)$ MSE(Affine, Causal MMSE Estimator)

MSE(Estimator in (b)) $\Bigg(\quad \Bigg)$ MSE(Unconstrained MMSE Estimator)

41. **Multi-User Communication** (Midterm, Fall 1994) The "mobile-to-hub" link of a two-user communication system is modeled as shown below:



All signals are modeled as real random processes defined on index set $\mathcal{T} = \{1, 2\}$, which represents real time. The random vectors $\mathbf{x}(u)$ and $\mathbf{y}(u)$ represent the signals of two different users. Since the users are mobile, the channels to the hub are different (i.e., $\mathbf{H} \neq \mathbf{G}$ and $\mathbf{n}(u) \neq \mathbf{v}(u)$). It is also known that $\mathbf{m_x} = \mathbf{m_y} = \mathbf{m_v} = \mathbf{m_n} = \mathbf{0}$, and that the random vectors $\mathbf{x}(u)$, $\mathbf{y}(u)$, $\mathbf{n}(u)$, and $\mathbf{v}(u)$ are all mutually independent. It may also be assumed that the two noise vectors are white, that is $\mathbf{K_n} = \sigma_n^2\mathbf{I}$ and $\mathbf{K_v} = \sigma_v^2\mathbf{I}$ (with $\sigma_v^2 \neq 0$ and $\sigma_n^2 \neq 0$).

(a) Determine the following second moment quantities: $\mathbf{m_r}$, $\mathbf{K_r}$, $\mathbf{K_{xr}}$, and $\mathbf{K_{yr}}$ as a function of $\mathbf{K_x}, \mathbf{K_y}, \mathbf{G}, \mathbf{H}, \sigma_n^2, \sigma_v^2$.

(b) Two possible tasks at the hub are estimation of $\mathbf{x}(u)$ or $\mathbf{y}(u)$. Determine the Linear Minimum Mean-Square Error Estimator of these signal based on observing $\mathbf{r}(u)$ - denote these by $\hat{\mathbf{x}}(u)$, and $\hat{\mathbf{y}}(u)$.

(c) Another task which may be of interest at the hub is the joint estimation of both users. In other words, an estimate of the partitioned vector

$$\mathbf{z}(u) = \begin{bmatrix} \mathbf{x}(u) \\ \mathbf{y}(u) \end{bmatrix}$$

is desired. Determine the "Joint" Linear Minimum Mean-Square Error Estimator of this signal based on observing $\mathbf{r}(u)$ - denoted by $\hat{\mathbf{z}}(u)$

Is this joint estimation approach the same as combining the results of the single-user estimates obtained in part (b)? – i.e., Is the following true? –

$$\hat{\mathbf{z}}(u) = \begin{bmatrix} \hat{\mathbf{x}}(u) \\ \hat{\mathbf{y}}(u) \end{bmatrix},$$

where $\hat{\mathbf{x}}(u)$ and $\hat{\mathbf{y}}(u)$ are the single user estimates from part (a).

(d) Consider the specific case where $\sigma_v^2 = 1$, $\sigma_v^2 = 4$, $\mathbf{G} = \mathbf{K_x} = \mathbf{I}$, and

$$\mathbf{K_y} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

For this case, find the best Joint *Causal* LMMSE Estimate of $\mathbf{z}(u)$.

42. **The Waiting Game** (Midterm, Fall 1994) Despite the fact that you've only finished half of EE562a, you land a lucrative job as a RADAR system engineer. You are given the following design problem on your first day: Decide if a target is present ($\mathcal{H}_1$) or absent ($\mathcal{H}_2$). Your new boss tells you that the problem may be modeled as a real-valued, binary hypothesis test

$$\begin{aligned} \mathcal{H}_1 : \qquad & x(u,t) = A + n(u,t) \qquad & t = 1,2,3\ldots \\ \mathcal{H}_2 : \qquad & x(u,t) = n(u,t) \qquad & t = 1,2,3\ldots \end{aligned}$$

where $A > 0$ is known constant, and the noise process $n(u,t)$ is a sequence of independent mean-zero random variables, each with variance σ^2.

Since you haven't seen a problem like this before, you decide to use only the first N observations; that is you decide to design a decision rule based on the observation vector

$$\mathbf{x}_N(u) = \mathbf{x}(u) = \begin{bmatrix} x(u,1) & x(u,2) & \cdots & x(u,N) \end{bmatrix}^{\mathrm{t}}.$$

(a) Conditioned on which hypothesis is true, give the second-moment description of this observation vector - i.e., find $\mathbf{m_x}$ and $\mathbf{K_x}$ under the assumption that $\mathcal{H}_1$ is true, then repeat assuming that $\mathcal{H}_2$ is true.

(b) Based on $\{x(u,i)\}_{i=1}^{N}$, design a good decision rule of the form

$$d(u,N) \underset{\mathcal{H}_2}{\overset{\mathcal{H}_1}{\gtrless}} T(N).$$

Specify the decision statistic $d(u,N)$ and the threshold $T(N)$.

Along with this decision rule, you decide to tell your boss what the minimum value of N is to ensure that your rule is correct at least 99% of the time. What is this minimum choice for N (denoted by $N_{\min}$) in terms of the signal-to-noise ratio: $\gamma = A^2/\sigma^2$?

