Probability and Statistics – Problem Set

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1 Preliminaries, Combinatorics, Set Probability

1.1. A number of bats are in a cave.

- 2 bats can see out of their left eye.
- 3 bats can see out of the right eye.
- At least 5 bats cannot see out of their left eye.
- At least 4 bats cannot see out of their right eye.

What is the smallest number of bats that can be in the cave?

1.2. Let the sample space associated with a certain random experiment be $\mathcal{U} = \{a, b, c, d, e\}$. You are interested in a probability model which will allow you to define the probability of the sets $\{a\}$ and $\{b, c, d\}$. Define the smallest σ-algebra of events which allows the probability of these sets to be measured.

1.3. How many events are in the smallest sigma algebra that contains the two events A_1 and A_2 which form a partition of the sample space?

1.4. **The Union Bound:** Prove the following results for arbitrary events A, B, C and A_i for $i = 1, 2 \ldots n$.

(a) $P(A \bigcup B) \leq P(A) + P(B)$. When does equality hold?

(b) $P(A \bigcup B \bigcup C) \leq P(A) + P(B) + P(C)$.

(c) $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$.

1.5. Show that if $A \subset B$ then $P(A) \leq P(B)$.

1.6. Two fair dice are rolled and the sum of the outcome is noted. A student declares that the probability that a 7 is rolled is 1/11. The reasoning is that there are 11 possible outcomes and only one results in a 7. Is this correct? Explain. If this is incorrect, provide the correct answer.

1.7. How many distinct permutations of 4 red balls, 2 white balls and 3 blue balls are possible?

1.8. A probability model for the USC football team's season is that each game is won or lost independently of the other games. The probability that a given game is won is 0.9. Given that USC has won their first 3 games, what is the probability that they will win all of their remaining 10 games?

1.9. A binary word is made up of 8 bits, each taking on the values 0 or 1. If a binary word is selected at random, what is the probability that it will have exactly three 1's? What is the probability that it will have fewer than four 1's?

1.10. A state lottery is played by picking six numbers in the range $\{1, 2 \ldots 49\}$ - each of the numbers should be different. The state then draws 6 balls at random from an urn containing 49 balls labeled from 1 to 49 without replacement. What is the probability of winning (getting all six numbers) with one ticket?

1.11. A hot dog vendor provides onions, relish, mustard, catsup, and hot peppers for your hot dog. Determine how many combinations of toppings (no "double" toppings) are possible assuming:

(a) You use exactly one topping.
(b) You use exactly two toppings.
(c) There are no restrictions on how many toppings you can use.

1.12. There are m teams in league A and n teams in league B. On game day there are k games played ($k \leq n$ and $k \leq m$); each game pits a team from league A against a team from league B. A given team can play at most once on game day. How many different sets of match-ups ("schedules") can be made for game day?

(a) If a '"schedule" includes the teams playing and a distinct time for each of the k games?
(b) If a "schedule" includes only the team match-ups and no information on time?

1.13. Consider the set of 4-tuples (w, x, y, z) where w, x, y, and z are non-negative integers and

$$w + x + y + z = 32$$

If a member of this set is drawn at random, what is the probability that the value of w in the drawn 4-tuple is 4?

1.14. There are 10 problems on a TRUE/FALSE exam. You fill out the exam randomly.

(a) What is the probability that you answer all 10 questions correctly?
(b) What is the probability that you answer all 10 questions incorrectly?
(c) A passing grade is 7 or more correct. What is the probability that you pass?
HINT: The events of 7, 8, 9 and 10 correct are mutually exclusive.

1.15. There are 100 computers in a given production run. The QA engineer suggests that, in practice, not all need to be tested. She argues that with high probability a bad lot can be detected by testing only $M < 100$ of the computers. To check this claim, consider the following:

(a) There are D defective computers among the 100; T of the 100 computers are selected at random and tested. What is the probability that F of the T tested fail?

(b) An entire lot should be rejected when 20 or more computers are defective. The following test is suggested: Randomly select T computers and test each; if more than B fail, the lot is rejected. What is the minimum value of T and a corresponding threshold B which ensures that bad lots will be rejected with probability greater than 0.9 and that lots containing 0 or 1 defects will be rejected with probability no more than 0.10? (You may need a computer to determine this answer).

1.16. **Population Estimation:** It is desired to estimate the number of foxes in a forest without catching all of them. Previously, 10 foxes have been captured, tagged and released. A month later 20 foxes are captured and 5 of these have tags.

(a) If the actual number of foxes in the forest is N (unknown to us), what is the probability of this event as a function of N? Denote this probability by $p(N)$.

(b) The estimate of the number of foxes is taken to be the value of N which maximizes $p(N)$. What is this estimate?
HINT: Compare $p(N)/p(N-1)$ against 1 to perform the maximization.

1.17. The state lottery was recently changed to decrease the probability of winning the grand prize. The new game is played by selecting a group of 6 numbers from $\{1, 2, 3 \dots 51\}$ (two more than previously). The state selects a group of 6 numbers from $\{1, 2, 3 \dots 51\}$ and you win the grand prize if all 6 of your numbers match the state's. The probability of winning became so low for the new game that people began to try different ways of "increasing their odds."

(a) Let p denote the probability that a given ticket wins. Determine p for the new game.

(b) (*The Determined Individual*) Joan decides that no matter what the probability of winning is, she will eventually win if she plays enough times. She buys 1 ticket every 24 hours of every 365 days per year for 80 years of her life. Each ticket costs \$5 and she selects her numbers independently from one ticket to the next. Determine:

i. $\Pr\{\text{Joan wins the Lottery at least one time}\}$
ii. Amount of money she spends trying

(c) (*The Group Effort*) Wayne decides that he would be happy splitting the grand prize with his friends if he can increase his chance of winning. Wayne organizes N of his friends and each (independently) buys exactly one ticket for this week's drawing. The minimum value of N ($N_{\min}(\epsilon)$) to insure that the probability that there is at least one winner among the group is $\geq \epsilon$: $N_{\min}(\epsilon)$. Calculate the value of $N_{\min}(\epsilon)$ for $\epsilon = 10^{-5}$, $\epsilon = 10^{-3}$, $\epsilon = 0.1$, $\epsilon = 0.5$.

(d) If Wayne has gathered enough friends to win with probability 0.5 using the above strategy, then can you figure out a way that they can increase their win probability above 0.5 without adding more people to the group?

(e) (*The Compromising Bureaucrat*) The state lottery commissioner realizes that, because of the tougher odds, sales are slumping since the the game was changed. Her remedy is to introduce a consolation prize: if exactly 3 of your 6 numbers match any 3 of the state's 6, then you win the consolation prize. Determine the probability that a single ticket wins the consolation prize.

1.18. Every person has been assigned a unique (i.e no two people have the same) ID number of the form $x_9x_8x_7 - x_6x_5 - x_4x_3x_2x_1$, where $x_i \in \{0, 1, 2 \dots 9\}$ for $i = 1, 2 \dots 9$. A class consists of M students selected at random from the general population. For compactness, the instructor keeps his records based only on the *shortened student ID of* $x_4x_3x_2x_1$. Let the size of the general population be n, and assume that n is also equal to the number of possible (long) ID's. Determine n and the probability that the instructor will have a conflict in his record (i.e. a conflict will occur if there are at least two students in the class with the same short student ID).

1.19. There are 15 pizza toppings available. You can order any of these toppings either as standard (*i.e.,* a single amount) or as extra (*i.e.,* a double amount). In other words, there are single and double toppings allowed, but no triple, quadruple, etc. How many different 4-topping pizzas are possible? If one 4 topping pizza of this type is selected at random, what is the probability that it has a double topping of at least one item?

1.20. Amy sees 3 quarters, 2 dimes, 1 nickel, and 3 pennies on a table. She will take some of this change and place it in her pocket. Coins of a given amount are not distinguishable.

(a) How many different (non-empty) combinations of coins can Amy take?

(b) How many different (non-zero) amounts of money can Amy take?

1.21. Jim has 20 *identical* shirts and 4 drawers in which to place them. How many ways can he do this?

1.22. Let A and B be given events with, $P(A) > 0$, $P(B) \in (0, 1)$ and $P(A|B)$ all known. Determine $P(A|B^c)$ in terms of these known quantities. If, in addition, A and B are *mutually exclusive*, what is the relationship between $P(A)$ and $P(A|B^c)$?

1.23. Suppose that $A \subset B$, $P(A) = 1/4$ and $P(B) = 1/3$. Determine $P(A|B)$ and $P(B|A)$.

1.24. Consider three events A, C, and D with $P(A|C) = 1/2$, $P(A|D) = 1/5$, $P(C) = 1/5$, and $P(D) = 2/5$. Also, events C and D are mutually exclusive. Provide an expression for $P(A|C \cup D)$ in terms of the quantities provided. Then, provide the numerical value for this conditional probability.

1.25. Consider a random experiment in which a student is selected at random from the USC student body. Let A be the event that a student is older than 20. Let B be the event that a student is male. Let C be the event that a student is an engineering major. Determine the set relations for the following events:

(a) A student is male or older than 20.

(b) A student is female and 20 or younger.

(c) A student is female, an engineer, and 20 or younger.

(d) A student is not male and not older than 20, but is an engineer.

(e) A student is an engineer or a female 20 or younger.

1.26. For the following events A and B, state whether they are independent, mutually exclusive or neither.

(a) A = flight 1712 departs LAX on time on 9/6/05, B = flight 1712 arrives in Denver on time on 9/6/05.

(b) A = a given person is a Democrat, B = a given person is a Republican.

(c) A = a given adult is over 6 feet tall, B = a given adult has IQ greater than 120.

(d) A = OPEC embargo, B = reduction is U.S. gasoline prices by more than 50%

(e) A = New England Patriots win Pro Football championship, B = USC Trojans win College Football Championship.

1.27. Two fair dice are rolled; conditioned on the event that the dice land on different numbers, what is the probability that at least one die lands on 6?

1.28. There are 10 problems on a TRUE/FALSE exam. 20% of the students are completely unprepared for the exam and answer randomly. The remaining students are prepared. Prepared students pass the exam (7 or more correct) with probability 0.95.

(a) What is the probability that a randomly selected student passes?

(b) Given that a student does not pass, what is the probability that he was prepared?

1.29. The Rose Bowl is played between the champions of the PAC-10 and BIG-10 conferences. USC is the PAC-10 champion with probability 0.6 and they win the rose bowl 90% of the times that they represent the PAC-10. UCLA wins the PAC-10 20% of the time and given that they make it to the Rose Bowl, they win it with probability 0.2. When one of the other PAC-10 teams win the conference championship, they are equally likely to win or lose in the Rose Bowl. There are 10 teams in the Pac-10.

(a) What is the probability that USC wins the Rose Bowl?

(b) What is the probability that the Big-10 representative wins the Rose Bowl. (assume that Rose Bowl ties occur with zero probability).

(c) Given that the BIG-10 wins the Rose Bowl in a certain year, what is the probability that UCLA represented the PAC-10? Under the same condition what is the probability that USC represented the PAC-10?

1.30. A study is being conducted in an attempt to correlate quality of education and income level. Engineers who have been in the workforce for at least ten years are categorized according to their income level being one of HIGH, MEDIUM, or LOW. Their undergraduate colleges are also noted and this population is limited to those who graduated from USC, Stanford, or Harvey Mudd College (HMC). Half of this population graduated from USC, ten percent from HMC and the rest graduated from Stanford. For USC graduates, 60% are HIGH income earners and 30% are MEDIUM income earners. Half of Stanford graduates earn HIGH income and 20% earn LOW income. HMC graduates make the most; 90% of HMC graduates earn high income and the rest are equally likely to be LOW or MEDIUM wage earners.

(a) What is the probability that an engineer is an HMC graduate and makes a HIGH salary?

(b) What is the probability that an engineer is an HMC graduate and does not make a LOW salary?

(c) Determine the probability that an engineer has a LOW, MEDIUM, or HIGH income:

i. P(LOW income)

ii. P(MEDIUM income)

iii. P(HIGH income)

(d) If an engineer has a LOW income, what is the probability that she graduated from Stanford?

(e) Suppose that you're given that a particular engineer has a HIGH income and asked to make your best decision as to which school she attended. Describe a strategy for making this decision and then use your strategy to make your best decision.

1.31. A computer memory chip fails between times t_1 and t_2 with probability

$$P(\text{Fails between } t_1 \text{ and } t_2) = \int_{t_1}^{t_2} \lambda e^{-\lambda z} dz, \tag{1}$$

where t_1 and t_2 are measured in units of hours after start-up and λ is a constant with units of $(\text{hours})^{-1}$.

(a) What is the probability that the chip does not fail in the first T hours?

(b) What is the probability that the chip does fail in the first T hours?

(c) Given that the chip has not failed in the first S hours, what is the probability that it will fail between time S and $S+T$?

1.32. A fair coin is flipped 10^6 times, determine:

$$\text{Pr}\left\{499,000 \leq \#\text{ of "heads" in } 10^6 \text{ flips} \leq 501,000\right\}$$

1.33. A simple method for detecting errors in a binary digital communications system is to use a parity check bit. A packet consists of $(n-1)$ data bits and 1 parity bit. The parity bit is selected so that an even number of "1's" contained in the transmitted packet of length n. The signal is then distorted and the receiver makes errors independently at each bit location with probability p. The number of 1's in the detected signal is then counted; if this number is even the packet is labeled good, otherwise it is labeled bad and the data is ignored.

(a) What is the probability that a packet is declared bad?

(b) Derive upper and lower bounds for the probability in (a) which can be made arbitrarily tight by including more terms.

(c) Use the family of bounds found in (b) to obtain a numerical answer for the probability of declaring the packet bad when $n = 1000$ and $p = 5 \times 10^{-3}$. Repeat for $n = 1000$ and $p = 1 \times 10^{-4}$.

1.34. The hypergeometric probability law governs the selection of k items from a set of n total where m of the n are 'marked'. For a specific example, suppose there are n people in a room, m of whom are left-handed (the rest are right-handed). If k people are drawn without order and without replacement, then the probability of having l left-handed people is

$$\textsc{Pr}\{l \text{ left-handers in } k \text{ draws}\} = \frac{\binom{m}{l}\binom{n-m}{k-l}}{\binom{n}{k}}, \quad l \leq k$$

(a) Show that if the ratio m/n is fixed that as n becomes large, this tends toward the binomial distribution with $p = m/n$.

(b) Consider a community of n people where 20% of the people are left-handed. Suppose a panel of 3 people is formed from this community by selecting at random. What is the probability that there will be no left-handed people on the panel? Do this for $n = 10$ and repeat for $n = 100$.

(c) Compare the large n approximation from the first part of this problem to the exact answer obtained in the second part. What is the intuitive reasoning why the binomial distribution approximates the hypergeometric distribution for large n?

1.35. Ms. Nev R. Passem is a new QA engineer for a large computer manufacturer. Her first task on the job is to review the company's "burn-in" testing procedures. She is told that there are two types of computers: good and bad. The company wants to sell the good computers and use bad computers for salvage. The computers fail between times t_1 and t_2 (both ≥ 0) with probability

$$\text{Good Computers:} \qquad \textsc{Pr}\{\text{fail in } (t_1, t_2]\} = \int_{t_1}^{t_2} \lambda_g e^{-\lambda_g x} dx$$

$$\text{Bad Computers:} \qquad \textsc{Pr}\{\text{fail in } (t_1, t_2]\} = \int_{t_1}^{t_2} \lambda_b e^{-\lambda_b x} dx.$$

The company's current test is to run the computers from time 0 to T, and if a given computer does not fail during this burn-in period it is accepted and shipped; if it fails it is rejected and used for salvage.

She is also told that the company is concerned with two measures of quality

$$\text{Detection Probability:} \qquad P_D = \textsc{Pr}\{\text{rejecting a computer given that it is bad}\}$$

$$\text{False Alarm Probability:} \qquad P_{FA} = \textsc{Pr}\{\text{rejecting a computer given that it is good}\}.$$

The company's quality standards are $P_{FA} \leq 0.01$ and $P_D \geq 0.99$.

(a) Determine the False Alarm and Detection probabilities as a function of T,λ_b,λ_g.

(b) Nev is not sure that P_D and P_{FA} are the best standards to use; instead she suggests determining the probability that a rejected computer is good or bad. She figures that the probability that a given computer is bad is 0.01. The company tells her that their current test meets the standards with $P_{FA} = 0.008$ and $P_D = 0.993$. Using the results of (a), determine Nev's new standards:

i. $\Pr\{\text{Computer is bad}|\text{rejected}\}$
ii. $\Pr\{\text{Computer is good}|\text{rejected}\}$

(c) The company is interested in adjusting the test duration and they have asked Nev to determine the effects of varying T. Ms. Passem is told that $\lambda_g = 1/600$ and $\lambda_b = 1$ in units of $(\text{hours})^{-1}$. Nev realizes that, in order to achieve the company's standards ($P_{FA} \leq 0.01$ and $P_D \geq 0.99$) with the given parameters, the test duration must be in a specific interval. Determine the minimum and maximum values of T ($T_{\min} \leq T \leq T_{\max}$) so that the company's standards are met.

(d) After performing the above calculations, Nev is concerned that if the company's values for λ_g and λ_b are inaccurate, then designing a test to meet the standards ($P_{FA} \leq 0.01$ and $P_D \geq 0.99$) will be impossible. She determines that the ability of a test of this type (with any choice of T) to distinguish between good and bad computers is a function of

$$\gamma = \frac{\lambda_b}{\lambda_g}.$$

Determine the minimum value of γ ($\gamma_{\min}$) which allows a test of this type to meet the false alarm and detection probability requirements.

1.36. Bill is being interviewed and is given one question that will determine whether he is hired. The interviewer brings two buckets into the room, bucket 1 and bucket 2, along with 50 green balls and 50 red balls. The interviewer tells Bill that she will leave the room and he is to put all 100 balls into the buckets such that neither bucket is empty. She will return to the room, randomly select a bucket and then draw a ball from that bucket. If the drawn ball is green, Bill is hired, if it is red, he is not hired.

(a) If Bill places r red balls and g green balls into bucket 1, what is the probability that he will be hired?

(b) What strategy should Bill take to maximize his chance of being hired – *i.e.,* what are the best choices for r and g?

