# Set Theory and Probability Spaces

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## 1 Set Theory Review

Probability spaces are based on sets and rules for manipulating sets. This material should be familiar to students, but this section serves as review of these facts and notational conventions.

**Definition 1** *A* set *is a collection of of objects (also referred to as elements or members).*

For example: $A = \{\text{car, bus, motorcycle, bike}\}$.

**Definition 2** *A set $B$ is a* subset *of a set $A$ if every element of $B$ is also an element of $A$. The set $A$ is referred to as a* superset *of the set $B$.*

This relation is denoted $B \subset A$. The set $B$ is a *proper subset* of $A$ if it is a subset of $A$ and there is at least one element in $A$ that is not in $B$. The notation $B \subset A$ allows for the possibility of equality. If this is to be emphasized the notation $B \subseteq A$ is often used, while the notation $B \subsetneq A$ is used to indicate that $B$ is a proper subset of $A$.

For example, $B = \{\text{car, bike}\}$ is a subset of the set $A$ given above. One important set is the empty set, denoted by $\emptyset = \{\}$. The empty set is a subset of every set and if $A \subset \emptyset$, then $A = \emptyset$. Also, for most problems in set theory, there is a largest set which is called the *universe*. The universe is a superset of all sets one would like to consider.

One may also consider a *set of sets*. For example, $\mathcal{A} = \{\emptyset, A, B\}$ with the above examples of $A$ and $B$ is a set.

**Definition 3** *The* intersection *of two sets $A$ and $B$, denoted by $A \cap B$, is the set of all objects that are contained in both $A$* and *$B$. This is sometimes denoted $AB$.*

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 5\}$, then $A \cap B = \{2\}$. If $A \cap B = \emptyset$, then $A$ and $B$ are *disjoint sets*.

**Definition 4** *The* union *of two sets $A$ and $B$, denoted by $A \cup B$, is the set of all objects that are contained in either $A$* or *$B$. This is sometimes denoted $A + B$.*

For the previous example, $A \cup B = \{1, 2, 3, 4, 5\}$.

**Definition 5** *The* difference *between the set $A$ and the set $B$, denoted by $A \backslash B$, is all elements in $A$ that are not in $B$. This is sometimes denoted $A - B$.*

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 5\}$, then $A \backslash B = \{1, 3, 4\}$.

**Definition 6** *The* complement *of a set $A$, denoted by $A^c$, is all elements not in $A$. This is sometimes denoted $\bar{A}$ or $A'$.*

Note that if $U$ denotes the universe, then $A^c = U \backslash A$.

From these definitions, a few facts follow for an arbitrary set $A$:

- The complement of the universe is the empty set: $U^c = \emptyset$.
- The complement of $A^c$ is $A$.
- $A \cup A^c = U$ (universe)
- $A \cap A^c = \emptyset$
- $A \backslash B = A \cap B^c$.

Two useful relations are DeMorgan's Law and the Distributive Law. The Distributive Law states that intersection distributes over unions:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad \text{Distributive Law} \tag{1}$$

DeMorgan's Law

$$(A \cup B)^c = A^c \cap B^c \qquad \text{DeMorgan's Law (I)} \tag{2}$$

$$(A \cap B)^c = A^c \cup B^c \qquad \text{DeMorgan's Law (II)} \tag{3}$$

**Definition 7** *A* partition *for the set $A$ is a collection of disjoint sets $\{B_i\}$ whose union is equal to $A$. Specifically*

$$A = \bigcup_i B_i \qquad B_i \cap B_j = \emptyset, \text{ for all } i \neq j$$