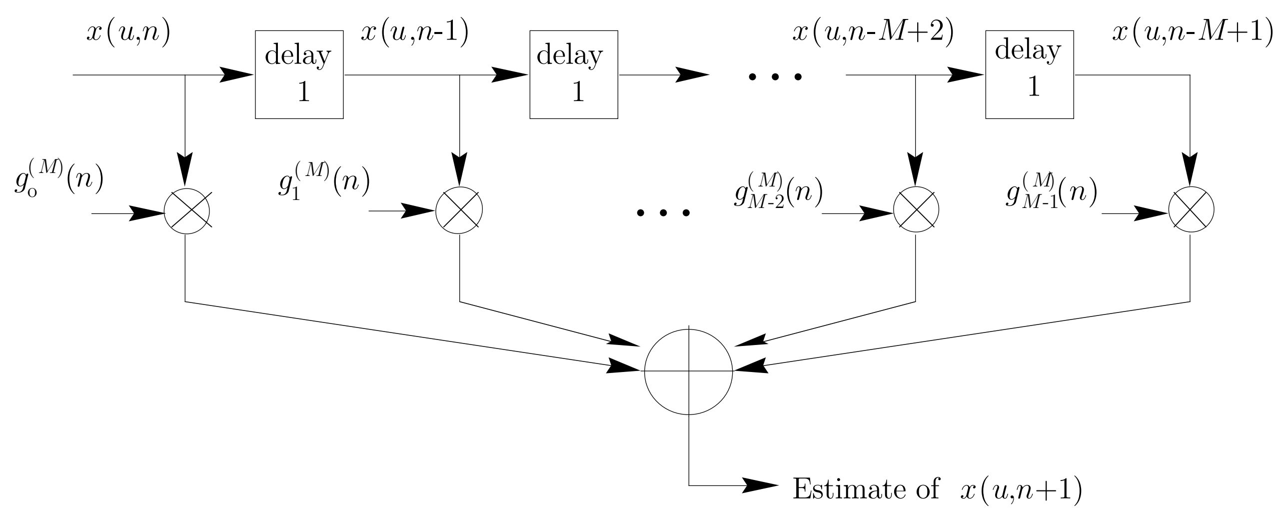
(c) Feeling great about your quick results, you report them to your boss. Your boss is not impressed; she tells you that the noise $n(u,t)$ is a Gaussian random process and asks you to reconsider your results.

You're pretty sure that your rule is still acceptable, but decide to rework the performance calculations based on the Gaussian noise assumption. To do so, you first compute the complete statistical description of $d(u,N)$ conditioned on the hypothesis. Find the following:

- $\mathbb{E}\{d(u,N)|\mathcal{H}_1\}$
- $\mathbb{E}\{d(u,N)|\mathcal{H}_2\}$
- $\text{var}[d(u,N)|\mathcal{H}_1]$
- $\text{var}[d(u,N)|\mathcal{H}_2]$
- $f_{d(u,N)|\mathcal{H}_1}(z)$
- $f_{d(u,N)|\mathcal{H}_2}(z)$

(d) Sketch the the pdf's $f_{d(u,N)|\mathcal{H}_1}(z)$ and $f_{d(u,N)|\mathcal{H}_2}(z)$ vs. z. Label the axis with $T(N)$ and shade the region under these curves corresponding to a decision error. Now find the probability of error $P(\mathcal{E};N)$ in terms of the Q-function, and the corresponding value of $N_{\min}$ to insure that the correct decision is made with probability at least 0.99.

43. **What's Next?** (Midterm, Fall 1994). The one-step linear prediction problem is to estimate a real discrete time random process one time sample into the future from the current and past values. The *order* of a linear predictor is the number of these current and past samples used to form the estimate. A linear prediction filter of order M is diagrammed below.



The optimal choices for the real M-th order linear prediction coefficients are denoted by $\{g_{\text{opt},i}^{(M)}(n)\}_{i=0}^{M-1}$, so that the predicted value is

$$\hat{x}^{(M)}(u,n+1) = \sum_{i=0}^{M-1} g_{\text{opt},i}^{(M)}(n)x(u,n-i) = \left[\mathbf{g}_{\text{opt}}^{(M)}(n)\right]^{\text{t}} \mathbf{x}_n^{(M)}(u),$$

with

$$\mathbf{g}_{\text{opt}}^{(M)}(n) = \begin{bmatrix} g_{\text{opt},0}^{(M)}(n) \\ g_{\text{opt},1}^{(M)}(n) \\ \vdots \\ g_{\text{opt},M-1}^{(M)}(n) \end{bmatrix} \qquad \mathbf{x}_n^{(M)}(u) = \begin{bmatrix} x(u,n) \\ x(u,n-1) \\ \cdots \\ x(u,n-M+1) \end{bmatrix}.$$

The value of the minimum mean-square error is denoted by

$$\text{MSE}^{(M)}(n+1) = \mathbb{E}\left\{[e^{(M)}(u,n+1)]^2\right\} = \mathbb{E}\left\{[x(u,n+1) - \hat{x}^{(M)}(u,n+1)]^2\right\}.$$

This problem concerns the design of linear predictors for a random process which has the following second moment description:

$$m_x(n) = 0 \qquad K_x(n_1,n_2) = \rho^{|n_1-n_2|} \qquad n, n_1, n_2 \in \{\ldots-2,-1,0,1,2,\ldots\}.$$

(a) Set-up the M-th order linear prediction problem for estimation of $x(u,n+1)$ from $\mathbf{x}_n^{(M)}(u)$; indicate the dimension of each quantity:

- $\mathbf{m}_{\mathbf{x}}^{(M)}(n) = \mathbb{E}\left\{\mathbf{x}_n^{(M)}(u)\right\}$
- $\mathbf{K}_{\mathbf{x}}^{(M)}(n) = \mathbb{E}\left\{\mathbf{x}_n^{(M)}(u)[\mathbf{x}_n^{(M)}(u)]^{\text{t}}\right\}$
- $\mathbb{E}\left\{x(u,n+1)\right\}$
- $\mathbf{k}_{\mathbf{x}}^{(M)}(n) = \mathbb{E}\left\{x(u,n+1)\mathbf{x}_n^{(M)}(u)\right\}$

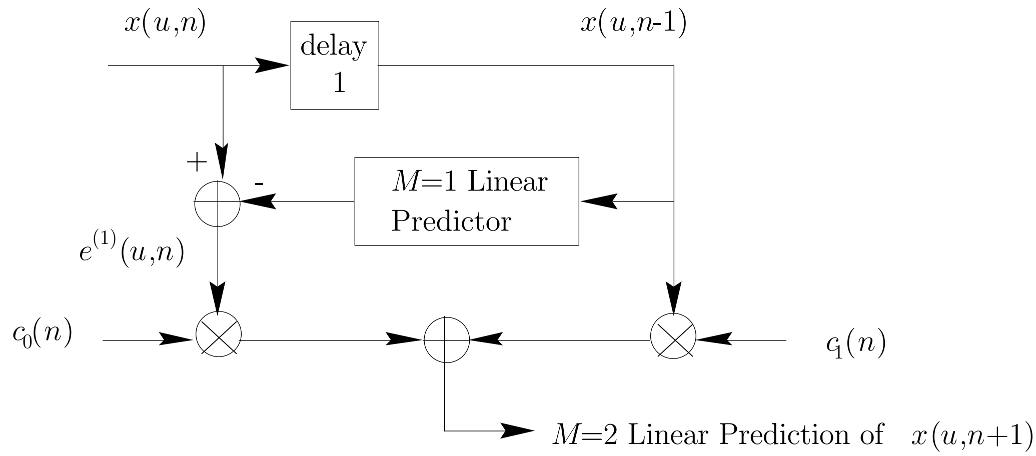
Give an expression for $\mathbf{g}_{\text{opt}}^{(M)}(n)$ (throughout this problem you may assume that the inverse of a square matrix exists).

Is this linear prediction filter time-variant (i.e., $\mathbf{g}_{\text{opt}}^{(M)}(n)$ is a function of n) or time-invariant (i.e., $\mathbf{g}_{\text{opt}}^{(M)}(n)$ is not a function of n)?

(b) Find the best order-one ($M = 1$), one-step linear predictor, and the associated mean-square error – i.e., $\hat{x}^{(1)}(u,n+1)$ and $\text{MSE}^{(1)}(n+1)$.

(c) Find the best order-two ($M = 2$), one-step linear predictor, and the associated mean-square error – i.e., $\hat{x}^{(2)}(u,n+1)$ and $\text{MSE}^{(2)}(n+1)$

(d) An equivalent method for finding the best order-two ($M = 2$), one-step linear predictor is based on the following:



In other words, the observation $\mathbf{x}_n^{(2)}(u)$ is replaced by

$$\mathbf{y}_n(u) = \begin{bmatrix} e^{(1)}(u,n) \\ x(u,n-1) \end{bmatrix} = \begin{bmatrix} x(u,n) - \hat{x}^{(1)}(u,n) \\ x(u,n-1) \end{bmatrix},$$

where $\hat{x}^{(1)}(u,n)$ is the $M = 1$ first-order prediction based on $x(u,n-1)$ and $e^{(1)}(u,n)$ is the corresponding prediction error. Before making any computations, explain why this method is equivalent to the prediction method in part (c).

Demonstrate the equivalence by finding the following:

- $\mathbf{m}_{\mathbf{y}}(n) = \mathbb{E}\{\mathbf{y}_n(u)\}$
- $\mathbf{K}_{\mathbf{y}}(n) = \mathbb{E}\{\mathbf{y}_n(u)[\mathbf{y}_n(u)]^{\mathrm{t}}\}$
- $\mathbb{E}\{x(u,n+1)\mathbf{y}_n(u)\}$
- $c_{\mathrm{opt},0}(n)$
- $c_{\mathrm{opt},1}(n)$

(e) Find the best order-thirty ($M = 30$), one-step predictor linear predictor, and the associated mean-square error – i.e., $\hat{x}^{(30)}(u,n+1)$ and $\mathrm{MSE}^{(30)}(n)$. Give a full explanation.

Now assume that, in addition, $x(u,n)$ is a Gaussian random process. Determine $f_{x(u,n+1)|x(u,n),x(u,n-1)\cdots,x(u,n-29)}(z|v_0,v_1,\ldots v_{29})$.

44. (Final, Summer 1993) Let $x(u,t)$ be a continuous time wide-sense stationary process with $m_x = 0$ and periodic covariance function: $K_x(\tau) = K_x(\tau + kT)$ for all integers k. Show that $x(u,t)$ is periodic with probability 1; i.e. prove that

$$x(u,t) \stackrel{\text{as}}{=} x(u,t+kT) \quad \forall\, k \in \{0,\pm 1,\pm 2,\ldots\}.$$

45. (Final, Summer 1993) Let $x(u,t)$ be a wide-sense stationary random process defined on $\mathcal{T} = \mathcal{R}$, with $R_x(\tau) = \frac{1}{\sqrt{\pi}}e^{-\tau^2}$.