1.37. Use a computer (*e.g.,* the program `binomial.c`, the spreadsheet, Matlab, etc.) to find example n, p and k where the following conditions hold:

(a) Both the Poisson and Gaussian approximations are valid.

(b) The Poisson approximation is valid, but the Gaussian approximation is invalid.

(c) The Gaussian approximation is valid, but the Poisson approximation is invalid.

(d) Both approximations are invalid.

(e) For at least one case where the Gaussian approximation is valid, demonstrate that it may not hold far from k_{max}.

1.38. Suppose that each child born to a couple is equally probable to be a girl or a boy. Also assume births are independent of the sex distribution of the previous children. For a couple having 6 children, determine the probability of the following events:

(a) All children are the same sex.

(b) The 3 eldest are boys and the 3 youngest are girls.

(c) There is at least one girl.

(d) The sixth child is a girl given that all 5 of their children are boys.

1.39. A certain loaded die produces a "1" with probability 0.5, and all other outcomes with equal probability. This loaded die is placed in a box with seven fair dice. A die is selected at random from the box and rolled 10 times. Determine the probability that the die selected was the loaded die given the follow events:

(a) Exactly 5 of the 10 rolls result in 1's.

(b) There are three 1's, zero 2's, one 3, two 4's, three 5's and one 6.

(c) No rolls come up 2.

1.40. This problem deals with the lottery game defined in problem 1.10 (*i.e.,* 49 pick 6 lottery). Assume that each ticket purchased wins independently of the all other tickets. The lottery is played once per week and 20 million tickets are sold each week. Determine the following:

(a) The probability that in a given week there are exactly k winners. Give numerical answers for $k = 0$, $k = 1$, and $k = 3$.

(b) The probability that nobody wins for the first three weeks of March in a specific year.

(c) The probability of at least 50 winners in 50 weeks.

1.41. Messages are sent through a satellite communication system. A message is received in error with probability 10^{-5}. Errors occur independently for different messages. Suppose 100,000 messages are sent across this this channel, find a good numerical answer for the probability that 4 or fewer errors occur.

1.42. Consider three events A, B, and C where A and C are mutually exclusive. Give the simplest expression for the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap C|B)$

(c) $P(A|B \cap C)$

1.43. Consider the probability space associated with rolling a fair die with sample space $\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$. Provide an example of two non-empty events associated with this experiment, A and B, that are mutually exclusive. Provide an example of two non-empty events associated with this experiment, C and D, that are statistically independent.

1.44. Consider a class with n students. What is the probability that at least two people in the class share the same birthday? Do not consider the year born or account for leap years.

1.45. A class has 39 students and no large rooms are available for the midterm exam. Instead, the exam will be given in 3 separate rooms. How many ways can the class be split into 3 equal-sized groups for the exam?

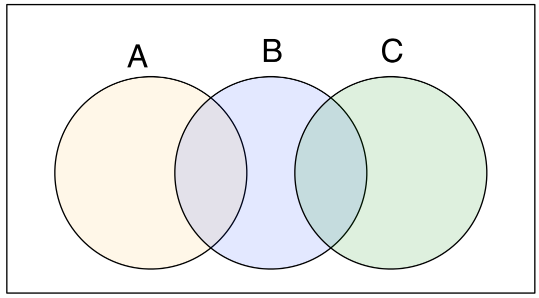

Figure 1: The relation between events A, B, and C.

1.46. Consider three events A, B, and C with set relations show in Fig. 1. Give the simplest expression for the following probabilities:

(a) $P(A \cup B \cup C)$
(b) $P(A \cap B \cap C)$
(c) $P(A|B \cup C)$
(d) $P(A|B \cap C)$
(e) $P(B \cup C|A)$

1.47. You and two friends drive separately to the movies. The movie parking lot has 20 spaces, all in a row. If 15 spaces are occupied at random, what is the probability that you and your two friends can find adjacent parking spots? In other words, what is the probability that there are at least 3 adjacent empty spaces? Can you verify your answer via Monte Carlo simulation? Can you generalize this to n spots with m occupied and searching for k in a row?

1.48. Two fair dice are rolled. Let D be the event that the sum of the dots on the two dice is an odd number and let F be the event that at least one die came up 4.

(a) Find $P(F)$, $P(D)$, and $P(F|D)$. Are F and D independent?
(b) Suppose you know that a 4 was rolled on at least one die and you are asked to bet whether D or D^c (even sum) occurred. Which would you bet?

1.49. Suppose Los Angeles is made up of 50% USC people and 50% UCLA people. Steve selects 8 people at random from this Los Angeles population to form a focus group. From this group of 8, Sarah selects 4 people at random for another experiment. Let A be the event that of the 4 people selected by Sarah, exactly 2 are USC people. Find the probability of this event.

1.50. An n-bit word is sent across a binary symmetric channel (BSC) with error probability ϵ. This means that each bit location in the n-bit word is flipped independently with probability ϵ.

(a) A word error occurs if any of the n bits is flipped. What is the probability of word error P_w?

(b) Reconsider part (a) when an error correcting code is used. Specifically, a t-error correcting code is used which will correct any pattern of t or fewer bit flips (*i.e.,* t or fewer channel errors). The code may also correct some error patterns of more than t flips, but not all. Let $P_{w,\text{decoded}}$ denote the probability of word error after decoding the code. Find a good upper bound on this probability.

(c) A simple Hamming code has $n = 7$ and $t = 1$. Evaluate the uncoded word error probability from (a) and the decoded word error probability from (b) for this code assuming $\epsilon = 0.03$.

1.51. You work at a company providing web services and are tasked to write a program to detect a denial of service (DoS) attack. Your company has a test to detect increased traffic, so the challenge you are faced with is to quickly distinguish a DoS attack from a burst of heavy user activity (heavy-use).

Let B_k be the event that k service requests are received during some prescribed unit of time. Your colleagues have collected data from past DoS attacks and past heavy-use periods so you have a good model for $P(B_k|H)$ and $P(B_k|A)$ where H is the event of valid, heavy user activity and A is the event corresponding to a DoS attack. You also know that valid heavy use periods are 9 times more probable than DoS attacks.

(a) Find an expression for the a posteriori probability of a DoS attack and a heavy-use period, given that k service requests were received –*i.e.,* $P(A|B_k$ and $P(H|B_k)$.

(b) Consider a specific model for the number of service requests. Specifically, there are n service request opportunities and a request occurs during each of these independently with probability p. When a DoS attack is present, $p = p_A$. During a period of heavy use $p = p_H$. Given that B_k has occurred ($k \in \{0, 1, 2 \dots n\}$) provide a good rule for declaring that a DoS attack has occurred (otherwise a period of heavy-use will be declared). Simplify this rule as much as possible.

(c) Consider the rule from part (b) with $n = 10$, $p_A = 0.6$ and $p_H = 0.2$. Describe your rule from (b) in this case for each value of k from $k = 0$ to $k = 10$ – *i.e.,* either "DoS" or "heavy" is declared for each value of k.

1.52. A team of social scientists has conducted a study of the role that personality plays in teams in the classroom. Specifically, they classify each student into one of two personality types: introverts and extroverts. They studied two-student teams drawn at random from a class and drew conclusions about teams comprising 2 introverts, 2 extraverts, or mixed teams.

You know that the researchers used just two classes in their study – an engineering class and a business class – and that they selected between these two class randomly. There were 12 students in the business class and 8 of them were extroverts. There were 4 extroverts and 10 introverts in the engineering class.

You are curious about the validity of the study, so you decide to put some of your probability knowledge to work.

(a) Find the probability of each type of team when the business class is used – *i.e.,* $P(2\text{ extraverts}|\text{biz})$, $P(2\text{ introverts}|\text{biz})$, and $P(1\text{ extravert, }1\text{ introvert}|\text{biz})$.

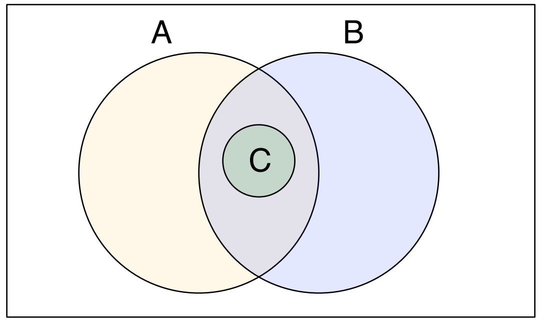

Figure 2: Three events and their set relations.

(b) Find the probability of each type of team when the engineering class is used.

(c) Given the type of the team, find the probability it was drawn from the business or engineering class.

(d) One of the researchers' main conclusions was that a team of two extroverts will not be effective, while a team of two introverts is highly effective. Do your results above make you question this conclusion? Explain.

1.53. Consider three events A, B, and C with set relations show in Fig. 2.

Give the simplest expression for the following probabilities

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A|B \cup C)$

(d) $P(A|B \cap C)$

(e) $P(A \cup B|C)$

1.54. In a digital communication system, a bit *interleaver* is often used. This interleaver reorders a block of N bits in some specific way. Specifically, if the input is $b[i]$ for $i \in \{1, 2, \ldots N\}$, then the output is $c[j]$ for $j \in \{1, 2, \ldots N\}$ where $j = I[i]$ is unique – here $I[\cdot]$ is the interleaving pattern. How many different interleavers of size N are there?

1.55. A single bit message is sent over a channel that causes errors with probability ϵ – *i.e.,* a 1 is flipped to a 0 and a 0 is flipped to a 1 with probability ϵ. In order to increase the reliability, it is decided that the message should be repeated 5 times. Errors are now introduced by the channel independently at each repetition with probability ϵ. The receiver decides whether the message was a 1 or a 0 by majority vote (*e.g.,* if it receives more 1s than 0s, it decides a 1 was transmitted). What is the probability that the receiver decodes the message erroneously?

1.56. It is estimated that 50% of the population of LA are female. It is also estimated that 20% of the population of LA are cyclists. Based on this, find good upper and lower bounds on the event that a randomly selected person from LA is a female or a cyclist.

1.57. A class of 50 students is made up of 30 EE majors and 20 CS majors. Consider k students are selected from this class at random. What is the probability that there will be an equal number of CS and EE majors in this group of k students (carefully specify this as a function of k)? What is this probability if $k = 20$?

1.58. Hooli is a large international corporation that makes the world a better place through ubiquitous cloud storage. Hooli has many data centers throughout the world. The data center closest to USC has 10,000 hard drives. Each of these hard drives has a probability of 2×10^{-4} of failing in any given day.

(a) If 4 or more hard drives fail in a given day, Hooli initiates a data transfer protocol to move data from their USC data center to another data center. What is the probability that Hooli initiates a data transfer in a given day?

(b) What is the probability of having t days in a 7 day week where data transfers occur? Evaluate this for $t = 3$.

(c) What is the probability that no data transfers are required in the month of February?

1.59. Consider a binary optical communication system. The transmitter sends either a 0 or a 1. The transmitter sends a 1 with probability p. This bit is sent by turning on (1) or off (0) a lazer at the transmitter. There is an optical detector at the receiver. Physicists provide a model for the detector: for each bit transmission there are one million opportunities to detect a photon, but only a small number of these opportunities result in successful detection of a photon (*i.e.*, produce current at the detector output). Given that a 1 was sent, there is a probability of 2×10^{-6} of successful detection for each opportunity. Given that a 0 was sent, this probability is 5×10^{-7}.

(a) Find a good model for the probability of detecting k photons given the transmitted data – *i.e.*, let B_0 be the event that a 0 is transmitted and B_1 be the event that a 1 was transmitted.

(b) Find the Maximum A Posteriori decision rule for deciding whether a 1 or 0 was sent given that k photons were detected. Specifically this can be expressed as: "decide a 1 was sent $\iff k > T$, otherwise decide a 0 was sent". Determine the value of T. For the specific case of $p = 1/2$, what is T (numerical)? If p is small enough, this threshold T will increase. What is the range of p that will cause the threshold to be larger than in your above answer for $p = 1/2$?

(c) Although the above MAP rule minimizes the probability of decision error, this probability of error is nonzero. For the case of $p = 1/2$, find the probability of error $P(E)$ for the MAP decision rule.

1.60. Snapchat has 10 million users. The servers are designed to handle 6 snaps in a given 1 second period. Users take snaps independently, with each user taking a snap in a 1 second interval with probability 3×10^{-7}. What is the probability that the server capacity is exceeded in a given 1 second interval?

1.61. A USC alumni panel will have 10 members. The panel will have alumni from the EE, CS, and Business departments. In terms of the departments of the alumni, what are the number of different types of panels?

1.62. A section of EE364 has 25 students in total, with 20 CS majors and 5 EE majors. If the EE364 basketball team is selected by choosing 5 of these students randomly, what is the probability that the team is made up of all CS majors? What is the probability that the team has exactly 1 EE member?

1.63. If events A and B are independent, are the compliment events A^c and B^c independent? State whether this is true or false. If true, provide a proof. If false, provide a counter-example.

1.64. If events A and B are independent, are A and B^c also independent? State whether this is true or false. If true, provide a proof. If false, provide a counter-example.

1.65. Consider the probability space associated with rolling a fair die with sample space $\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$. Provide an example of two non-empty events associated with this experiment, A and B, that are mutually exclusive. Provide an example of two non-empty events associated with this experiment, C and D, that are statistically independent.

1.66. If events A and B are independent, conditioned on event C, are A and B independent, conditioned on C^c? State whether this is true or false. If true, provide a proof. If false, provide a counter-example.

1.67. Alice and Bob are junior CS majors and they are both registering for the same 6 classes next semester. Three of these classes have 2 sections and three have 3 sections. What is the probability that Alice and Bob share exactly 3 sections next semester if they choose their sections randomly?

You are to design a email spam filter that will filter spam emails to a junk folder and not filter non-spam emails. Your associate has estimated the probabilities associated with this task. Let S denote the event that a given email is spam and let $H = S^c$ denote the event that a given email is "ham" (not spam). Your colleague estimates that 30% of emails are spam.

Your colleague has also designed two features to use for the filtering task. Let F_0 be the event that a given email contains feature 0 and F_1 be the event that a given email contains feature 1. Use the shorthand notation for the 4 possible feature observations:

$$\begin{aligned} R_0 &= F_1^c \cap F_0^c \\ R_1 &= F_1^c \cap F_0 \\ R_2 &= F_1 \cap F_0^c \\ R_3 &= F_1 \cap F_0 \end{aligned}$$

You colleague has provided the following model for these observations:

	R_0	R_1	R_2	R_3
spam (S)	0.10	0.25	0.15	0.50
ham (H)	0.50	0.10	0.30	0.10

where the elements of the table are the conditional probabilities of the column, given the row. For example, $P(R_0|H) = 0.5$ is in the second row of the first column.

(a) Give an **expression** for the a posteriori probability $P(H|R_0)$ in terms of quantities given in the problem statement. Compute numerical values for the a posteriori probabilities of spam and ham and enter them in the table below. For example, in the second row of column 1 put the value of $P(H|R_0)$:

	R_0	R_1	R_2	R_3
spam (S)				
ham (H)				

(b) Find the Maximum A Posteriori decision (filtering) rule for deciding whether an email is spam or ham. Specify your rule by filling in the table below – i.e., each entry should be either "filter" or "don't filter" and should be the MAP decision for whether the email is spam or ham given the observation in the column header.

	R_0	R_1	R_2	R_3
filter OR don't filter:				

(c) You google email spam filtering and read the wikipedia page. This discusses a Naive Bayes Classifier (NBC) for this task. You wish to compare this NBC to your filter. The naive assumption is that the features are independent given spam and also independent given ham. For example, in place of $P(R_0|H) = P(F_1^c \cap F_0^c|H)$ you would use $P(F_1^c|H)P(F_0^c|H)$.

Give an **expression** for $P(F_0|H)$ in terms of the quantities given in the problem statement.

Fill in the table below with the conditional feature probabilities – *e.g.,* the second row of the first column will contain the value of $P(F_0|H)$.

	F_0	F_0^c	F_1	F_1^c
spam (S)				
ham (H)				

(d) Find the NBC filtering rule for deciding whether an email is spam or ham. Specify your rule by filling in the table below – i.e., each entry should be either "filter" or "don't filter" and should be the NBC decision for whether the email is spam or ham given the observation in the column header.

	R_0	R_1	R_2	R_3
filter OR don't filter:				

(e) Find the probability of error (mis-filtering) for the MAP rule and the NBC rule. Let $\mathcal{E}$ denote the event that an email is mis-filtered – *i.e.,* an error occurs if spam is not filtered to junk or ham is filtered to junk. Find $P_{\text{MAP}}(\mathcal{E})$ and $P_{\text{NBC}}(\mathcal{E})$. Which of these is smaller (hint: you can answer this even if you did not get the numerical answers)?

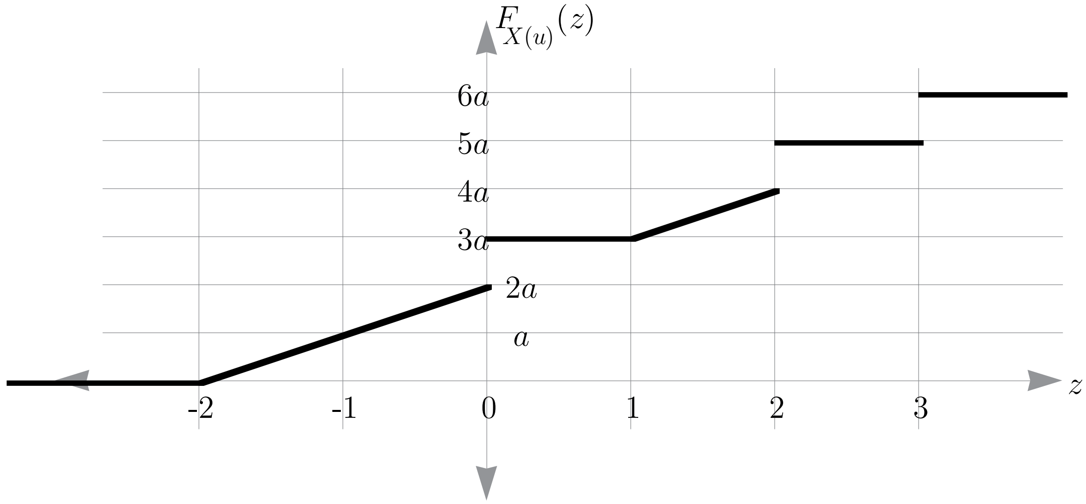

Figure 3: The cdf of $X(u)$.