(a) Determine m_x and $S_x(f)$

(b) Answer the following:

- Is $x(u,t)$ a real (i.e. $\Im\{x(u,t)\} = 0$)? – YES NO MAYBE
- Is $x(u,t)$ continuous in the mean square sense? – YES NO MAYBE
- Is $x(u,t)$ differentiable in the mean square sense? – YES NO MAYBE
- Does $\frac{d^2}{dt^2}x(u,t)$ exist in the mean square sense? – YES NO MAYBE
- Is it possible to design a causal, stable filter with impulse response $h(t)$ so that $R_x(\tau) = h(\tau) * h^*(-\tau)$?

46. (Final, Summer 1993) Consider the following relation regarding the second moment description of two jointly-wss continuous time processes:

$$R_{xy}(\tau)R_{yx}(-\tau) \leq R_x(0)R_y(0).$$

Explain why this both sides of this expression are real numbers. Is this relation true? –Always Sometimes Never.

47. (Final, Summer 1993) Let $\{x_n(u)\}_{n=1}^{\infty}$ be a sequence of finite variance real random variables with second moment description:

$$\mathbb{E}\{x_n(u)\} = m_n \qquad \text{var}[x_n(u)] = \sigma_n^2.$$

Show that

$$\text{mss-}\lim_{n\to\infty} x_n(u) = 0 \text{ if and only if } \lim_{n\to\infty} m_n = \lim_{n\to\infty} \sigma_n^2 = 0.$$

48. (Final, Summer 1993) A certain gambling game has the following mathematical model:

$$W_n(u) = \sum_{k=1}^{n} b_k x_k(u),$$

where $W_n(u)$ represents the total winnings (losses) after the n^{th} bet, b_k denotes the amount of the k^{th} bet and $x_k(u)$ is the outcome the k^{th} trial:

$$x_k(u) = \begin{cases} +1 & \text{with probability } p \text{ (a win)} \\ -1 & \text{with probability } 1-p \text{ (a loss).} \end{cases}$$

Each trial is independent: $x_k(u)$ and $x_l(u)$ are independent for all $k \neq l$.

For each of the following betting strategies determine the values of p (the win probability) for which the limit of $W_n(u)$ exists in the mean square sense. When the mss limit exists, find its expected value. <u>(You must show your work and explain your reasoning).</u>

(a) $b_k = \frac{1}{k}$

(b) $b_k = \frac{10^{10}}{k^2}$ <u>HINT:</u> $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

49. (Final, Summer 1993) Let $x(u,t)$ be a wide sense stationary Gaussian random process with

$$S_x(f) = \mathrm{rect}\left(\frac{f}{2B}\right) = \begin{cases} 1 & |f| \leq B \\ 0 & |f| > B. \end{cases}$$

(a) Determine the following:

- Is $x(u,t)$ strictly stationary? –YES NO MAYBE
- $f_{x(u,t)}(z)$
- $f_{x(u,t),x(u,t+\tau)}(z_1, z_2)$
- $f_{x(u,t+\tau)|x(u,t)}(w|v)$
- $\Pr\{x(u,0) > x(u,10/B)\}$

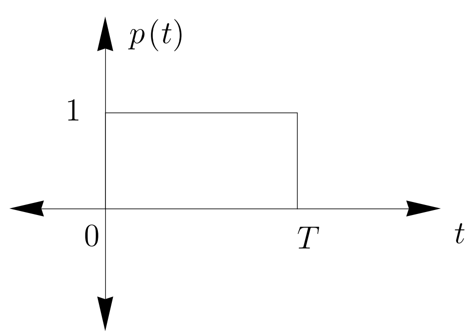
(b) Determine the probability density function of the random vector

$$\mathbf{y}(u) = \left[\begin{array}{cccc} x(u,0) & x(u,1/(2B)) & x(u,1/B) & x(u,200/B) \end{array}\right]^{\mathrm{t}}.$$

50. (Final, Summer 1993) Consider the random process $y(u,t) = p(t - \theta(u))$, where $\theta(u)$ is an exponential random variable with parameter $\lambda > 0 \Rightarrow$

$$f_{\theta(u)}(z) = \lambda e^{-\lambda z} \mathrm{u}(z) = \begin{cases} \lambda e^{-\lambda z} & z \geq 0 \\ 0 & z < 0, \end{cases}$$

and $p(t)$ is as sketched below:



Determine the mean of $y(u,t)$ (be sure to consider all values of $t \in \mathcal{R}$) and answer the following questions:

- Is $y(u,t)$ wide-sense stationary? – YES NO MAYBE
- Is $y(u,t)$ strictly stationary? – YES NO MAYBE
- Is $y(u,t)$ ergodic in the mean? – YES NO MAYBE

51. (Final, Summer 1993) Consider the following continuous time random process:

$$y(u,t) = \operatorname{sgn}\left(A\sin(2\pi t + \theta(u))\right) = \begin{cases} +1 & \text{if } A\sin(2\pi t + \theta(u)) > 0 \\ 0 & \text{if } A\sin(2\pi t + \theta(u)) = 0 \\ -1 & \text{if } A\sin(2\pi t + \theta(u)) < 0, \end{cases}$$

where $\theta(u)$ is uniformly distributed on $[0, 2\pi]$. Then $y(u,t)$ has the following representation:

$$y(u,t) = \sum_{k=-\infty}^{\infty} Y_k(u) e^{j2\pi kt} \quad t \in [0,1].$$

Determine the following:

- $Y_k(u)$
- $\mathbb{E}\left\{Y_k(u)\right\}$
- $\mathbb{E}\left\{Y_k(u) Y_l^*(u)\right\}$
- $\operatorname{var}\left[Y_k(u)\right]$

52. **Circular Reasoning** (Final, Summer 1993) Consider the standard (real) binary hypothesis testing problem defined on the index set $\mathcal{T} = \{0,1,2,3,4,5\}$:

$$\mathcal{H}_i : \quad \mathbf{x}(u) = \mathbf{s}_i + \mathbf{v}(u) \quad i = 0, 1,$$

where

$$\mathbf{s}_0 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{s}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \end{bmatrix} \quad \mathbf{m}_\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{K}_\mathbf{v} = \frac{1}{6}\begin{bmatrix} 12 & 1 & -3 & -2 & -3 & 1 \\ 1 & 12 & 1 & -3 & -2 & -3 \\ -3 & 1 & 12 & 1 & -3 & -2 \\ -2 & -3 & 1 & 12 & 1 & -3 \\ -3 & -2 & -3 & 1 & 12 & 1 \\ 1 & -3 & -2 & -3 & 1 & 12 \end{bmatrix}$$

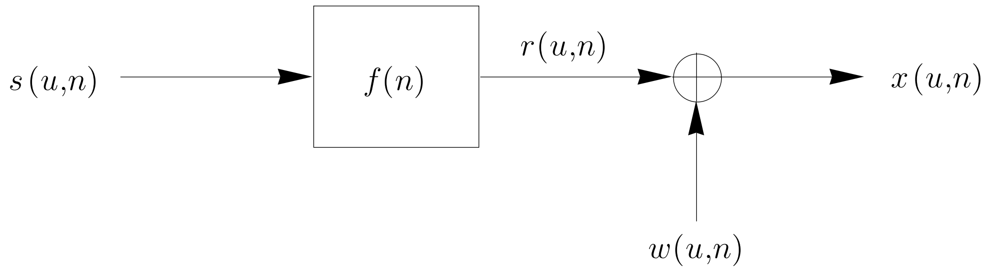
(a) Determine the minimum distance decision rule.

(b) Assuming that the noise is Gaussian, compute the probability of error given that $\mathcal{H}_1$ is true. (I am not asking for an upper bound!).

53. **The Great (?) Equalizer** (Final, Summer 1993). Consider the problem of estimating a discrete time random signal from a filtered, noisy observation:

$$x(u,n) = r(u,n) + w(u,n) = f(n) * s(u,n) + w(u,n).$$

This model is illustrated below:



In addition to this model we know that the signal, $s(u,n)$, and the noise, $w(u,n)$, are independent processes and both are wide sense stationary. The relevant information is

$$F(\nu) = \mathbb{DTFT}\left\{f(n)\right\} = \frac{\frac{1}{2} - e^{-j2\pi\nu}}{1 - \frac{1}{2}e^{-j2\pi\nu}}$$
$$R_s(m) = \frac{9}{8}\left(\frac{1}{3}\right)^{|m|}$$
$$S_w(\nu) = \frac{3}{\frac{10}{9} - \frac{2}{3}\cos(2\pi\nu)}.$$

(a) Determine the rest of the second moment description of $x(u,n)$ and $s(u,n)$ – i.e., $S_x(\nu)$ and $S_{sx}(\nu)$.

(b) Find the frequency response of the best causal LTI (Wiener) filter for estimating $s(u,n)$ from $x(u,n)$: $G_{\mathrm{opt},C}(\nu)$.

54. (Final, Summer 1993) You have access to a noisy version of the desired continuous time random signal, $s(u,t)$:

$$x(u,t) = s(u,t) + n(u,t).$$

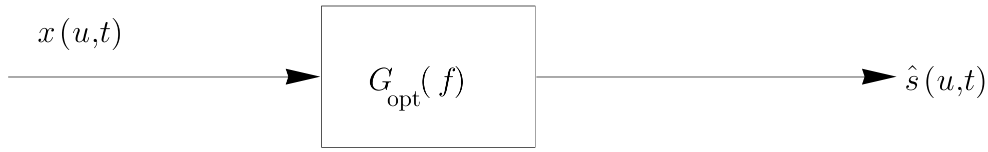
The signal and noise are orthogonal random processes and both are stationary in the wide sense. The second moment descriptions are

$$R_s(\tau) = 2e^{-|\tau|} \qquad S_n(f) = \frac{2}{4 + (2\pi f)^2}.$$

(a) Find the rest of the second moment description of $s(u,t)$ and $x(u,t)$ – i.e., m_x, m_s, $R_x(\tau)$, $S_x(f)$, $R_{sx}(\tau)$ and $S_{sx}(f)$.