1.68. For each of the following types of random variables find $p = \text{Pr}\{X(u) > m_X\}$:

(a) $X(u)$ is Normal with mean m_X, variance σ_X^2

(b) $X(u)$ is Exponential with parameter λ

(c) $X(u)$ is Poisson with parameter $\alpha = 3.3$

(d) $X(u)$ is Uniform on the continuum $[a, b]$

2 Random Variables

2.1. Consider the cdf of $X(u)$ shown in Fig. 3. If $\text{Pr}\{X(u) \leq 3\} = 1$, then determine:

(a) a

(b) $\text{Pr}\{X(u) \leq 2\}$

(c) $\text{Pr}\{X(u) = 1\}$

(d) $\text{Pr}\{2.5 \leq X(u) < 3\}$

(e) $\text{Pr}\{X(u) = 3\}$

2.2. A fair coin is tossed 4 times. Determine and sketch the cumulative distribution function for the following random variables:

(a) $X(u)$, the number of "Heads" observed.

(b) $Y(u)$, the number of "Tails" observed.

(c) $D(u) = X(u) - Y(u)$, the difference between the number of heads and number of tails.

2.3. The geometric random variable is defined as the number of coin flips required to get the first "Head." If heads occur with probability p, determine $\text{Pr}\{X(u) = k\}$ and the cdf.

2.4. Show that if $X(u)$ is a Gaussian (also known as "normal") random variable with pdf

$$f_{X(u)}(z) = \mathcal{N}(z; m, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(z-m)^2}{2\sigma^2}\right], \tag{2}$$

then

$$\text{Pr}\{X(u) \geq x\} = Q\left(\frac{x-m}{\sigma}\right). \tag{3}$$

2.5. $X(u)$ is a Gaussian random variable with mean $m_X = 2$ and variance $\sigma_X^2 = 25$. Determine the probability that $X(u)$ is greater than 10.

2.6. The random variable $X(u)$ has pdf given by

$$f_{X(u)}(z) = \begin{cases} cz(1-z) & 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(a) Find c.

(b) Find $\text{Pr}\{1/2 \leq X(u) \leq 3/4\}$.

(c) Find $F_{X(u)}(z)$.

2.7. The probability mass function (pmf) of $X(u)$ is

$$p_{X(u)}(k) = \text{Pr}\{X(u) = k\} = Ka^{|k|} \qquad k = 0, \pm 1, \pm 2 \ldots \tag{5}$$

where $K > 0$ is a constant. Determine the following:

(a) Possible range for a

(b) K

(c) $\Gamma_{X(u)}(z) = \mathbb{E}\left\{z^{X(u)}\right\}$

(d) The mean of $X(u)$.

2.8. A Rayleigh random variable has cdf

$$F_{X(u)}(z) = \begin{cases} 1 - e^{-z^2/(2\sigma^2)} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{6}$$

Determine the following

(a) The pdf of $X(u)$.

(b) The $\text{Pr}\{\sigma \leq X(u) \leq 2\sigma\}$.

(c) The $\text{Pr}\{X(u) \geq 3\sigma\}$.

2.9. Let $X(u)$ be an exponential random variable with parameter λ. Segment the real line into 5 equiprobable disjoint intervals.

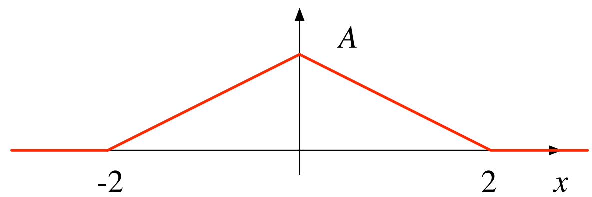

Figure 4: The pdf considered in problem 2.15.

2.10. The time that Jim arrives in his office is modeled as a random variable, $Y(u)$, measured in minutes after eight AM today. For each of the cases below determine

- $F_{Y(u)}(z|Y(u) > t)$ where t is a fixed number
- The probability that Jim arrives in the next minute given that he has not arrived at time t: $\Pr\{Y(u) \in (t, t+1] | Y(u) > t\}$.

(a) $Y(u)$ is an exponential random variable with parameter λ in $(\text{minutes})^{-1}$.

(b) Jim arrives sometime between 8 and 9 o'clock, with the probability uniformly distributed in this region. (i.e. $Y(u)$ is uniform on $[8, 9)$). Let $Y(u)$ model the arrival time in minutes after 8:00.

2.11. Show that the Geometric random variable (see problem 2.3) has the memoryless property:

$$\Pr\{X(u) \leq m + k | X(u) > m\} = \Pr\{X(u) \leq k\}, \quad m,\ k \geq 0 \text{ integers,} \tag{7}$$

or $F_{X(u)}(m + k | X(u) > m) = F_{X(u)}(k)$.

2.12. Let $X(u)$ have pdf $\mathcal{N}(x; 0; \sigma^2)$, find the pdf of $X(u)$ conditioned on the event $\{X(u) > 0\}$.

2.13. If $X(u)$ is a Bernoulli random variable, equal to 1 with probability 0.25 and equal to 0 with probability 0.75, determine the mean and variance of $X(u)$ and the probability that $X(u)$ is greater than 0.1.

2.14. The pmf of $X(u)$ is given by

$$p_{X(u)}(k) = \Pr\{X(u) = k\} = \frac{1}{3} 2^{-|k|} \qquad k = 0, \pm 1, \pm 2 \ldots \tag{8}$$

Determine the mean and variance of $X(u)$.

2.15. The random variable $X(u)$ has pdf as shown in Fig. 4, where A is a positive constant.

(a) Determine the constant A

(b) Find and sketch the cdf of $X(u)$

(c) Determine the probability that $X(u)$ is greater than 1

(d) Determine the mean and variance of $X(u)$

(e) Determine $\mathbb{E}\{\sin(10X)\}$

2.16. There are two boxes of light bulbs. A bulb is selected from a box at random and used. The time to failure is an exponential random variable $X(u)$ (conditioned on the failure rate), for light bulbs from either box. The failure rate (in units of 1/hours) is modeled as a random variable $R(u)$. The pdf of $R(u)$ depends on which box the bulb is drawn from

$$\text{Box 1:} \quad f_{R(u)}(\lambda) = \begin{cases} 1/M & \lambda \in [0, M) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Box 2:} \quad f_{R(u)}(\lambda) = \frac{\exp(-\lambda/500)}{500}\mathrm{U}(\lambda).$$

Determine and sketch the following for the two cases of $M = 20$ and $M = 5$:

(a) The pdf of the failure rate $R(u)$

(b) The a-posteriori pdf of $R(u)$ given that the selected bulb fails during the first 10 hours of use.

2.17. Let $X(u)$ be a continuous random variable and $Y(u) = [X(u)]^k$. Find the pdf of $Y(u)$ for $k = 1, 2, 3 \ldots$.

2.18. Let $Y(u) = g(X(u))$ with

$$f_{X(u)}(x) = \frac{1}{2}e^{-x}\mathrm{U}(x) + \frac{1}{4}\delta(x) + \frac{1}{4}\delta(x-2)$$

$$g(x) = \begin{cases} x^2 & -\infty < x < 1 \\ 1 & 1 \le x < 2 \\ 4 - x & 2 \le x < \infty. \end{cases}$$

Determine and sketch the pdf of $Y(u)$.

2.19. Consider the random variable $X(u)$ with pdf

$$f_{X(u)}(x) = \begin{cases} \frac{1+x}{2} & -1 \le x \le +1 \\ 0 & \text{otherwise} \end{cases}$$

If $Y(u) = 3X(u) - 2$, determine and sketch the pdf of $Y(u)$.

2.20. As a newly hired systems engineer, it is your job to analyze the cascaded system shown in Fig. 5.

The corresponding definitions are

$$g(x) = \frac{1}{|x|} \qquad Y(u) = g(X(u))$$

$$h(y) = \ln(y) \qquad Z(u) = h(Y(u))$$

$$r(z) = \begin{cases} 1 & z \ge 1/4 \\ 0 & z < 1/4 \end{cases} \qquad W(u) = r(Z(u)).$$

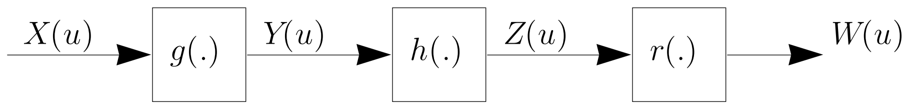

Figure 5: The system considered in problem 2.20.

The pdf of the input random variable is

$$f_{X(u)}(x) = \begin{cases} 4x^3 & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases} \tag{9}$$

Determine:

(a) The means of $X(u)$ and $Y(u)$

(b) The mean and pdf of $Z(u)$

(c) The mean and pdf of $W(u)$

2.21. A "dead-zone" nonlinearity is defined as

$$g(x) = \begin{cases} 0 & -a < x < a \\ x - a & x \geq a \\ x + a & x \leq -a, \end{cases}$$

where a is a positive constant. Determine and sketch the pdf of $Y(u) = g(X(u))$ when $X(u)$ is a Gaussian random variable with mean m_X and variance σ_X^2.

2.22. A computer routine called `rand(a,b)` returns a uniform random number between `a` and `b`. If you want to generate a mean zero, unit variance uniform random number, what values of `a` and `b` would you use in the function call?

2.23. Let $X(u)$ be an integer valued, non-negative random variable ($\text{Pr}\{X(u) < 0\} = 0$), with $\text{Pr}\{X(u) = k\} = p_{X(u)}(k)$ for $k = 0, 1, 2 \ldots$.

(a) Show that in this case

$$m_X = \mathbb{E}\{X(u)\} = \sum_{k=0}^{\infty} \text{Pr}\{X(u) > k\}. \tag{10}$$

Use this result to find the mean of the geometric random variable.

(b) A similar result holds for the case of a continuous, non-negative random variable:

$$m_X = \mathbb{E}\{X(u)\} = \int_0^{\infty} (1 - F_{X(u)}(x))dx. \tag{11}$$

To see how this can be useful, compute the mean of the exponential r.v. using this formula.

2.24. Let $\Theta(u)$ be a random variable uniformly distributed on $[0, 2\pi)$. Find the mean and variance of the following random variables

(a) $X(u) = \sin\Theta(u)$

(b) $Y(u) = \cos\Theta(u)$

(c) $R(u) = \sqrt{[X(u)]^2 + [Y(u)]^2}$

(d) $Z(u) = X(u)Y(u)$.

2.25. USC plays 13 football games this season and wins each, independently, with probability 0.9. What is the expected number of wins for the season?

2.26. Let $X(u)$ be a mean zero Gaussian random variable with variance σ_X^2. Find the expected value of $X(u)$ conditioned on the event $\{X(u) > 0\}$. (Hint: see problem 2.12).

2.27. If the random variable $\Theta(u)$ is uniformly distributed on the interval from 0 to 2π, then $X(u) = \cos(\Theta(u))$ has arc-cos pdf:

$$f_{X(u)}(x) = \begin{cases} \frac{1}{2\pi\sqrt{1-x^2}} & -1 \le x \le +1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the mean and variance of $X(u)$.

2.28. Find the following probabilities:

- $\Pr\{X(u) > m_X\}$
- $\Pr\{X(u) \le m_X\}$

for the cases of $X(u)$ Gaussian (variance σ_x^2), uniform on $[a, b]$, and exponential with mean $1/\lambda$. What is a sufficient condition for these two probabilities to be equal?

2.29. Let $X(u)$ be a Gaussian random variable with mean zero and variance σ_X^2. Determine:

(a) $\mathbb{E}\left\{e^{3X(u)}\right\}$

(b) $\mathbb{E}\left\{e^{-3X(u)}\right\}$

(c) $\mathbb{E}\{\cosh(3X(u))\}$

2.30. If the normalized time that a professor requires a Ph.D. student to study before graduating is x, then the number of students studying under the professor may be modeled as a random variable $N(u)$ with

$$\Pr\{N(u) = k \text{ for grad. time } x\} = \frac{x^k}{k!}e^{-x} \quad k = 0, 1, 2 \dots \tag{12}$$

For a particular professor, Prof. I.M. Tubuzy, the time required for a student to graduate is best modeled as a random variable $X(u)$ with

$$f_{X(u)}(x) = \lambda e^{-\lambda x}\mathrm{U}(x), \tag{13}$$

where $\lambda > 0$ is a constant. In other words, the probability that Dr. Tubuzy has k students conditioned on his graduating time is

$$\textsc{Pr}\left\{N(u) = k | X(u) = x\right\} = \frac{x^k}{k!} e^{-x} \quad k = 0, 1, 2 \ldots \tag{14}$$

(a) Determine the unconditional pmf of $N(u)$.

(b) What type of random variable is $M(u) = N(u) + 1$? Recognizing this, determine the second moment description of $N(u)$.

(c) Prof. Tubuzy has n students. Find the pdf and the average value of $X(u)$ conditioned on this information.

(d) Determine the condition on n so that

$$\mathbb{E}\left\{X(u) | N(u) = n\right\} > \mathbb{E}\left\{X(u)\right\} \tag{15}$$

In other words, when does learning that Dr. Tubuzy has n students increase the expected graduating time?

2.31. Find the mean and variance of the geometric random variable defined in problem 2.3.

2.32. Let $X(u)$ be Gaussian with mean m and variance σ^2. Find $\mathbb{E}\left\{|X(u)|\right\}$. (Hint: first find $\mathbb{E}\left\{|X(u) - m|\right\}$.

2.33. The pdf of $Y(u)$ is

$$f_{Y(u)}(y) = \frac{y}{\sigma^2} \exp\left(\frac{-y^2}{2\sigma^2}\right) \mathrm{U}(y). \tag{16}$$

Determine the mean and third moment of $Y(u)$

2.34. Using the characteristic function (or moment generating function), find $\mathbb{E}\left\{[X(u)]^4\right\}$ when $X(u)$ is Gaussian with zero mean and variance σ_X^2. What is the third moment in this case?

2.35. A fair coin is flipped 100 times. Find a lower bound, based on Chebychev's bound, on the probability of the event that the number of heads observed is in $\{45, 46, \ldots 55\}$. Compare this to the the exact probability is 0.73.

2.36. Show that, for $a > 0$,

$$\textsc{Pr}\left\{X(u) > a\right\} \leq \textsc{Pr}\left\{|X(u)| > a\right\}.$$

In the case where the pdf of $X(u)$ is symmetric around zero, show that

$$\textsc{Pr}\left\{X(u) > a\right\} = \frac{1}{2} \textsc{Pr}\left\{|X(u)| > a\right\}$$

2.37. The height of a randomly selected USC student is measured. The average height of a USC student is 65 inches.

(a) Determine a good upper bound for $\textsc{Pr}\{$the student is at least 74 inches tall$\}$.

(b) If, in addition, it is known that the standard deviation of the height is 4 inches, specify a and b so that

$$\Pr\{\text{student is between } a \text{ and } b \text{ inches tall}\} > 0.9$$

and give a good lower bound on $\Pr\{\text{the student is between 60 and 72 inches tall}\}$.

2.38. The purpose of this problem is to gain a feel for tail-probability bounds discussed in class. Let the random variable of interest be Gaussian with zero mean and variance σ_x^2. Develop (if necessary) and investigate the following bounds for $\Pr\{X(u) \geq a\sigma_x\}$:

- Specialized Markov bound: Use the second result from problem 2.36 and the result of problem 2.32.
- Chebychev bound.
- Fourth moment bound: Use the results of problem 2.34.
- Chernoff bound.
- The over bound given in the Q-function handout.
- The exact value of the probability.

Plot these expressions against a.

2.39. Consider a national election between two candidates, one Republican and one Democrat. The voting population is 40% Democrat and 20% Republican with the remaining voters being independent (*i.e.,* neither Republican nor Democrat). Both Republican and Democrat voters vote for their party's candidate with probability 0.9.

(a) In this part, consider the probability that an independent voter votes for the Republican candidate to be p. For what range of p will a Republican victory be more probably than a Democrat victory?

Assume that $p = 0.3$. If a randomly selected ballot has been cast for the Democrat candidate, what is the probability that it was cast by a Republican voter?

(b) What is the pdf of $P(u)$ given that you observe one ballot and it was cast for a Republican?

(c) What is the pdf of $P(u)$ given a Republican wins the election? You can assume that a Republican win is equivalent to the condition that a Republican victory is more probable than a Democrat victory (*i.e.,* from the condition in part (a)).

2.40. Consider the random variable $Y(u) = g(X(u))$ where the periodic function $g(x)$ is shown in Fig. 6. If $X(u)$ is uniform on $[0, 4)$, determine and sketch the pdf of $Y(u)$. Repeat for the case of $X(u)$ is uniform on $[0, \frac{5}{2})$.

2.41. Consider a game played in lecture. A student can choose to bet on a game or to abstain from betting. Students decide to bet with probability 0.8. If a student decides to bet, she wins with probability 0.6. When a student wins a bet, her point total is increased by 10; when she loses, it is decremented by 10.

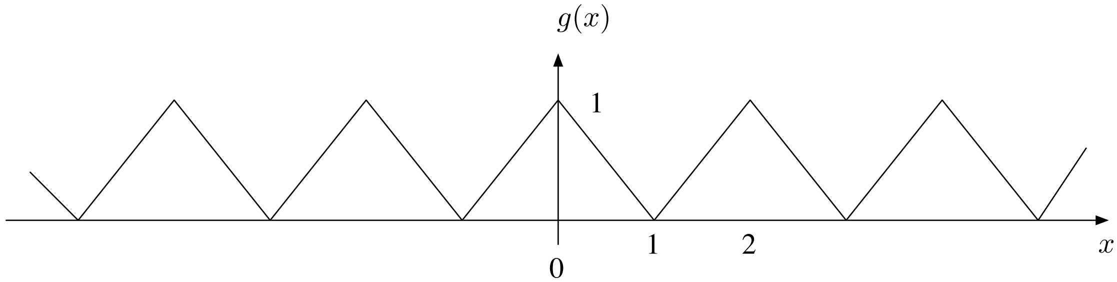

Figure 6: The periodic function considered in Problem 1.40.