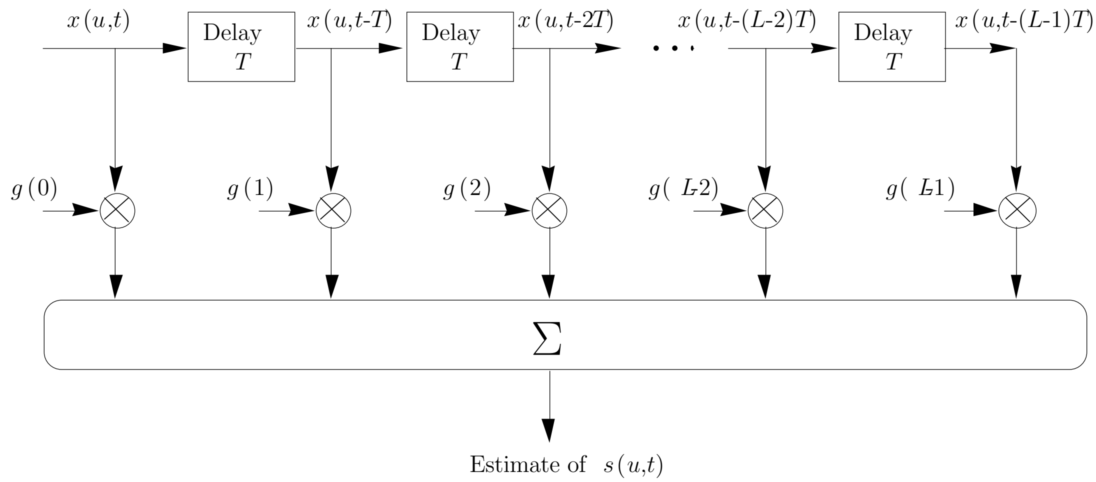
(b) Find the optimal (Wiener) filter for estimating $s(u,t)$ from $x(u,t)$ – $G_{\mathrm{opt}}(f)$.

Indicate how to modify this estimator provide the Wiener filtered estimate of the noise, $\hat{n}(u,t)$, in addition to the Wiener filtered estimate of the signal, $\hat{s}(u,t)$. *Your modification cannot be the addition of another filter.* Indicate the modification on the figure below:



(c) Find the optimal causal LTI (Wiener) filter for estimating $s(u,t)$ from $x(u,t)$ – $G_{\mathrm{opt},C}(f)$.

(d) Consider the estimator with the following *Tapped Delay Line* (TDL) structure:



Where the estimate of $s(u,t)$ is of the form

$$\sum_{k=0}^{L-1} g(k)x(u, t-kT),$$

where the tap coefficients, $\{g(k)\}_{k=0}^{L-1}$ are complex numbers.

Give an equation which the optimal (MMSE) choice of the tap coefficients, $\{g_{\mathrm{opt},TDL}(k)\}_{k=0}^{L-1}$, must solve. Carefully define all new terms needed in terms of the known quantities of the previous parts of this problem.

<u>Hint:</u> One method of solution is to think of t as fixed and try to formulate this as a random vector problem.

For the special case of $L = 2$ and $T = \ln 2$ (i.e $e^{-T} = 1/2$), solve for the optimal TDL coefficients

55. (Final, Summer 1994) Let $x(u,t)$ be a continuous-time wide-sense stationary process with

$$S_x(f) = \frac{2}{1 + (2\pi f)^2} + 9\delta_D(f) \qquad K_x(\tau) = e^{-|\tau|}.$$

What can be said about the mean m_x?

56. (Final, Fall 1994). A (2×1) random vector $\mathbf{x}(u)$ is the input to the following system:

$$\mathbf{x}(u) \longrightarrow \boxed{\mathbf{M}_1} \longrightarrow \boxed{\mathbf{M}_2} \longrightarrow \cdots \longrightarrow \boxed{\mathbf{M}_L} \longrightarrow \mathbf{y}(u)$$

The (2×2) matrices are defined by:

$$\mathbf{M}_i = \begin{bmatrix} i & -2i \\ -2i & i \end{bmatrix}.$$

Suppose that $\mathbf{x}(u)$ has second moment description:

$$\mathbf{K}_{\mathbf{x}} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \qquad \mathbf{m}_{\mathbf{x}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Determine $\mathbf{m}_{\mathbf{y}}$ and $\mathbf{K}_{\mathbf{y}}$.

57. (Final, Summer 1994) Let $y(u)$ be uniformly distributed on $[0, 1]$:

$$f_{y(u)}(z) = \begin{cases} 1 & 0 \le z \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define a sequence of random variables $\{x_n(u)\}$ as follows:

$$x_n(u) = \begin{cases} n & y(u) < 1/n \\ 0 & \text{otherwise.} \end{cases}$$

Describe the convergence properties of $\{x_n(u)\}$ – provide an explaination:

- Does $x_n(u)$ converge in the mean-square sense?
- Does $x_n(u)$ converge almost surely?
- Does $x_n(u)$ converge in probability?
- Does $x_n(u)$ converge in distribution?

58. (Final, Summer 1994) Consider a real Gaussian random process $x(u, n)$ which is wide-sense stationary on $\mathcal{Z}_N$, with mean $m_x = 0$, and spectrum $\lambda_x(k) = \mathbb{DFT}\{K_x(m)\}$. A new random process $X(u, k)$, $k \in \mathcal{Z}_N$ is defined by taking the DFT of $x(u, n)$

$$X(u, k) = \mathbb{DFT}\{x(u, n)\} = \sum_{n=0}^{N-1} x(u, n) e^{-j2\pi \frac{k}{N} n} \qquad k \in \mathcal{Z}_N.$$