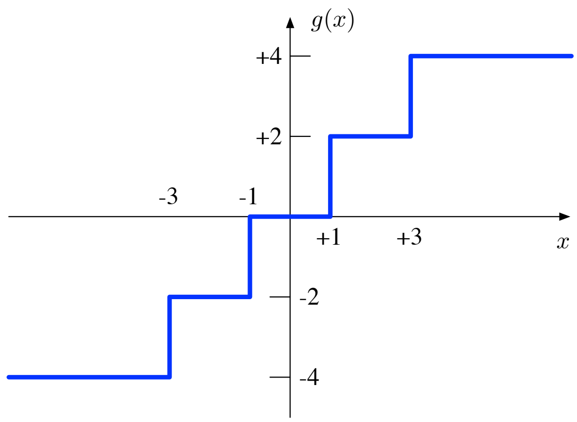

Figure 7: An ideal model for a 2-bit analog to digital converter.

(a) Let $X(u)$ model the net change in the student's point total for one game as described above. Find the pmf of $X(u)$ and determine the second moment description of $X(u)$.

(b) Suppose that one of these games is played each week for 10 weeks. Each week the game is played the same and the games are not related (*i.e.,* independent trials). Let the event A be that a student has 5 wins, 2 losses, and 3 abstains in this 10 game series. Determine $P(A)$. If A occurs, what is the net change in the student's point total?

(c) Consider the case where 3 of these games have been played and let $Y(u)$ model the net change to the student's point total. Find the pmf of $Y(u)$ – first specify the values $\{y_k\}_k$ that $Y(u)$ can take and then specify the corresponding probabilities.

2.42. A simple model for a 2-bit analog to digital converter (ADC) is given by the function in Fig. 7. This problem is concerned with the distortion introduced by this A/D conversion when the input is modeled as a zero mean Gaussian random variable. Specifically, let $X(u)$ be the ADC input, $V(u) = g(X(u))$ be the ADC output and $Z(u) = X(u) - V(u)$ be the approximation error. The input $X(u)$ is modeled as Gaussian with zero mean and variance σ^2.

(a) Determine the pdf of $Z(u)$

(b) Determine the following probabilities: $\Pr\{|Z(u)| > 1\}$ and $\Pr\{0 < Z(u) < 1\}$

2.43. A class of 50 students has met for 9 lectures. Suppose that the instructor randomly selected 5 students at the beginning of each lecture and excused them from lecture. What is the probability that a specific student, say Jane, would have been excused from exactly 2 lectures thus far?

2.44. The IT Department at Ajax, Inc. is made up of 3 employees: Alice, Bob, and Sue. Ajax forms committees for various tasks by drawing from each department. In 2014, there will be 20 committees that require exactly one IT Department employee to participate. Which IT employee serves on a given committee is determined by drawing straws (*i.e.,* randomly selecting between the 3).

(a) Consider the event C that Alice, Bob, and Sue serve on 3, 7, 10 committees, respectively. Also, let the event D be that Alice, Bob, and Sue serve on 1, 0, 19 committees, respectively. Are these two events equally probable? Explain.

(b) Find the probability that Alice serves on exactly m committees. What is the most probable number of committees Alice will serve on?

(c) Repeat part (b) conditioned in the event E that Bob serves on exactly 10 committees. What is the most probable number of committees Alice will serve o given E?

(d) Repeat part (b) conditioned in the event F that Bob serves on less than 3 committees. What is the most probable number of committees Alice will serve on given F?

2.45. A study is being done by the legal department of an amusement park in order to assess potential liability. In particular, they are concerned with how well the roller-coaster's restraining system works for people of various heights.

They have divided their potential customers into three categories, each of which they estimate is the same size: Adult Females, Adult Males, and Children. The height of each of these populations is modeled as Gaussian (in inches):

- Adult females have a mean height of 63 and a standard deviation of 4.
- Adult males have a mean height of 69 and a standard deviation of 6.
- Children have a mean height of 48 and a standard deviation of 12.

The park is considering possible rules for prohibiting people from riding based on height. Each of the prohibitions below should be considered individually.

(a) If the park prohibits people who are less than 36 inches tall, what is the probability that a randomly selected child will be prohibited from riding?

(b) If the park prohibits people who are outside of the range (55, 71) inches tall, what is the probability that a randomly selected Adult Female will be prohibited from riding?

(c) If the $X(u)$ is the random variable modeling height, provide an expression for the pdf of $X(u)$ and sketch this pdf.

(d) The most cautious of the lawyers suggests that only people between 60 and 71 inches in height should be allowed to ride. Find the probability that a randomly selected person will be prohibited from riding.

(e) A person is selected at random and their height is 67 inches. Given this, what is the probability that the selected person is a Child, Adult Female, or Adult Male?

2.46. The game of “Two Heads” is played by flipping a coin until two heads have occurred. The probability of heads on a given flip is p and the flips are modeled as independent Bernoulli trials.

Let $X(u)$ be the random variable that models the number of flips required to obtain exactly 2 heads.

(a) Specify the values $\{x_k\}$ that $X(u)$ can take and determine the probability mass function (pmf) $p_{X(u)}(k) = \text{Pr}\{X(u) = x_k\}$.

(b) If a fair coin is used, what is the probability that the two-heads condition is reach in 5 or fewer flips? Again, with a fair coin, what is the most probable value of $X(u)$?

(c) Find the mean and variance of $X(u)$ for general p. If a fair coin is used, what is the mean number of flips to reach the two-heads condition?

2.47. Let $X(u)$ have pdf

$$f_{X(u)}(x) = \frac{\alpha}{2}e^{-\alpha|x|}$$

where $\alpha > 0$ is a parameter. Determine and sketch the cdf for $X(u)$. You have access to a random number generator that produces realizations of $V(u)$ which is uniformly distributed on $[0, 1]$ and you wish to generate $Y(u) = g(V(u))$ so that $F_{Y(u)}(z) = F_{X(u)}(z)$. Define and sketch the function $g(v)$.

2.48. Consider the following compound random experiment. First, one of two coins is randomly selected from a bin containing one fair coin and one unfair coin. The fair coin comes up heads with probability 1/2 and the unfair coin comes up heads with probability p. Once a coin has been selected, it is flipped. If tails occurs, then a fair die is rolled once. If heads occurs, two fair dice are rolled. Let $X(u)$ model the total number of dots on the die or dice. Determine and sketch the following:

(a) $\text{Pr}\{X(u) = d|\text{tails}\}$

(b) $\text{Pr}\{X(u) = d|\text{heads}\}$

(c) $\text{Pr}\{X(u) = d\}$

2.49. A laboratory laser intensity measurement has been determined to be well-modeled by a Rayleigh random variable. Specifically, this is modeled by the random variable $R(u)$ with cdf given by

$$F_{R(u)}(r) = (1 - e^{-r^2/2})\text{U}(r)$$

It is desired to use this source of randomness to generate a Gaussian random variable. Specifically, a function $g(\cdot)$ is sought so that $X(u) = g(R(u))$ is standard Gaussian (*i.e.,* zero mean and variance one).

(a) Determine $\Pr\{R(u) < 1.2\}$ and $\Pr\{R(u) > 1.2\}$

(b) Determine and sketch the desired function $g(r)$ so that $X(u) = g(R(u))$ has a standard Gaussian random distribution.

(c) If the value of $r = 2.145$ is measured what is $g(r)$?

2.50. You wish to learn about the probability of some event A. Initially you do not have any idea what this probability is so you model it as being uniformly distributed on $[0, 1]$. Specifically, you model the probability of A occurring as a random variable $P(u)$ with uniform distribution on $[0, 1]$.

The goal is learn about this probability by accessing some data sets and determining in which sets A has occurred and in which sets A has not occurred.

(a) Determine the mean and variance of $P(u)$ and $\Pr\{|P(u) - 1/2| > 1/4\}$

(b) Suppose that there are n observations available and A occurs in exactly k of these ($k \leq n$) – let this event be denoted by $B_{n,k}$. Find an expression for the pdf of $P(u)$ given $B_{n,k}$. Simplify as much as possible.

(c) Consider the specific case of $n = 6$ and $k = 4$. Simplify and sketch the conditional pdf from the previous part.

(d) Consider the same problem with $n = 600$ and $k = 400$. Use reasonable and accurate approximations to obtain a tractable expression for the result.

2.51. Consider scheduling a meeting where n people must attend. Assume that there are T possible meeting times to consider. Assume that each of the required attendees is available in each meeting time with probability p (*i.e.,* otherwise they are already scheduled for another meeting). Also assume that availability across possible meeting times for a given person is independent and that all attendees have independent schedules. What is the probability that a meeting can be successfully scheduled? In other words, what is the probability that all required attendees are available for at least one meeting time? What is this probability when $p = 3/5$, $n = 6$, and $T = 10$?

2.52. Let $V(u)$ be uniformly distributed on $(0, 1)$. Conditioned on $V(u) = v$, $X(u)$ is uniform on $(0, v)$. Find $f_{X(u)}(x)$ and $f_{V(u)|X(u)}(v|x)$.

2.53. An investing club has n members. The club has decided to vote on whether or not to jointly invest in each of M different funds. For each fund, all n members vote. If no more than 1 member votes NO, then the club will invest in the fund. Investor preferences are modeled as independent from fund to fund with each investor voting to invest in a given fund with probability p. Investors are also assumed to vote independently of each other. What is the probability mass function for the number of funds in which the club invests?

2.54. Consider the following game: a coin is flipped until 3 heads and 2 tails are observed – denote 3 heads and 2 tails as a "full house". The probability of a heads for each flip is p. Let $X(u)$ model the number of flips required to first obtain a full house.

(a) Determine the probability mass function for $X(u)$ – *i.e.,* $p_{X(u)}(k) = \Pr\{X(u) = z_k\}$ for each of the values z_k from above.

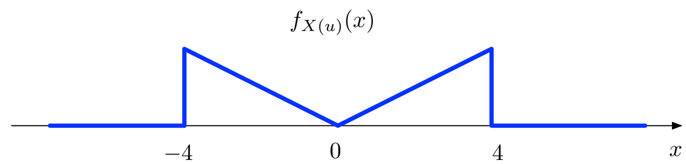

Figure 8: The pdf of a random variable $X(u)$.

(b) If you were allowed to select the coin, what value of p would you select to maximize the probability that a full house is obtained in the first 5 flips? What is the corresponding probability?

(c) Consider this game using a *fair* coin. Observers can wager on the outcome of the game – *e.g.,* bet that a full house is obtained in n or fewer coin flips. You are asked to design a strategy by a conservative bettor. The bettor wants to select n so that the probability that he wins is at least 0.75. Determine this value of n and the corresponding probability of a full house in n or fewer flips.

2.55. Consider the "full house" game from Prob. 2.54. Recall that one flips a coin, that lands on heads with probability p, until 3 heads and 2 tails are observed. Consider this game with a fair coin and with the following betting rules. A player bets \$1 and she is returned:

- \$1.50 is she flips a full house in 5 flips,
- \$1.25 is she flips a full house in 6 flips,
- \$1.00 is she flips a full house in 7 flips,
- \$0 is she flips a full house in 8 or more flips.

Let $Z(u)$ model the *net winnings* for playing this game once – *e.g.,* if she obtains a full house in 5 flips, her net winnings are \$0.50 since she gets her dollar back, plus 50 cents.

Find the probability of a positive net winnings and the mean and variance of $Z(u)$. Does it make sense (financially) to play this game, repeatedly, if you have a large amount of money to bet?

2.56. The continuous random variable $X(u)$ has pdf as shown in Fig. 8. Determine the following:

(a) m_X

(b) σ_X^2

(c) $\Pr\{X(u) > 0\}$

(d) $\Pr\{|X(u)| > 2\}$

(e) $\mathbb{E}\{[X(u)]^3\}$

2.57. A coin is flipped n times and the number of heads is modeled by $X(u)$. The probability that a heads occurs on each flip is p.

(a) What is the probability mass function of $X(u)$?

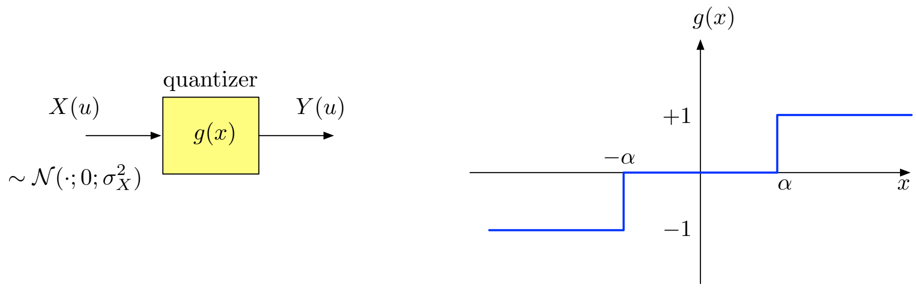

Figure 9: A 3-level quantizer.

(b) Let $\hat{P}(u) = X(u)/n$ be the random variable that models the fraction of flips that are heads. What is the mean and variance of $\hat{P}(u)$?

(c) Assuming that $np(1-p) \gg 1$, determine approximate values for $\Pr\left\{|\hat{P}(u) - p| < \epsilon\right\}$. Assuming that the coin is fair and $\epsilon = 0.1$, evaluate the above expression for the following values of n: 16, 100, 1024.

2.58. You have access to a random number generator that generates realizations of a random variable $W(u)$ that is uniform on $[-\pi, +\pi]$. It is desired to convert this to a random number generator that generates realizations of a Rayleigh random variable. A Rayleigh random variable $X(u)$ has pdf given by

$$f_{X(u)}(x) = \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} \mathrm{U}(x)$$

where $\mathrm{U}(x)$ is the unit step function – *i.e.,* the pdf is zero for negative values of x.

Find a function $h(\cdot)$ such that $h(W(u))$ has the above Rayliegh distribution.

2.59. How much information is there in the roll of a fair die (in bits/roll)? How much info is there in a roll of a loaded die with: $p(1) = 0.4$, $p(2) = 0.1$, $p(3) = 0.01$, $p(4) = 0.09$, $p(5) = 0.25$, $p(6) = 0.15$?

2.60. What is the maximum rate for error free communication (capacity) of a binary symmetric channel with error probability $\epsilon = 0.4$? How about ϵ of 0.2, 0.1, 0.01, 0.0001?

2.61. A continuous random variable $X(u)$ has pdf

$$f_{X(u)}(x) = \frac{1}{2} e^{-|x|}$$

Find the mean, variance, and cdf of $X(u)$. Let $Y(u) = 10 - 4X(u)$. Find the mean, variance, and pdf of $Y(u)$

2.62. This problem considers a simple quantizer as illustrated in Fig. **??**.

Specifically, the input to the 3-level quantizer is $X(u)$, modeled as a zero-mean Gaussian random variable with variance σ_X^2. The output of the quantizer $Y(u)$ takes 3 values as defined by the sketch of $g(x)$.

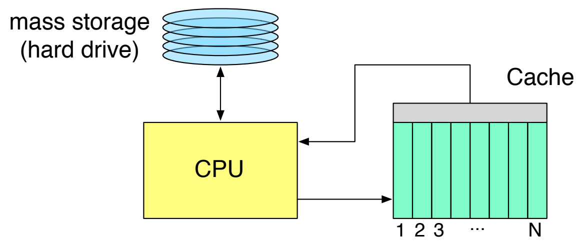

Figure 10: Simple model for a memory cache system.

(a) Determine the pmf of $Y(u)$. What is the mean and variance of $Y(u)$?

(b) If it is desired to design the quantizer so that $Y(u)$ is uniform on its three values, what value of α should be used?

2.63. This problem considers a simple model for a computer cache memory system. This model is as illustrated in Fig. 10.

The computer uses two types of memory: large, slow mass storage and small, fast cache memory. When the CPU executes an instruction, it checks to see if the required data is in the cache. If so, it pulls it out of the cache; if not it pulls the data from mass storage. After the instruction is executed, the data used is written into the cache. The cache holds N of these data items – *i.e.,* the N most recently used by the CPU.

The cache is a FIFO (first-in, first out) memory. That is, the oldest item is removed to make room for the newest item. Thus, a specific data item leaves the cache in one of two ways: it is pulled by the cache for use or it is pushed out of the cache after N instruction cycles – *i.e.,* in this latter case, we say that the data item "aged-out".

(a) Let $X(u)$ model the time, in instruction cycles from it being used, until a given data item is removed from the cache for the first time. Specifically, $X(u) = 1$ corresponds to the data being used in the next cycle after it was first used, $X(u) = N$ corresponds to the data being pulled by the CPU when it is the oldest item in the cache, and $X(u) = N+1$ corresponds to the data item aging out.

The probability that the CPU needs the given data item is p for each cycle and is modeled as independent over cycles. Determine the probability mass function for $X(u)$. Sketch this pmf for the case of $p = 0.25$ and $N = 4$.

(b) The cache hit probability P_{hit} is the probability that the given data item is in the cache when the CPU needs it – *i.e.,* the probability that the given data item has not aged after being written to the cache before the CPU needs it. For a minimum desired hit probability $P_{\text{hit,min}}$ and a given p, find the minimum cache size $N_{\min}$ so that $P_{\text{hit}} \geq P_{\text{hit,min}}$. For the specific case of $p = 10^{-4}$ and $P_{\text{hit,min}} = 0.8$, what is $N_{\min}$ (numerical)?

(c) What is the mean of $X(u)$ from part (a)?

(d) If the cache is designed to satisfy the $P_{\text{hit}} \geq P_{\text{hit,min}}$ condition in (c) – *i.e.,* $N = N_{\min}$, what is the mean as a function of p, and $P_{\text{hit,min}}$? Specifically, provide a simple approximation of the mean in terms of $P_{\text{hit,min}}$ and p. For the specific case of $p = 10^{-4}$ and $P_{\text{hit,min}} = 0.8$, and $N = N_{\min}$ what is the mean (numerical)?

2.64. $X(u)$ is Gaussian with mean 2 and variance 4. Consider $Y(u) = 2X(u) - 1$ and determine the mean and variance of $Y(u)$ as well as $\Pr\{Y(u) > 11\}$.