Determine the following regarding $X(u, k)$ and its vector representation

$$\mathbf{X}(u) = [X(u, 0)\ X(u, 1)\ \cdots\ X(u, N-1)]^{\mathrm{t}}.$$

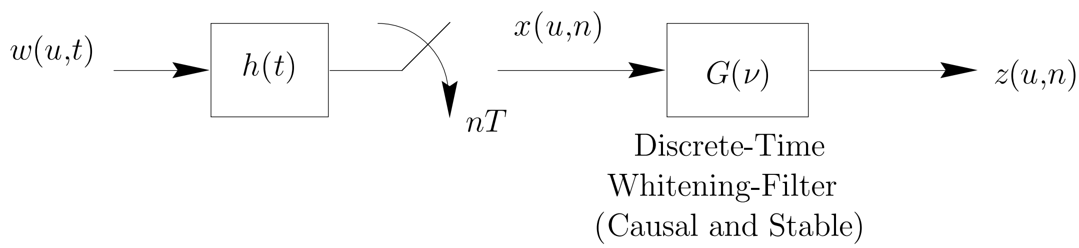
(a) $m_X(k) = \mathbb{E}\{X(u,k)\}$

(b) $K_X(k_1,k_2) = \mathbb{E}\{X(u,k_1)X^*(u,k_2)\}$

(c) $f_{\mathbf{X}(u)}(\mathbf{z})$

(d) Is $X(u,k)$ WSS on $\mathcal{Z}_N$?

59. (Final, Summer 1994) It is desired to convert a broad-band continuous-time process into discrete-time white noise. Let $w(u,t)$ be modeled as continuous-time white noise: $S_w(f) = 1$. Consider the following system:



Here $x(u,n) = w(u,t) * h(t)|_{t=nT}$ and $z(u,n) = g(n) * x(u,n)$. It is desired to design the discrete-time whitening-filter $G(\nu)$ so that $S_z(\nu) = 1$. The whitening filter must be causal and stable.

(a) What is the condition on $h(t)$ for $x(u,n)$ to be white?

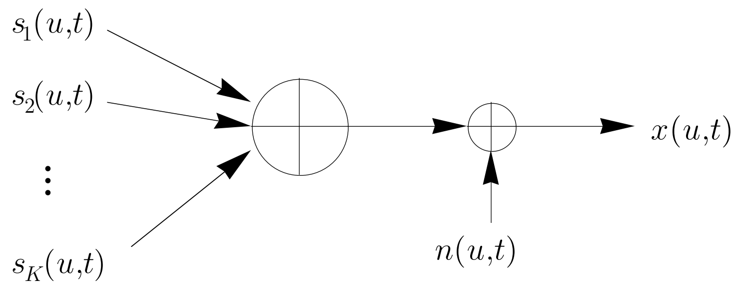
(b) Assume that the above condition is not meet (i.e., $x(u,n)$ is colored); what is the condition which must hold for the whitening procedure to be possible? State this condition in terms of $H(f) = \mathbb{FT}\{h(t)\}$.

60. **Multiple Access** (Final, Fall 1994). A communication system with $K > 1$ users is modeled as

$$x(u,t) = \sum_{k=1}^{K} s_k(u,t) + n(u,t),$$

where $n(u,t)$ is continuous-time white Gaussian noise with PSD level $N_0/2$. The K user signals $\{s_k(u,t)\}$ are mutually independent random processes, each with power σ^2, and PSD $\sigma^2 S(f)$. All user signals are zero mean and independent of $n(u,t)$.

This system is diagrammed below:



Based on the observation $x(u,t)$, it is desired to estimate user 1.

(a) Determine $S_{s_1x}(f)$ and $S_x(f)$.

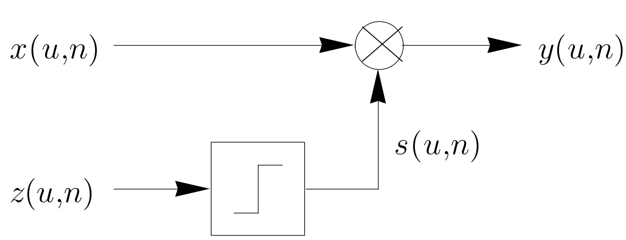
(b) Find the optimal (Wiener) filter for estimating $s_1(u,t)$ from $x(u,t)$; i.e., $\hat{s}_1(u,t) = g_{\text{opt}}^{(1)}(t) * x(u,t)$ – denote this by $G_{\text{opt}}^{(1)}(f)$.
Give an expression for the Minimum Mean-Square Error (MMSE) obtained by the above estimator. Express your answer in terms of $\gamma = \frac{2\sigma^2}{N_0}$ (a measure of signal-to-noise ratio) – denote this by $\text{MMSE}^{(1)}(\gamma) = \mathbb{E}\left\{|s_1(u,t) - \hat{s}_1(u,t)|^2\right\}$.
Determine the effects of the *multiple access interference* by finding the limit of the MMSE as the noise vanishes (express your answer as a function of *only* K and σ^2).

(c) What is the best estimator of $s_k(u,t)$ for $k \neq 1$? Again specify this estimate by giving the optimal filter frequency response – denoted by $G_{\text{opt}}^{(k)}(f)$
Answer the following questions (circle the best answer):
- Is $\hat{s}_1(u,t) = \hat{s}_k(u,t)$ for $k \neq 1$? – YES NO MAYBE
- Is $s_1(u,t) = s_k(u,t)$ for $k \neq 1$? – YES NO
- Is the estimator defined by $G_{\text{opt}}^{(k)}(f)$ causal? – YES NO MAYBE
- Would you say that this Wiener filter provides"good" estimates of the user signals? – explain.