2.65. $X(u)$ has mean 10 and variance 4, find an upper bound for $\Pr\{-6 < X(u) < 26\}$

2.66. What is the probability that a Gaussian random variable with mean m and variance σ^2 is between one and two standard deviations away from its mean?

2.67. Let $X(u)$ be a discrete random variable taking on integer values with pmf given by:

$$p_{X(u)}(k) = \frac{1}{3}2^{-|k|} \qquad k = 0, \pm 1, \pm 2, \pm 3 \ldots$$

(a) Determine $\Pr\{|X(u)| < 10\}$

(b) Let $Y(u) = [X(u)]^2$. Are $X(u)$ and $Y(u)$ are independent? Are $X(u)$ and $Y(u)$ are uncorrelated?

2.68. $X(u)$ and $Y(u)$ have the same variance, σ^2 and correlation coefficient ρ_{xy}. Let $S(u) = X(u) + Y(u)$ and $D(u) = X(u) - Y(u)$. What is the normalized correlation coefficient for $S(u)$ and $D(u)$?

2.69. The speed that cars travel on the 110 freeway at 3:00 during a weekday is 30 mph on average with a standard deviation of 5 mph. Consider the probability of a car traveling more than 60 mph during this time – i.e., denote this as P. What is a good bound for the largest value that this probability can take?

3 Pairs of Random Variables and Random Vectors

3.1. Sketch the region in the $(X(u), Y(u))$-plane corresponding to the following events:

(a) $\{X(u) - Y(u) \leq 2\}$

(b) $\{e^{X(u)} < 6\}$

(c) $\{\max(X(u), Y(u)) < 6\}$

(d) $\{\min(X(u), Y(u)) < 6\}$

(e) $\{|X(u) - Y(u)| \leq 2\}$

(f) $\{|X(u)| > |Y(u)|\}$

(g) $\{X(u)/Y(u) < 1\}$

(h) $\{[X(u)]^2 \leq Y(u)\}$

(i) $\{X(u)Y(u) \leq 2\}$

3.2. Determine $\Pr\{X(u) < Y(u)\}$ and $\Pr\{X(u) - Y(u) \leq 10\}$ when

$$f_{X(u)Y(u)}(x, y) = 2e^{-(x+2y)}\mathrm{U}(x)\mathrm{U}(y). \tag{17}$$

3.3. Determine the marginal pdf's of $X(u)$ and $Y(u)$ if

$$f_{X(u)Y(u)}(x, y) = xe^{-x(1+y)}\mathrm{U}(x)\mathrm{U}(y). \tag{18}$$

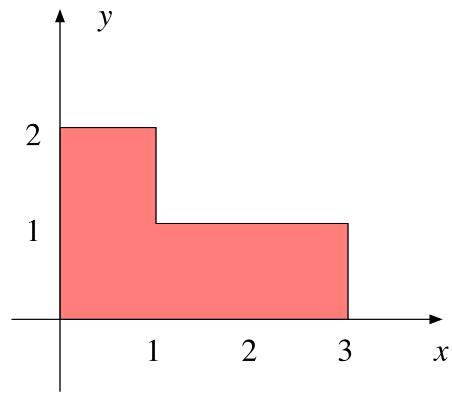

Figure 11: The joint pdf considered in problem 3.5.

3.4. A fair coin is flipped 5 times. Let $X(u)$ be the number of "heads" observed and let $Y(u) = 4[X(u)]^2$. Find the joint probability mass function for $X(u)$ and $Y(u)$.

3.5. Consider random variables $X(u)$ and $Y(u)$ with joint pdf function $f_{X(u),Y(u)}(x,y) = 1/4$ in the shaded area and zero outside the shaded as shown in Fig. 11.

(a) Find $\Pr\{X(u) > Y(u)\}$

(b) Find and sketch $f_{X(u)|Y(u)}(x|y)$.

(c) Find $\Pr\{1 < X(u) < 2|Y(u) = 0.5\}$ and $\Pr\{1 < X(u) < 2|Y(u) = 1.5\}$

(d) Find $\Pr\{1/4 < X(u) < 2|Y(u) = 0.1\}$ and $\Pr\{1 < X(u) < 2|Y(u) = 2.5\}$

3.6. The joint probability density function of $X(u)$ and $Y(u)$ (i.e., $f_{X(u),Y(u)}(x,y)$) is equal to a constant K in $\{(x,y) : 0 \le x < 2, 0 \le y < 2\}$. Determine

(a) K

(b) The pdf of $X(u)$

(c) $\Pr\{0 \le Y(u) \le 1|X(u) = 0.5\}$

3.7. Determine the pdf of $Z(u) = X(u)+Y(u)$ when $X(u)$ and $Y(u)$ are independent and uniformly distributed on $[0,1]$.

3.8. The joint pdf of $X(u)$ and $Y(u)$ is 2 in the shaded region shown in Fig. 12 and 0 outside this region.

(a) Find the marginal pdf's: $f_{X(u)}(x)$ and $f_{Y(u)}(x)$

(b) Are $X(u)$ and $Y(u)$ statistically independent?

(c) Let $Z(u) = X(u) + Y(u)$, and find the pdf of $Z(u)$. HINT: This part is a little tedious; start by drawing the lines $y = z - x$ on the above plot for different values of z. Then perform the integration by considering the following cases separately:

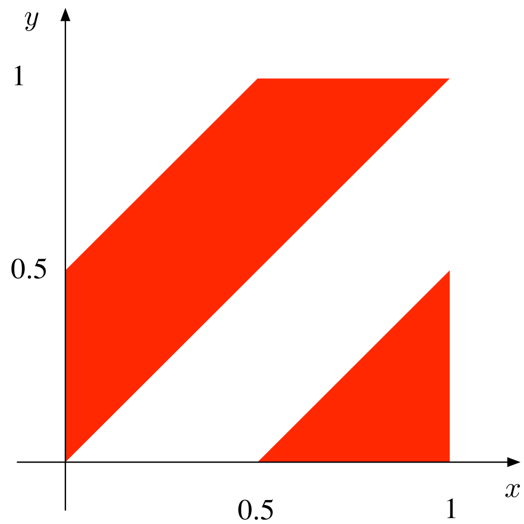

Figure 12: The joint pdf for problem 3.8.

- $z \in [0, 0.5)$
- $z \in [0.5, 1)$
- $z \in [1, 1.5)$
- $z \in [1.5, 2)$

(d) Comparing your solution with problem 3.7, what is the conclusion?

3.9. USC maintenience stores light bulbs in either shed 1 or shed 2 with equal probability. Let $S(u)$ be a random variable modeling which shed a bulb comes from (i.e. $S(u) = i \iff$ the bulb is from shed i). Let $X(u)$ be the the time until the bulb burns out after installation. Bulbs from the different sheds fail with different probability:

$$f_{X(u)|S(u)}(x|1) = 2e^{-2x}\mathrm{U}(x)$$
$$f_{X(u)|S(u)}(x|2) = \begin{cases} 1/10 & x \in [0, 10] \\ 0 & \text{otherwise.} \end{cases}$$

Determine the following:

(a) The mean of $X(u)$

(b) $\Pr\{S(u) = 1|X(u) = 5\}$

(c) $\Pr\{S(u) = 1|X(u) = 12\}$

3.10. The purpose of this problem is to verify some of the notation and results concerning jointly-Gaussian random variables. $X(u)$ and $Y(u)$ are jointly-Gaussian if their joint-pdf is of the form

$$f_{X(u)Y(u)}(x, y) = \frac{\exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sigma_X\sigma_Y} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)\right]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}.$$

(a) Verify that the marginal densities are

$$\begin{aligned} f_{X(u)}(x) &= \mathcal{N}(x; m_X; \sigma_X^2) \\ f_{Y(u)}(y) &= \mathcal{N}(y; m_Y; \sigma_Y^2). \end{aligned}$$

Note that once you have verified one of the above equations the other follows by symmetry.

(b) Verify that the above density can be written as

$$\begin{aligned} f_{X(u)Y(u)}(x,y) &= \mathcal{N}_2(\mathbf{z}; \mathbf{m}; \mathbf{K}) \\ &= \frac{1}{2\pi\sqrt{\det(\mathbf{K})}} \exp\left[\frac{-1}{2}(\mathbf{z}-\mathbf{m})^{\mathrm{t}}\mathbf{K}^{-1}(\mathbf{z}-\mathbf{m})\right], \end{aligned}$$

where

$$\begin{aligned} \mathbf{z} &= \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{m} &= \begin{pmatrix} m_X \\ m_Y \end{pmatrix} \\ \mathbf{K} &= \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix} \end{aligned}$$

and $(\cdot)^{\mathrm{t}}$ denotes the transpose of a matrix/vector. What is the form of $\mathbf{K}$ when $X(u)$ and $Y(u)$ are independent?

(c) Verify that the pdf of $Y(u)$ conditioned on $X(u)$ is

$$f_{Y(u)|X(u)}(y|x) = \mathcal{N}(y; m_{Y|X}(x); \sigma_{Y|X}(x)^2), \tag{19}$$

where

$$\begin{aligned} m_{Y|X}(x) &= m_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - m_X) \\ \sigma_{Y|X}(x)^2 &= \sigma_{Y|X}^2 = (1-\rho^2)\sigma_Y^2. \end{aligned}$$

HINT: For parts (a) and (c) you'll need to complete a square. Recall:

$$z^2 - 2yz = (z-y)^2 - y^2. \tag{20}$$

3.11. Let $X_1(u)$ and $X_2(u)$ be independent Gaussian random variables with zero mean variance 1. The random variables $Y_1(u)$ and $Y_2(u)$ are defined by

$$\begin{aligned} Y_1(u) &= 2X_1(u) + X_2(u) \\ Y_2(u) &= 3X_1(u) + 4X_2(u). \end{aligned}$$

Determine the following:

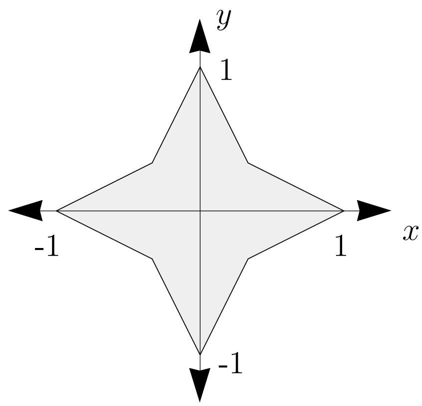

Figure 13: The joint pdf considered in problem 3.12.

(a) The marginal pdf of $Y_1(u)$

(b) The joint pdf of $Y_1(u)$ and $Y_2(u)$.

(c) $f_{Y_1(u)|Y_2(u)}(y_1|y_2)$

3.12. The joint probability density function (pdf) of $X(u)$ and $Y(u)$, $f_{XY}(x, y)$, is nonzero, but not necessarily constant, *only* in the shaded region in Fig. 13.

(a) Are $X(u)$ and $Y(u)$ independent? Is there enough information provided to answer this?

(b) Are $X(u)$ and $Y(u)$ are uncorrelated? Is there enough information provided to answer this?

(c) Is $\mathbb{E}\left\{|X(u)[Y(u)]^3|\right\}$ equal to, greater than, or less than 1? Is there enough information provided to answer this?

3.13. Listed below are random variables and the quantities which they are intended to model. In each case, determine whether the correlation coefficient ρ is positive, negative, or zero.

(a) Dental Hygene:

$$B = \text{number of minutes per day a person spends brushing their teeth}$$
$$C = \text{number of cavities a person has in a year}$$

(b) Performance Evaluation

$$E = \text{an employee's performance evaluation rating: 1 to 10, 10 being best}$$
$$S = \text{employee salary}$$

(c) Personal Numbers

$$\begin{aligned} T &= \text{a person's telephone number} \\ W &= \text{a person's weight} \end{aligned}$$

3.14. For each of the joint-densities in problems 3.2 and 3.3, find $f_{Y(u)|X(u)}(y|x)$ and $f_{X(u)|Y(u)}(x|y)$. In each case also state whether the two random variables are independent or not.

3.15. Let $X(u)$ and $Y(u)$ be independent Exponential random variables, each with mean $1/\lambda$, and let $Z(u) = X(u) + Y(u)$. Determine the mean and the n^{th} moment of $Z(u)$.

3.16. Show that the sum of two independent Cauchy random variables is a Cauchy random variable.

3.17. The joint density of $X(u)$ and $Y(u)$ is

$$f_{X(u)Y(u)}(x, y) = 2x^2 e^{-xy} e^{-x^2} \mathrm{U}(x)\mathrm{U}(y). \tag{21}$$

Determine the joint-pdf of the two random variables

$$W(u) = \frac{X(u)}{Y(u)} \qquad\qquad Z(u) = X(u)Y(u).$$

3.18. Let $X(u)$ and $Y(u)$ be jointly Gaussian random variables with means and variances: $m_X = 2$, $\sigma_X^2 = 4$, $m_Y = -2$, $\sigma_Y^2 = 2$, respectively. Also, let the correlation coeffiecent be $\rho_{X,Y} = -0.5$. Let

$$Z(u) = 3X(u) + 4Y(u) - 5$$

Determine the following the mean and variance of $Z(u)$ and $\Pr\{0 \leq Z(u) \leq 1\}$.

3.19. The normalized homework score (from 0 to 100%) can be modeled as a random variable $H(u)$. The normalized test scores (from 0 to 100%) can be modeled as a random variable $T(u)$. Based on past results, the second moment description of these random variables as

$$\begin{aligned} m_H &= 48.1 & \sigma_H &= 28.9 \\ m_T &= 36.4 & \sigma_T &= 17.6 \\ \mathbb{E}\left\{(H(u) - m_H)(T(u) - m_T)\right\} &= 273 \end{aligned}$$

Assuming that these are good estimates -

(a) What is the correlation coefficient between $H(u)$ and $T(u)$? Explain the meaning.

(b) Given your performance on the homework, predict your exam score.

(c) How close is the exam score estimate to your actual midterm exam scores? Discuss the error - i.e., if your actual exam score is higher than the estimate, what does that say about your study habits?

(d) What is the MSE of the exam score estimator?

3.20. The performance of a particular communication system is limited by interference from adjacent channels. The received signal is modeled as

$$X(u) = S_1(u) + \alpha S_2(u), \tag{22}$$

where $S_1(u)$ is the signal in the channel of interest and $S_2(u)$ is the interfering signal from the adjacent channel. The constant α represents an attenuation factor, so that $|\alpha| < 1$.

The signals on adjacent channels are not independent and are modeled as jointly-Gaussian random variables. Both $S_1(u)$ and $S_2(u)$ are zero mean and have variance σ^2. The normalized correlation coefficient for $S_1(u)$ and $S_2(u)$ is ρ.

Your task is design the best (Minimum Mean-Squared-Error) biased-linear estimator of $S_1(u)$ based on observing $X(u)$.

(a) Determine the mean and variance of $X(u)$

(b) Determine COV$[X(u)S_1(u)]$

(c) Find the best linear estimate of $S_1(u)$ based on observing $X(u)$ - denoted by $\hat{S}_1(u)$ - and the corresponding minimum MSE.

(d) While implementing the estimator you designed in part (b), a technician suggests that you can also obtain an estimate of the adjacent channel signal by intuition (the technician doesn't know any probability theory). The technician's estimate is

$$\hat{S}_2^T(u) = \frac{1}{\alpha}\left[X(u) - \hat{S}_1(u)\right]. \tag{23}$$

If you designed the best linear estimate of $S_2(u)$ based on $X(u)$ (denoted by $\hat{S}_2(u)$), would you get the same estimate as the technician?

3.21. Let the joint pdf of $X_1(u)$ and $X_2(u)$ be

$$f_{\mathbf{x}(u)}(x_1, x_2) = \frac{1}{2\pi\sqrt{3}} \exp\left(\frac{-1}{3}\left[(x_1-1)^2 - (x_1-1)(x_2-2) + (x_2-2)^2\right]\right). \tag{24}$$

(a) Find $\mathbf{m}_{\mathbf{x}}$ and $\mathbf{K}_{\mathbf{x}}$. Hint: This is a joint-Gaussian pdf.

(b) Any pair of jointly-Gaussian random variables can be transformed into two independent jointly-Gaussian random variables using a biased linear transformation. In this case show that $Y_1(u)$ and $Y_2(u)$ are independent Gaussians, when

$$\mathbf{y}(u) = \begin{bmatrix} Y_1(u) \\ Y_2(u) \end{bmatrix} = \mathbf{A}(\mathbf{x}(u) - \mathbf{m}_{\mathbf{x}})$$
$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(c) Can you find a matrix $\mathbf{B}$ so that

$$\mathbf{z}(u) = \mathbf{B}\mathbf{y}(u) \tag{25}$$

is a Gaussian random vector with $\mathbf{K}_{\mathbf{z}} = \mathbf{I}$?

3.22. Let $X(u)$ and $Z(u)$ be jointly-Gaussian random variables, each with mean zero and variance 1. Also, let ρ be the correlation coefficient between $X(u)$ and $Z(u)$. Given that $X(u)$ is observed to be 0.91, what is the best MMSE estimate of $Z(u)$? Given this same realization of $X(u)$, what is the best estimate of $Z(u)$ based on a linear function of $X(u)$?

3.23. Consider random variables $X(u)$ and $Y(u)$ with joint pdf function $f_{X(u),Y(u)}(x,y) = 1/4$ in the shaded area and zero outside the shaded as shown shown in Fig. 11.

(a) Find the MMSE linear estimate of $X(u)$ given $Y = y$.

(b) Find the MMSE estimate of $X(u)$ given $Y = y$.

(c) Sketch the functions $g(y)$ and $h(y)$.

3.24. Let $X(u)$ and $Y(u)$ have variance $\sigma^2 = \sigma_X^2 = \sigma_Y^2$. Find the variances and covariance for the random variables $W(u) = (X(u) + Y(u))/\sqrt{2}$ and $Z(u) = (X(u) - Y(u))/\sqrt{2}$. Note, you can assume that $X(u)$ and $Y(u)$ have zero means.

3.25. Let $X(u)$ and $Y(u)$ be independent, zero mean Gaussian random variables, each with variance σ^2. Determine $\Pr\left\{\sqrt{X(u)^2 + Y(u)^2} \leq r\right\}$. Discuss the relation to problem 2.8.

3.26. The joint pdf of $X(u)$ and $Y(u)$ is

$$f_{X(u)Y(u)}(x,y) = \frac{1}{2\pi} \exp\left(\frac{-(x^2+y^2)}{2}\right). \tag{26}$$

Determine the following:

(a) $\Pr\{|X(u)| \leq 1, |Y(u)| \leq 1\}$

(b) $f_{X(u)}(x)$

(c) m_X and σ_X^2

3.27. This problem addresses a method of converting two independent uniform random variable to two independent Gaussian random variables. Consider the independent uniformly distributed random variables $X_1(u)$ and $X_2(u)$

$$f_{X_1(u)}(z) = f_{X_2(u)}(z) = \begin{cases} 1 & z \in (0,1) \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

The purpose of this problem is to demonstrate that the following are independent Gaussian random variables:

$$\begin{aligned} Y_1(u) &= \sqrt{-2\ln(X_1(u))}\cos(2\pi X_2(u)) \\ Y_2(u) &= \sqrt{-2\ln(X_1(u))}\sin(2\pi X_2(u)). \end{aligned}$$

(a) Determine the following: $f_{X_1(u)X_2(u)}(x_1,x_2)$, $\mathbb{E}\{Y_1(u)\}$ and $\mathbb{E}\{Y_2(u)\}$, and $\mathbb{E}\{Y_1(u)Y_2(u)\}$

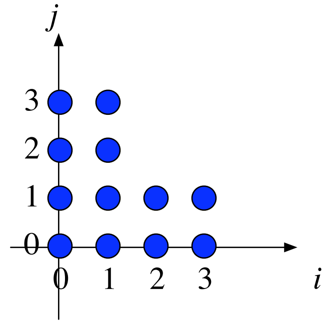

Figure 14: The region of nonzero joint pmf for problem 3.28.

(b) Consider the random variable $R(u) = \sqrt{-2\ln(X_1(u))}$. Determine the pdf $f_{R(u)}(r)$ and mean of this random variable.

(c) Determine the joint density of $Y_1(u)$ and $Y_2(u)$, generated as described above

(d) Answer the following questions:

- Are $X_1(u)$ and $X_2(u)$ uncorrelated?
- Are $X_1(u)$ and $X_2(u)$ orthogonal?
- Are $Y_1(u)$ and $Y_2(u)$ uncorrelated?
- Are $Y_1(u)$ and $Y_2(u)$ orthogonal?
- Are $Y_1(u)$ and $Y_2(u)$ independent?
- Are $R(u)$ and $X_2(u)$ independent?

3.28. The discrete random variables $X(u)$ and $Y(u)$ have joint pmf $p_{X(u),Y(u)}(i,j)$ that is nonzero only for the integer values of i and j as shown in Fig. 16. Furthermore, the pmf is equal to a constant C for these values of i and j.

(a) Determine the constant C and $\Pr\{Y(u) \geq X(u) - 1\}$

(b) Determine and sketch the marginal pmfs of $X(u)$ and $Y(u)$.

(c) Find the mean and variance of $X(u)$ and $Y(u)$:

(d) Determine the correlation between $X(u)$ and $Y(u)$ and the normalized correlation coefficent

3.29. For each of the joint-pdf's, find the following:

- The best (unconstrained) MSE estimate of $X(u)$ based on observing $Y(u)$, and the corresponding MSE.
- The best biased-linear estimate of $X(u)$ based on observing $Y(u)$, and the corresponding MSE.

(a) The joint-pdf of problem 5.28 in Leon-Garcia.

(b) The joint-pdf of problem 3.21 with $X(u) = X_1(u)$ and $Y(u) = X_2(u)$.

(c) What conclusions can you draw from these results?

3.30. The continuous random variables $X(u)$ and $Y(u)$ have joint pdf given by

$$f_{X(u),Y(u)}(x,y) = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1, \text{and } x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the region in the (x,y)-plane where $f_{X(u),Y(u)}(x,y) \neq 0$.

(b) Find and sketch the marginal pdfs

(c) Find and sketch the pdf of $Y(u)$ conditioned on $X(u) = x$.

(d) Conditioned on $X(u) = x$, the minimum mean-squared error (MMSE) estimate of $Y(u)$ is $\hat{y}_{MMSE} = g_M(x)$. Determine and sketch the function $g_M(x)$. For each of the realizations of x, provide the estimate $\hat{y}_{MMSE}$: $x = 0$, $x = 1/2$, and $x = 1$.

(e) Conditioned on $X(u) = x$, the linear minimum mean-squared error (LMMSE) estimate of $Y(u)$ is $\hat{y}_{LMMSE} = g_L(x) = ax + b$, for some constants a and b. Determine and sketch the function $g_L(x)$.

(f) Determine the second moment description of $X(u)$ and $Y(u)$ (means, variances and correlation coefficient).

(g) Consider the following two statements:

A: Random variables $W(u)$ and $Z(u)$ are jointly Gaussian.

B: The LMMSE and MMSE estimates of $Z(u)$ from W are the same.

State whether the following are TRUE or FALSE:

i. If **A**, then **B**

ii. If **B**, then **A**

3.31. The joint pdf of $X(u)$ and $Y(u)$ is non-zero only inside the "bow-tie" shaped region shown in Fig. 15

Inside this shaded region, the joint pdf is

$$f_{X(u),Y(u)}(x,y) = Kxy$$

where K is a constant.

(a) Determine K and find the following probability: $\text{Pr}\{0 < X(u) < 1/2, 0 < Y(u) < 1/2\}$.

(b) Determine and sketch the marginal pdf of $X(u)$

(c) Determine and sketch the marginal pdf of $Y(u)$

(d) Find the following probability: $\text{Pr}\{0 < Y(u) < 1/2\}$.

(e) Are $X(u)$ and $Y(u)$ independent?

(f) Determine the conditional pdf of $Y(u)$, given $X(u)$ and sketch this for $X(u) = 1/4$.

(g) Find the following probability: $\text{Pr}\{0 < Y(u) < 1/2 | X(u) = 1/4\}$.

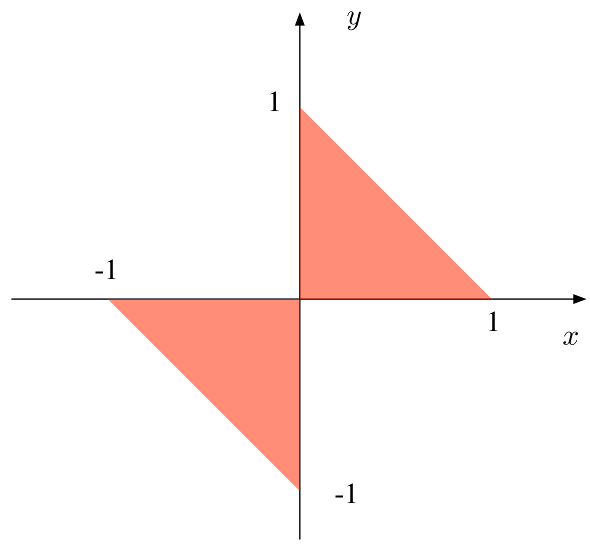

Figure 15: The pdf considered in Prob. 3.31.

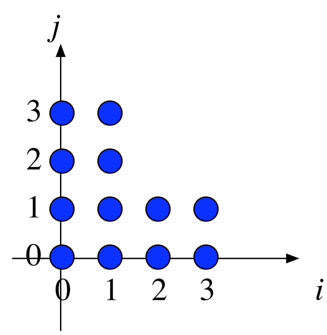

Figure 16: The joint pmf considered in problem 3.33.

3.32. Let $Z(u) = (X(u)+V(u)) \bmod 1$, where "mod 1" means modulo 1. For example, $3.25 \bmod 1 = 10.25 \bmod 1 = 0.25 \bmod 1 = 0.25$. Here $X(u)$ and $V(u)$ are independent and $V(u)$ is uniformly distributed on $[0, 1)$.

If $X(u)$ is Gaussian with mean m and variance σ^2, find the pdf of $Z(u)$. Repeat this for the case of $X(u)$ exponentially distributed with mean $1/\lambda$.

3.33. Consider the pair of discrete random variables with joint pmf $p_{X(u),Y(u)}(i, j)$ that is a constant in for the shaded values illustrated in Fig. 16.

(a) Determine the correlation coefficient for this pair of random variables

(b) Determine and sketch the conditional pmf of $X(u)$, given that $Y(u) = 1$

3.34. The joint pdf of $X(u)$ and $Y(u)$ is a non-zero constant K only inside the region shown in Fig. 17.

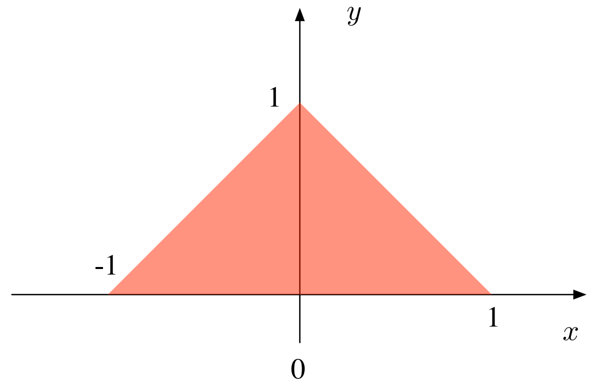

Figure 17: The joint pdf used in problem 3.34. .

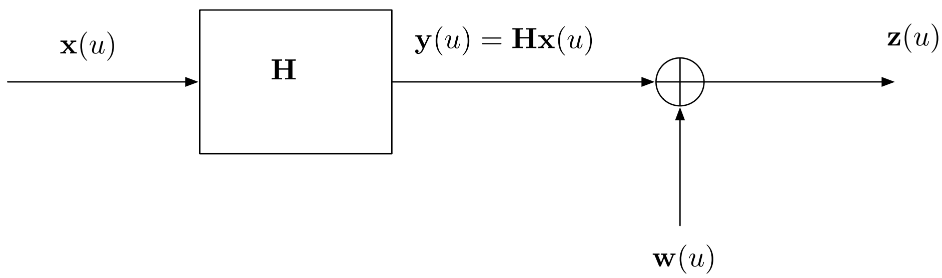

Figure 18: The MIMO system consider in problem 3.35.

(a) Determine the constant K.

(b) Determine and sketch the marginal pdf of $X(u)$ and find the mean and variance of $X(u)$.

(c) Determine the conditional pdf of $Y(u)$, given $X(u)$ and sketch this for $X(u) = 1/4$.

(d) Let the best MMSE estimate of $Y(u)$ from $X(u)$ be $\hat{Y}_{MMSE}(u) = g_U(X(u))$ and the best Affine estimate of $Y(u)$ from $X(u)$ be $\hat{Y}_{AMMSE}(u) = g_A(X(u))$. Determine and sketch these two functions.

3.35. Consider the multiple-input/multiple output (MIMO) communication system shown in Fig. 18. The random vector

$$\mathbf{x}(u) = \begin{pmatrix} X_1(u) \\ X_2(u) \end{pmatrix}$$

is the input. The two components of $\mathbf{x}(u)$ are modeled as independent Gaussians, each with mean zero and variance one. The MIMO channel is modeled by a deterministic matrix and additive noise. Specifically, the matrix is given by

$$\mathbf{H} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

and the additive noise has the same statistical model as $\mathbf{x}(u)$. The noise (*i.e.*, $W_1(u)$ and $W_2(u)$) and the signal (*i.e.*, $X_1(u)$ and $X_2(u)$) are independent.

The signal at the receiver is

$$\begin{pmatrix} Z_1(u) \\ Z_2(u) \end{pmatrix} = \mathbf{z}(u) = \mathbf{H}\mathbf{x}(u) + \mathbf{w}(u)$$

(a) Find the mean vector and covariance matrix of $\mathbf{z}(u)$.

(b) Suppose one has the measurement from one receiver terminal and desires to estimate the signal on the other. Find the best MMSE estimate of $Z_2(u)$ based on $Z_1(u)$ and give the associated MMSE.

(c) For the best estimate of $Z_2(u)$ from the previous part, let the estimation error be $E(u) = Z_2(u) - \hat{Z}_2(u)$. Find the following probability: $\text{PR}\left\{E^2(u) > 6\right\}$.

(d) Given a realization of $\mathbf{z}(u) = \mathbf{z}$, what is the MMSE estimate of $\mathbf{x}(u)$? What is the associated MMSE with this estimate?

3.36. Let $\mathbf{x}(u) = (X_1(u)\ X_2(u))^t$ be a (2×1) Gaussian random vector with

$$\mathbf{m}_{\mathbf{x}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbf{K}_{\mathbf{x}} = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$$

Also, let $\mathbf{y}(u) = \mathbf{A}\mathbf{x}(u)$ with

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

Determine the following:

(a) $f_{X_1(u)}(z)$

(b) ρ_{X1,X_2}

(c) $\mathbb{E}\left\{X_2(u)|X_1(u) = x_1\right\}$

(d) $\mathbb{E}\left\{3X_2(u) + 2X_1(u) - 2|X_1(u) = x_1\right\}$

(e) $f_{\mathbf{y}(u)}(\mathbf{z})$

(f) $\mathbb{E}\left\{Y_2(u)X_1(u)\right\}$

(g) $\text{PR}\left\{Y_2(u) > 2\right\}$

3.37. Prove the following statement: $g(X(u))$ and $h(Y(u))$ are independent for all choices of engineering functions $g(\cdot)$ and $h(\cdot)$ *if and only if* $g(X(u))$ and $h(Y(u))$ are uncorrelated for all choices of engineering functions $g(\cdot)$ and $h(\cdot)$.

3.38. A simple model for a binary communication system in a fading channel is

$$R(u) = X(u)D(u) + W(u)$$

where $W(u)$ is zero mean Gaussian with variance σ_w^2, $X(u)$ is Gaussian with mean $m_X = 1$ and unit variance, and $D(u)$ takes values $+1$ and -1, each with probability $1/2$. The random variables $X(u)$, $D(u)$, and $W(u)$ are mutually independent.

This problem concerns estimation of the data bit $D(u)$ from a realization of the received signal $R(u)$.

(a) Find the second moment description of $R(u)$ and $D(u)$.

(b) Given a realization of the received signal r, find the LMMSE estimate of $D(u)$. Specifically, this best estimate can be written as $g_L(r)$, specify $g_L(r)$. What is the resulting mean squared estimation error?

(c) Given a realization of the received signal r, find the MMSE estimate of $D(u)$. Specifically, this best function $g(r)$ such that $\hat{D}(u) = g(R(u))$ has MMSE. Simplify as much as possible.

(d) Sketch the functions $g_L(r)$ and $g(r)$ from parts (b) and (c) on the same plot. Discuss these two plots and the implications for the MMSE performance of these two estimators:

3.39. Consider the random variable

$$Z(u) = 3X(u) + 2Y(u) + 2$$

where $X(u)$ and $Y(u)$ are jointly-Gaussian random variables with $m_X = 2$, $\sigma_X^2 = 1$, $m_Y = 1$, $\sigma_Y^2 = 4$ and correlation coefficient $\rho = -3/8$. Determine:

(a) m_Z, σ_Z^2 and $\mathrm{cov}\left[Z(u), Y(u)\right]$

(b) $\mathrm{Pr}\left\{|Z(u) - m_Z| > 4\right\}$

(c) $\mathbb{E}\left\{Z(u)|X(u) = 1\right\}$.

3.40. The continuous random variables $X(u)$ and $Y(u)$ have joint pdf given by

$$f_{X,Y}(x, y) = \begin{cases} K & x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

where $r > 0$ is a parameter of the distribution.

(a) Sketch the region in the (x, y)-plane where $f_{X(u),Y(u)}(x, y) \neq 0$ and determine

i. K

ii. $\mathrm{Pr}\left\{X^2(u) + Y^2(u) \leq (r/2)^2\right\}$

iii. Are $X(u)$ and $Y(u)$ independent? Are $X(u)$ and $Y(u)$ uncorrelated?

(b) Find and sketch the marginal pdfs.

(c) Find and sketch the pdf of $Y(u)$ conditioned on $X(u) = x$.

(d) Conditioned on $X = x$, the minimum mean-squared error (MMSE) estimate of Y is $\hat{y}_{\text{MMSE}} = g_M(x)$. Determine and sketch the function $g_M(x)$. Provide the estimate $\hat{y}_{\text{MMSE}}$ for $x = 0$, $x = r/2$ and $x = -r/2$.

(e) Consider the following two statements:

A: Random variables $W(u)$ and $Z(u)$ are independent.

B: The conditional mean of $Z(u)$ given $X(u) = x$ is m_X – *i.e.,* $\mathbb{E}\{Z(u)|X(u) = x\} = m_Z$.

Answer the following true or false questions about these statements by circling the best answer:

i. **TRUE/FALSE:** If **A**, then **B**

ii. **TRUE/FALSE:** If **B**, then **A**

3.41. There are four balls in a bag, each is numbered – *i.e.,* one vll is labeled 1, one labeled 2, etc. Two balls are draw at random without replacement and with order. Let $X(u)$ be the number of the ball drawn on the first draw. Let $Y(u)$ be the number of the ball drawn on the second draw.

(a) Give the joint probability mass function for $X(u)$ and $Y(u)$ by specifying $p_{X(u),Y(u)}(i, j)$ for $i, j \in \{1, 2, 3, 4\}$.

(b) Determine the marginal probability mass functions, means, and variances of $X(u)$ and $Y(u)$.

(c) Find and sketch the pmf of $Y(u)$ conditioned on $X(u) = 2$. Find and sketch the conditional mean of $Y(u)$ conditioned on $X(u) = i$.

(d) What is the correlation coefficient for $X(u)$ and $Y(u)$?

(e) What is the minimum mean-squared error for the best estimate of $Y(u)$ from $X(u)$?

(f) What is the minimum mean-squared error for the best estimate of $X(u)$ from $Y(u)$?

3.42. Use a computer to generate 500 realizations of a uniform random variable – the underlying random variable $V(u)$ will be uniform on $[0, 1]$.