61. **Mixed-Up** (Final, Fall 1994). This problem deals with the discrete time random process $y(u,n) = s(u,n)x(u,n)$, where

$$s(u,n) = \text{sgn}(z(u,n)) = \begin{cases} +1 & \text{if } z(u,n) \geq 0 \\ -1 & \text{if } z(u,n) < 0. \end{cases}$$

This system is illustrated below:



You may assume that $x(u,n)$ and $z(u,n)$ are uncorrelated jointly-Gaussian random processes with

$$S_z(\nu) = 1 \qquad S_x(\nu) = \frac{3/4}{5/4 - \cos(2\pi\nu)}.$$

(a) Determine the following second moment quantities:

- $R_y(m)$, $S_y(\nu)$, and m_y
- $R_s(m)$, $S_s(\nu)$ and m_s
- $R_{yx}(m)$ and $R_{yz}(m)$.

(b) Determine the following probability density functions:

- $f_{x(u,n),x(u,n+m)}(v_1, v_2)$
- $f_{z(u,0),z(u,3)z(u,10)}(v_1, v_2, v_3)$
- $f_{s(u,n)}(v)$
- $f_{y(u,n)}(v)$
- $f_{y(u,n),y(u,n+m)}(v_1, v_2)$

(c) Answer the following (YES or NO):

- Are $x(u,2)$ and $y(u,2)$ both Gaussian random variables?
- Are $x(u,2)$ and $y(u,2)$ jointly-Gaussian random variables?
- Is $y(u,n)$ a Gaussian random process?
- Are $z(u,n)$ and $s(u,n)$ equivalent in the wide-sense?
- Are $z(u,n)$ and $s(u,n)$ statistically equivalent? (i.e., Do they have the same complete statistical description?)
- Are $x(u,n)$ and $y(u,n)$ uncorrelated random processes?
- Are $x(u,n)$ and $y(u,n)$ independent random processes?
- Are $y(u,n)$ and $y(u,n+m)$ ($m \neq 0$) uncorrelated random variables?
- Are $y(u,n)$ and $y(u,n+m)$ ($m \neq 0$) independent random variables?

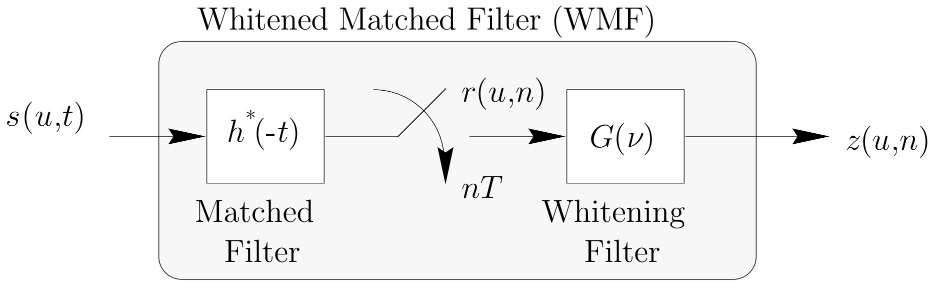
62. **Whitened-Matched Filter** (Rejected Final Exam Problem, Fall 1994). A communication signal which exhibits finite-length *intersymbol interference* (ISI) may be modeled as

$$s(u,t) = \underbrace{\sum_{k=-\infty}^{\infty} a(u,k)h(t-kT)}_{y(u,t)} + w(u,t),$$

where $a(u,k)$ is the data symbol sequence and $w(u,t)$ is modeled as continuous time white Gaussian noise (i.e., $S_w(f) = N_0/2$).

The ISI is modeled by the channel impulse response, which is nonzero only for $t \in [0, LT)$, with L an integer (i.e., the channel response to each symbol interferes with the next $L-1$ symbols).

The following receiver is suggested:



The sampled matched-filter output signal is $r(u,n) = [h^*(-t) * s(u,t)]|_{t=nT}$:

$$r(u,n) = x(u,n) + v(u,n),$$

where $x(u,n)$ is the contribution due only to the signal $y(u,t)$ and $v(u,n)$ is the contribution from the noise.

In general, $v(u,n)$ is not white, and it is convenient to pass this signal through a discrete-time whitening filter. This problem deals with the signal at the output of this matched-filter, whitening-filter cascade (i.e., the Whitened-Matched Filter (WMF)).

(a) Determine the parameters of the model for $r(u,n)$. The notation $E_h(\tau) = h(\tau) * h^*(-\tau)$ is useful.

- $x(u,n)$ as a function of $a(u,k), E_h(\tau)$
- m_v as a function of $E_h(\tau), N_0/2$
- $K_v(m) = \mathbb{E}\{v(u,n+m)v^*(u,n)\}$ as a function of $E_h(\tau), N_0/2, n, m$

(b) After whitening the noise $v(u,n)$, it is possible to model the signal as

$$z(u,n) = \underbrace{\sum_{m=0}^{L-1} a(u,n-m)f(m)}_{p(u,n)} + q(u,n),$$

where $q(u,n)$ is discrete time white noise with power $N_0/2$ – representing the contribution of $v(u,n)$. The signal $p(u,n)$ is the net result of the WMF on the signal.

Determine the relation between this *equivalent ISI channel* $f(m)$ and the parameters used to achieve this model

- $G(\nu)$ as a function of $F(\nu)$
- $E_h(mT)$ as a function of $f(m)$

(c) Define $F(z) = f(0) + f(1)z^{-1} + \cdots + f(L-1)z^{L-1}$, where z is a complex variable. Describe the poles/zeros of $F(z)$ if the whitening filter $G(\nu)$ is assumed to be minimum phase (i.e., causal, stable and causally invertible).

When is the whitening filter unnecessary – i.e., when is $v(u,n)$ white? State the condition on $h(t)$ for $G(\nu) = 1$.