Use this to generate the following:

- A histogram of these realizations (this should approximate the uniform pdf).
- Define $X(u) = aV(u) + b$ and select a and b so that $X(u)$ is uniform with mean zero and variance 1. Produce a histogram of the realizations of $X(u)$, as generated using the realizations of $V(u)$.
- Define $Y(u) = g(V(u))$ so that the pdf of $Y(u)$ is exponential with mean 3. Specify $g(.)$ and produce a histogram of realizations of $Y(u)$, as generated using the realizations of $V(u)$.
- Generate a second set of 500 realizations of uniform random number on the interval $[0, 1]$ – think of this as realizations of $W(u)$. Add each of these to the first set of 500 realizations to produce another set of 500 realizations of $(V(u)+W(u))$. Plot a histogram of the result.
- Using the realizations of $V(u)$ and $W(u)$ above, form the following random variables

$$Z_1(u) = \sqrt{-2\ln[V(u)]}\cos(2\pi W(u))$$
$$Z_2(u) = \sqrt{-2\ln[V(u)]}\sin(2\pi W(u))$$

Produce a histogram for the corresponding relations of $Z_1(u)$ and $Z_2(u)$. Also, produce a scatter plot of 500 points in the plane corresponding to (z_1, z_2).

3.43. Using the 500 realizations of $X(u)$ obtained above – *i.e.,* uniform with zero mean an variance 1 – form sums of these realizations of various sizes. Specifically, let the realizations of $X(u)$ be $x_0, x_1, \ldots x_{499}$. Form the following:

$$r_N(j) = \frac{1}{\sqrt{N}} \sum_{i=jN}^{jN+(N-1)} x_i \qquad j = 0, 1, \ldots (500/N) - 1$$

In words, this just means form non-overlapping sums of size N and scale them by $1/\sqrt{N}$.

Do this for $N = 2$, $N = 5$, $N = 10$ – this will yield 250, 100, and 50 realizations $\{r_N(j)\}$, respectively. Plot a histogram of these realizations for each value of N.

3.44. Consider a sequence of jointly Gaussian random variables $X_n(u)$ for which

$$\mathbb{E}\{X_n(u)\} = 0$$
$$\mathbb{E}\{X_i(u)X_j(u)\} = \rho^{|i-j|}$$

(a) Determine the following pdfs:
   i. $f_{X_{10}(u)}(x_{10})$
   ii. $f_{X_{10}(u)|X_9(u)}(x_{10}|x_9)$
   iii. $f_{X_{10}(u)|X_9(u),X_8(u)}(x_{10}|x_9, x_8)$

(b) Let $Y_n(u) = X_n(u)+W_n(u)$ where $W_n(u)$ is an iid sequence of Gaussian random variables with zero mean and variance σ_w^2. The sequences $X_n(u)$ and $W_n(u)$ are statistically independent (*i.e.,* any collection of X's are independent of any collection of W's).

   Determine the following pdfs:
   i. $f_{Y_{10}(u)}(y_{10})$
   ii. $f_{Y_{10}(u)|Y_9(u)}(y_{10}|y_9)$

(c) With the models from above, consider estimating $X_n(u)$ at a specific index n from $Y_{n-1}(u)$ and $Y_{n-2}(u)$. In words, predict $X_n(u)$ from noisy versions of the previous two values. More precisely, given the realizations y_{n-1} and y_{n-2}, form an estimate of $X_n(u)$ of the form

$$\hat{x}_n = g(y_{n-1}, y_{n-2}) \tag{28}$$

   Find the function $g(\cdot)$ that minimizes the MSE between $X_n(u)$ and $\hat{X}_n(u)$ and provide the associated MMSE. Also, find the limit of this MMSE as $\sigma_w^2 \to \infty$ and as $\sigma_w^2 \to 0$.

3.45. Discrete random variables $X(u)$ and $Y(u)$ both take values in $\{-1, 0, +1\}$. The joint probability mass function for $X(u)$ and $Y(u)$ is

$$p_{X(u),Y(u)}(i, j) = K(|i + j| + 1) \qquad i, j \in \{-1, 0, +1\}$$

where K is a constant.

(a) Determine K and provide and fill-in the table below showing the values of $p_{X(u),Y(u)}(i,j)$

	$i=-1$	$i=0$	$i=1$
$j=+1:$			
$j=0:$			
$j=-1:$			

(b) Provide a labeled sketch of the marginal pmfs:

(c) Determine the mean, variance and covariance for $X(u)$ and $Y(u)$.

(d) Find and provide a label sketch of the conditional probability mass functions:

- $p_{Y(u)|X(u)}(k|-1)$
- $p_{Y(u)|X(u)}(k|0)$
- $p_{Y(u)|X(u)}(k|+1)$
- $p_{X(u)|Y(u)}(k|-1)$
- $p_{X(u)|Y(u)}(k|0)$
- $p_{X(u)|Y(u)}(k|+1)$

(e) Given that $X(u)=i$, find the best estimate of $Y(u)$ under the Linear MMSE, MMSE, Maximum Likelihood, and Maximum A posteriori Probability criteria. Fill in the table below to describe your answer:

	$i=-1$	$i=0$	$i=1$
Linear MMSE:			
MMSE:			
ML:			
MAP:			

(f) What is the residual mean-squared error for the Linear MMSE estimator?

3.46. A "16-QAM" digital modulation used in Wi-Fi can be modeled as a random vector $(X(u),Y(u))$ that takes on the values shown in Fig. 19, each with equal probability. The energy in this type of modulation signal is

$$E=\mathbb{E}\left\{X^2(u)+Y^2(u)\right\}$$

What is the numerical value of energy for this 16-QAM modulation?

3.47. Consider the random vector

$$\mathbf{x}(u)=\begin{pmatrix} X_1(u) \\ X_2(u) \end{pmatrix}$$

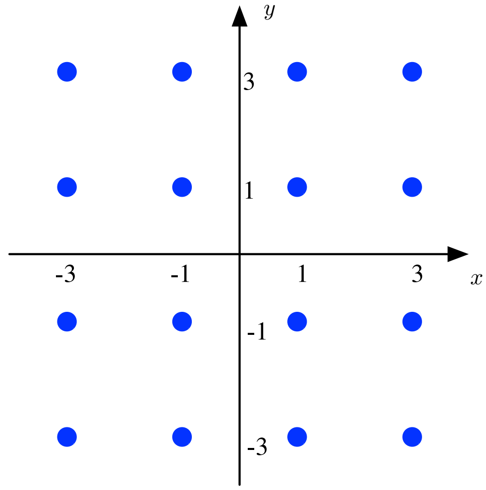

Figure 19: The 16QAM signal constellation.

with $X_1(u)$ and $X_2(u)$ jointly-Gaussian and each having zero mean. The covariance matrix for $\mathbf{x}(u)$ is

$$\mathbf{K}_{\mathbf{x}} = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$$

The random vector $\mathbf{y}(u)$ is related to $\mathbf{x}(u)$ via

$$\begin{pmatrix} Y_1(u) \\ Y_2(u) \end{pmatrix} = \mathbf{y}(u) = \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \mathbf{x}(u)$$

(a) Determine the following:

- $f_{X_2(u)|X_1(u)}(x_2|x_1)$
- m_{Y_1} and m_{Y_2}
- $\sigma^2_{Y_1}$ and $\sigma^2_{Y_1}$
- ρ_{y_1,y_2}
- $f_{Y_1(u),Y_2(u)}(y_1, y_2)$

(b) Determine the probability that $\mathbf{y}(u)$ is in the region C where C is shown in Fig. 20.

3.48. A signal $X(u)$ is observed with a known, deterministic, nonzero gain γ in additive noise $W(u)$

$$Z(u) = \gamma X(u) + W(u)$$

The noise $W(u)$ is Gaussian and it is statistically independent from the signal $X(u)$, which is also modeled as Gaussian. Both the signal and noise are mean zero and have variance σ^2_X and σ^2_W, respectively.

The goal of this problem is to design and analyze a good estimator for $X(u)$ when the observation $Z(u) = z$ is made. In this problem, you will use the Minimum Mean Squared

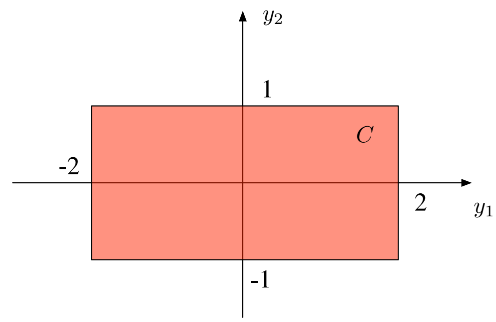

Figure 20: Find the probability that $\mathbf{y}(u)$ is in the shaded region.

Error (MMSE) criterion for designing the estimator. An important quantity in the analysis of this estimator is the Signal power to Noise power Ratio (SNR) given by

$$\text{SNR} = \frac{\gamma^2 \sigma_X^2}{\sigma_W^2}$$

(a) Determine the remaining quantities in the second moment description of the observation $Z(u)$ and the desired signal $X(u)$. Specifically, the mean and variance of $Z(u)$, the covariance for $Z(u)$ and $X(u)$, and the normalized correlation coefficient. These could all be functions of γ, σ_X^2, and σ_W^2.

(b) Find the best Linear MMSE estimator of $X(u)$ from $Z(u) = z$ and the normalized residual MSE error ($\frac{1}{\sigma_X^2}\text{MMSE}_{\text{Linear}}$).

(c) Express the results in (a) and (b) in terms of the SNR (the estimate/normalized-MMSE may depend on other variables).

(d) For the Linear MMSE solution what happens as the SNR goes to 0? What happens as the SNR goes to infinity? Describe the trending behavior for both the estimator of the normalized residual MMSE.

(e) For the specific case of $\gamma = 3$ and SNR$= 4$, evaluate the estimator and $\frac{1}{\sigma_X^2}\text{MMSE}_{\text{Linear}}$.

(f) Repeat part (b) without the linear constraint. In other words, find the normalized-MMSE estimator of $X(u)$ from $Z(u) = z$. Express the results as a function of the SNR.

3.49. Let $S_1(u)$ and $S_2(u)$ represent the midterm 1 and midterm 2 scores, respectively, in EE364. From past data good estimates of the second moment description are available:

$$\begin{aligned} m_{S_1} &= 60 & \sigma_{S_1} &= 10 \\ m_{S_2} &= 80 & \sigma_{S_2} &= 20 \\ \text{cov}\left[S_1(u), S_2(u)\right] &= 100 \end{aligned}$$

Given that a student scored s_1 on the first exam, what would be a good estimate for his/her score on midterm 2? As a concrete example, if $s_1 = 70$, what is the numerical value of the estimate of the second midterm score?

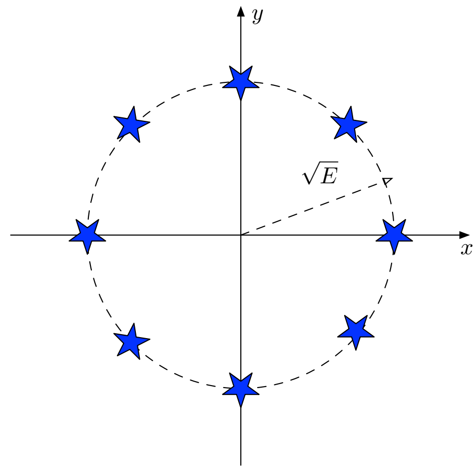

Figure 21: The signal set for 8-ary Phase Shift Keying (8PSK).

3.50. The random variables $X(u)$ and $Y(u)$ have the following properties

$$\text{var}\left[X(u)+Y(u)\right]=\sigma_+^2 \qquad\qquad \text{var}\left[X(u)-Y(u)\right]=\sigma_-^2$$

What is the covariance between $X(u)$ and $Y(u)$?

3.51. The signal set for an 8 Phase Shift Keying (8PSK) digital communication signal is shown in Fig. 21.

The random variables $X(u)$ and $Y(u)$ model the Cartesian coordinates for these points when the probability mass is uniform over the 8 points which have equal angular separation around the circle.

(a) Find and sketch the marginal pmfs for $X(u)$ and $Y(u)$.

(b) For each value of x having nonzero probability, sketch the conditional value of $Y(u)$ given $X(u)=x$:

(c) Determine the second moment description for $X(u)$, $Y(u)$ (means and variance of $X(u)$ and $Y(u)$ and correlation coefficient).

(d) Are $X(u)$ and $Y(u)$ are independent? Are they uncorrelated?

3.52. The scatter plot in Fig. 22 shows 1000 realizations of a Gaussian random vector $(X(u), Y(u))$. Given this scatter plot, approximate: m_X, m_Y, and ρ.

3.53. A fair die is rolled once. If the outcome is in $\{5,6\}$ then a fair coin is flipped 6 times. If the outcome is not in $\{5,6\}$ then an unfair coin is flipped 6 times – this coin having heads probability 2/3. What is the probability that 3 heads are flipped? Given that 3 heads are flipped, what is the probability that the die roll was in $\{5,6\}$?

3.54. Let $X(u)$ have pdf $f_{X(u)}(x)=\frac{1}{2}e^{-|x|}$ and $Y(u)=X^2(u)$. Determine the mean and variance of $X(u)$ and $\mathbb{E}\left\{X(u)Y(u)\right\}$. Are $X(u)$ and $Y(u)$ uncorrelated? Are $X(u)$ and $Y(u)$ independent?

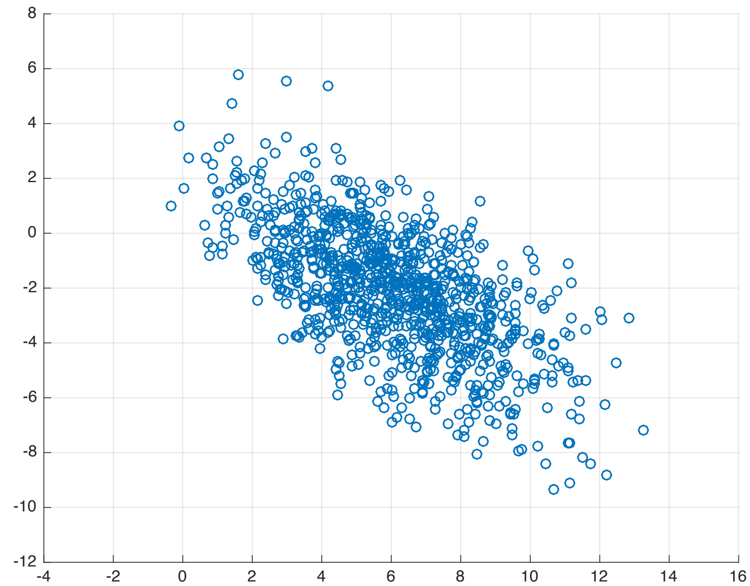

Figure 22: A scatter plot of 1000 realizations of a Gaussian random vector.

3.55. The run-time of a specific computer program has a normal distribution with mean 120 and standard deviation 2 seconds. If two independent measurements of the run-time are made, what is the probability that both measurements will lie between 116 and 118 seconds?

3.56. Suppose that the heights, in inches, of the women in a certain population follow a normal distribution with mean 65 and standard deviation 1, and that the heights of the men follow a normal distribution with mean 68 and standard deviation 2. Suppose also that one woman is selected at random and, independently, one man is selected at random. What is the probability that the woman will be taller than the man?

3.57. The continuous random variables $X(u)$ and $Y(u)$ have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} K & x^2+y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

where $r > 0$ is a parameter of the distribution.

(a) Sketch the region in the (x,y)-plane where $f_{X(u),Y(u)}(x,y) \neq 0$.
- What is $\Pr\left\{X^2(u)+Y^2(u) \leq (r/2)^2\right\}$?
- Are $X(u)$ and $Y(u)$ independent?
- Are $X(u)$ and $Y(u)$ uncorrelated?

(b) Find and sketch the marginal pdfs.

(c) Find and sketch the pdf of $Y(u)$ conditioned on $X(u) = x$.

(d) Conditioned on $X = x$, the minimum mean-squared error (MMSE) estimate of Y is $\hat{y}_{\text{MMSE}} = g_M(x)$. Determine and sketch the function $g_M(x)$. Given specific values of x, provide the estimate $\hat{y}_{\text{MMSE}}$: $x = -r/2$, $x = 0$, $x = r/2$.

(e) Consider the following two statements:

A: Random variables $W(u)$ and $Z(u)$ are independent.

B: The conditional mean of $Z(u)$ given $W(u) = w$ is m_Z – *i.e.*, $\mathbb{E}\{Z(u)|W(u) = w\} = m_Z$.

Determine whether the following statements are true or false:

- If **A**, then **B**.
- If **B**, then **A**,

(f) Consider the joint pmf $p_{X(u),Y(u)}(i, j)$ which is nonzero only at the points shown in Fig. 23. At these points, we have $p_{X(u),Y(u)}(0, 0) = K$ and $p_{X(u),Y(u)}(i, j) = Ki^j$ for those points indicated.

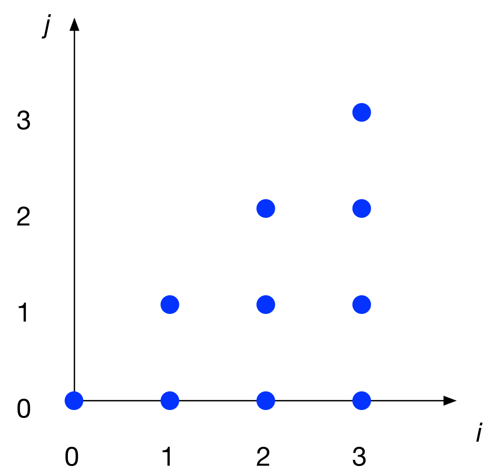

Figure 23: The points in the plane where the joint pmf is non-zero for Problem 2.57f.

i. What is the value of K? Find and sketch the marginal pmfs.

ii. For each value of i having nonzero probability, sketch the conditional value of $Y(u)$ given $X(u) = i$.

iii. Determine the second moment description for $X(u)$, $Y(u)$ – *i.e.*, means, variances and normalized correlation coefficient. Are $X(u)$ and $Y(u)$ are independent? Are $X(u)$ and $Y(u)$ are uncorrelated?

iv. What is the MMSE estimator for $Y(u)$ given that $X(u) = i$ has been observed? What is the Linear MMSE estimator for $Y(u)$ given that $X(u) = i$ has been observed? What is the associated minimum mean-squared error associated with this estimator?

4 Stochastic Convergence and Statistics

4.1. Let $X_1(u), X_2(u), \ldots X_n(u)$ be i.i.d. Gaussian random variables, each with mean m and variance σ^2. Consider the random variable

$$Y(u) = \frac{1}{n}\sum_{i=1}^{n} X_i(u)$$

(a) What is the pdf of $Y(u)$, $f_{Y(u)}(y)$?

(b) Determine $\Pr\{|Y(u) - m| \geq \alpha\sigma\}$ and plot this probability vs. α on a log-log chart (i.e., plot the $\log_{10}$ of this probability vs. $\log_{10}(\alpha)$).

4.2. For each data file provided, plot a (relative-frequency) – i.e., this can be viewed as an estimate of the pdf or pmf.

4.3. For each data file provided, compute

$$\hat{m}(n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where x_i is the i^{th} entry of the file and plot $\hat{m}(n)$ vs. n. What happens to this sample mean for larger values of n?

4.4. For each data file provided, convert the data by forming the sum of L data points in non-overlapping blocks. More precisely, form

$$\begin{aligned}
y_1 &= \frac{1}{L}(x_1 + x_2 + \cdots + x_L) \\
y_2 &= \frac{1}{L}(x_{L+1} + x_{L+2} + \cdots + x_{2L}) \\
y_3 &= \frac{1}{L}(x_{2L+1} + x_{2L+2} + \cdots + x_{3L}) \\
&\vdots \\
y_j &= \frac{1}{L}\left(x_{(j-1)L+1} + x_{(j-1)L+2} + \cdots + x_{jL}\right)
\end{aligned}$$

and produce a relative frequency plot for the $\{y_j\}$ data set obtained. Do this for $L = 4, 10, 25, 100$. Discuss the results and their relation to the previous two problems.

4.5. Let $X_i(u)$ be a sequence of independent, identically distributed (iid) random variables, each with zero mean and variance σ^2 and let

$$Y_n(u) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i(u)$$

Determine the mean and variance of $Y_n(u)$ and the and the limiting distribution for $Y_n(u)$ as n tends toward infinity.

4.6. A collection of data is modeled as a random sampling of a population, each member with uniform distribution. The uniform distribution is over an interval of length 4, but the center of this interval is unknown. Let c denote this unknown center.

This data is processed by forming arithmetic averages of size $L = 10$ using disjoint subsets of the data. Let $X_i(u)$ denote the i^{th} output of this processing.

(a) Determine the mean, variance and approximate pdf for $X_i(u)$:

(b) In one particular example, there is a sample of size 40 from the original uniformly-distributed population. After processing, this yields four averaged data points (i.e., realizations of $X_i(u)$). These are:

$$x_1 = 5.2$$
$$x_2 = 6.2$$
$$x_3 = 6.1$$
$$x_4 = 6.0$$

Based on this, find a good 95% confidence region for the center parameter c and explain your rationale – i.e., find A and B such that $A \leq c \leq B$ is a good 95 % confidence region.

4.7. Consider the sum $S_n(u) = \sum_{i=1}^{n} X_i(u)$ where each $X_i(u)$ has mean m and variance σ^2. Also, the correlation coefficient for $X_i(u)$ and $X_j(u)$ for $i \neq j$ is 1/2.

(a) Find the mean and variance of $S_n(u)$.

(b) The Weak Law of Large Numbers (WLLNs) states that the sample mean $\bar{X}_n(u) = \frac{1}{n}S_n(u)$ converges in probability to the constant m when the $X_i(u)$ are uncorrelated. Does the WLLN hold in this case where $\rho_{ij} = 1/2$ $(i \neq j)$?

4.8. Consider the $(n \times 1)$ random vector $\mathbf{x}(u)$ with components $X_i(u)$ for $i = 1, 2, \ldots n$. The mean of this random vector is zero and $\mathbb{E}\left\{X_i(u)X_j(u)\right\} = \rho^{|i-j|}$. Consider $n = 4$, and provide the covariance matrix for $\mathbf{x}(u)$. Let $S_n(u) = \sum_{i=1}^{n} X_i(u)$. For $n = 4$, find the mean and variance of this sum. Does the sample mean converge to m (in probability) for this case? In other words, does the Weak Law of Large Numbers hold for this type of correlation?

4.9. You have 5 values sampled randomly from a Gaussian population:

$$x_1 = 3.2$$
$$x_2 = -1.2$$
$$x_3 = 0$$
$$x_4 = 2$$
$$x_5 = 1$$

(a) Determine the sample mean and variance.

(b) The 90% confidence region for the mean based on these data is $[a, b]$. Specify a and b.

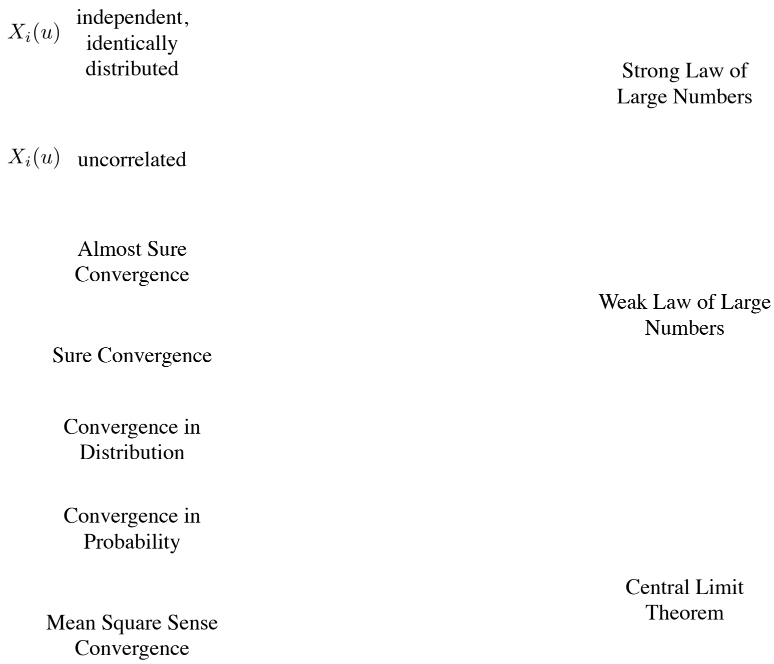

Figure 24: Connect the related concepts by drawing lines.

4.10. College students were asked "would you rather fight 100 duck-sized horses or 1 horse-sized duck?" Let the events

$$\begin{aligned} D &: \quad \text{1 horse-sized duck} \\ H = D^c &: \quad \text{100 duck-sized horses} \end{aligned}$$

500 students were polled and 16 preferred option D. From this data, $p = P(D)$ can be estimated as $\hat{p}$ with a 95% confidence region given by $[\hat{p} - \mathrm{MOE}_{95}, \hat{p} + \mathrm{MOE}_{95}]$ where MOE_{95} is the margin of error. Determine $\hat{p}$ and MOE_{95}.

4.11. Draw lines on Fig. 24 to illustrate the assumptions and mode of stochastic convergence (left) associated with each of the convergence theorems (right).

4.12. A political consultant wants to predict the outcome of a very close election between two candidates. She wants to have a marginal of error of no more than 1% (0.01) with 99% confidence. How many voters must she poll?

4.13. A set of two-dimensional data points $\{(x_i, y_i)\}_{i=1}^4$ is available:

$$\begin{aligned} x_1 &= -7 & \qquad y_1 &= 34 \\ x_2 &= 2 & y_2 &= 9 \\ x_3 &= 8 & y_3 &= -9 \\ x_4 &= 20 & y_4 &= -34 \end{aligned}$$

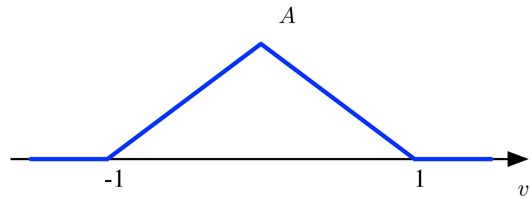

Figure 25: The pdf of $V_i(u)$.

(a) Determine the sample mean and sample variance for $\{x_i\}$ and for $\{y_i\}$. Also find the sample covariance for $\{x_i\}, \{y_i\}$.

(b) Assuming a Gaussian sample, find the 90% confidence region for m_X and m_Y based on the data above.

(c) If a linear regression if performed to estimate $\{y_i\}$ from $\{x_i\}$, what percentage of the variation in $\{y_i\}$ can be explained by a linear function of $\{x_i\}$?

(d) Qualitatively, how would you characterize this data set (circle one)?

i. $\{y_i\}$ is very well approximated as a linear function of $\{x_i\}$.

ii. $\{y_i\}$ is reasonably approximated as a linear function of $\{x_i\}$.

iii. $\{y_i\}$ is poorly approximated as a linear function of $\{x_i\}$.

4.14. Let $V_1(u), V_2(u), V_3(u), \ldots V_n(u)$ be a sequence of iid random variables with pdf $f_{V_i(u)}(v)$ as shown in Fig. 25.

(a) Determine the the value of A and the mean and variance of $V_i(u)$.

(b) Define

$$Y_n(u) = \frac{1}{n} \sum_{i=1}^{n} V_i(u)$$

Determine the mean and variance of $Y_n(u)$ and also the region B_n where the pdf of $Y_n(u)$ is nonzero.

(c) For large values of n, state an approximate pdf for $Y_n(u)$.

(d) Using this approximation find total probability mass in the approximation that is outside the region B_n – denote this by P_n.

(e) Evaluate P_n for $n = 2, 4, 6, 25$. Briefly discuss these values in the context of the Central Limit Theorem.

4.15. You are given a coin and asked to conduct an experiment to determine the probability of heads p. You decide to flip the coin until you have a relative margin of error of no more than 20% with 95% confidence – *i.e.,* the MOE should be no more than $0.2\hat{p}$. Given the following experimental statuses, state whether you should stop (have high enough confidence) or keep flipping:

- 100 flips, 50 heads

- 40 flips, 32 heads
- 500 flips, 20 heads

4.16. Suppose the one thing you remember from your probability class is that you can run a Monte Carlo simulation to estimate the probability of an event. You use a computer to run n independent trials of an experiment with the goal of estimating $p = P(A)$ where A is some event of interest associated with the experiment. You remember that "collecting 100 successes" is a good rule of thumb, so you start your simulation with that goal.

However, after running n trials, you see no successes. As a concrete example, you flip a coin 10 times and observe no heads. This holds even for large n (long simulations).

The goal of this problem is to develop a $100(1-\alpha)\%$ confidence region on p in this case and an associated rule of thumb for large n and 95% confidence – i.e., not seeing any successes tells us something about p.

(a) Under the above assumptions, what is the probability of observing zero occurrences of A (successes) in n independent trails?

(b) Find the largest value of p, denoted by $p_{\max}$, so that the above probability from (a) is at least α. Specifically, we seek the largest value of p so that

$$P(\text{"no successes in } n \text{ trials"}) \geq \alpha$$

Note that this defines a $100(1-\alpha)\%$ confidence region for p given by $[0, p_{\max}]$.

(c) Evaluate the 95% confidence region developed in (b) for the cases when no successes (occurrences of A) are observed in n trials where $n = 3, 10, 100$ and 1 million.

(d) For large values of n, when you observe no successes, what is a good rule of thumb for the 95% confidence region for the success probability p?

4.17. Consider two digital clocks that display hours and minutes, but no seconds. Clock A leads Clock B by some fixed number of seconds (less than 60). The offset in seconds is Δ_{sec}. If n randomly timed measurements are made and k disagreements are observed, what is a good estimate of Δ_{sec} and what is a good 95% confidence region for this estimate? (You can assume that k is much larger than 1).

If $n = 500$ and $k = 100$ find the estimate of Δ_{sec} and the corresponding MOE; Repeat this for $n = 500$ and $k = 30$.

4.18. Let $X_1(u), X_2(u), \ldots X_n(u)$ be a sequence of independent, identically distributed random variables, each with mean m and variance σ^2. Consider the sample mean and variance

$$\hat{M}_n(u) = \frac{1}{n}\sum_{i=1}^{n} X_i(u)$$

$$S_n^2(u) = \frac{1}{n-1}\sum_{i=1}^{n}\left[X_i(u) - \hat{M}_n(u)\right]^2$$

Show that $\mathbb{E}\left\{S_n^2(u)\right\} = \sigma^2$.

4.19. Let $V(u)$ be uniformly distributed on $(0,1)$. Conditioned on $V(u) = v$, $X(u)$ is uniform on $(0, v)$. Find and sketch $f_{X(u)}(x)$ and $f_{V(u)|X(u)}(v|x)$. Conditioned on $X(u) = x$, find the MMSE estimate of $V(u)$. Evaluate this estimator for $x = 0.2$ and $x = 0.8$.

4.20. A statistical analysis of a proposed cache system architecture is desired. Although the exact architecture and control algorithm are specified, it is too difficult to perform a closed form analysis, so you are asked to run a Monte Carlo simulation. The time that a particular memory item spends in the cache is modeled by a random variable $X(u)$ which takes values in $\{1, 2, 3, 4\}$. In your simulation, you run 10,000 trials and observe k_i realizations of $X(u)$ that take the value i. The specific data obtained are

$$\begin{aligned} k_1 &= 6000 \\ k_2 &= 50 \\ k_3 &= 0 \\ k_4 &= 3950 \end{aligned}$$

(a) Based on the results of this experiment find and estimate of the pmf of $X(u)$. You present a plot of this pmf estimate to your supervisor and she asks you to add error bars to indicate the 95% confidence region for these estimates. Provide the 95% confidence region for each of these 4 estimates – denote the 95% confidence region for $\hat{p}_{X(u)}(i)$ as R_i.

(b) Using the data collected, find the sample mean and sample variance of $X(u)$.

(c) In the existing cache system, in has been well established that the average time in the cache is 3.3. Your supervisor asks if you can reliably conclude from your data that the proposed cache system reduces the average time in cache. What is your response? Briefly explain.

4.21. A professor teaches a large calculus class. The following is the percentage of the class that has received A grades during the past 4 offerings.

$$\begin{aligned} y_1 &= 18.2\% \\ y_2 &= 28\% \\ y_3 &= 9\% \\ y_4 &= 22.5\% \end{aligned}$$

Determine the sample mean and variance. What is a 95% confidence region for the mean based on these data?

4.22. Students are asked to reach into a bucket and select a coin. The bucket contains a large number of quarters and pennies and no other coins. 89 students draw from this bucket and 25 of these students draw a penny. Based on these data, what is a good estimate for the probability that a student will draw a penny? What is the corresponding 95% confidence region for this probability?

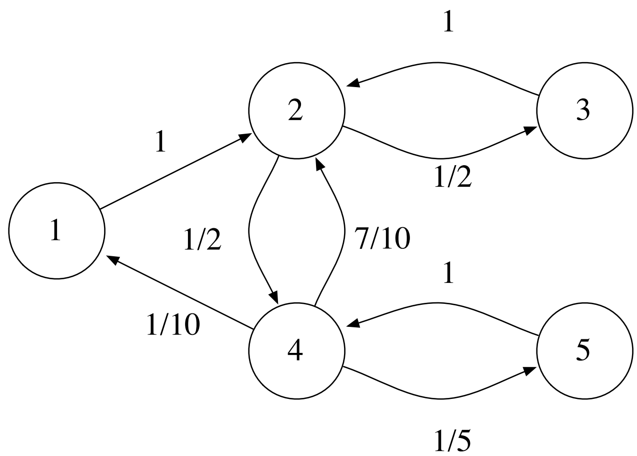

Figure 26: A five state DTMC.

5 Markov Chains

5.1. Consider the DTMC shown in Fig. 26.

(a) Is this chain irreducible? periodic? ergodic?

(b) Specify how many solutions to the stationary equation exist? Specify all of these.

5.2. For the B/B/1 queue considered in class.

(a) Plot the state distribution π_k vs. k for the following cases:

i. $\lambda = 1/2$, with $\mu = 5/8,\ 3/4,\ 7/8$

ii. $\lambda = 1/4$, with $\mu = 1/3,\ 1/2,\ 3/4$

(b) Plot the mean time to return to the empty queue state given that the system starts with an empty queue versus λ for the following cases (note: for $\lambda \in (0, \mu)$)

i. $\mu = 1/4$

ii. $\mu = 1/2$

iii. $\mu = 3/4$

5.3. Consider a variation of the B/B/1 queue considered in class, but modified to have a finite buffer size of 3. In other words, if a new job arrives and no service opportunity occurs at time n, and the queue currently has 3 jobs in it, then a job is dropped.

(a) Develop a DTMC model for this system that includes a "drop state" D. State D is entered as described above.

(b) Find the stationary distribution for this DTMC. Compare this to the stationary distribution of the infinite length B/B/1 queue – *i.e.,* how do these distributions compare for the numerical values of λ and μ.

(c) What percentage of time does the queue spend dropping jobs?

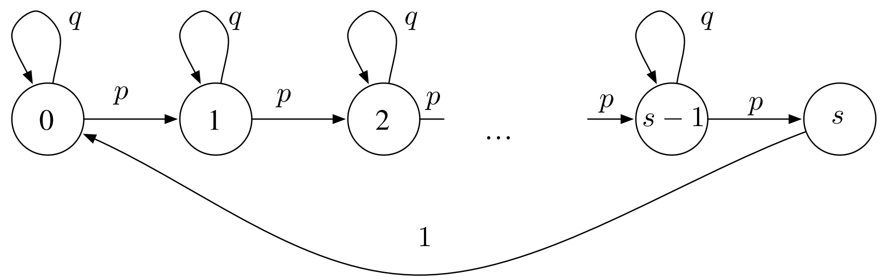

Figure 27: The Markov Chain associated with a negative binomial random variable.

(d) What percentage of time is the queue full?

5.4. Consider the DTMC with state transition diagram shown in Fig. 27 where $p \in (0, 1)$ and $q = 1 - p$:

(a) What is the steady state probability distribution for this DTMC?

(b) What is the mean time to reach state s given that the current state is 0?

(c) Discuss the relationship between this DTMC and the negative binomial distribution. In particular, can you use the result of part (b) to find the mean of the negative binomial random variable? (See the text for the definition of a negative binomial random variable).

5.5. Consider a DTMC that evolves by rolling a fair die. Specifically, assume that the current state of the DTMC is $X_n(u) = i$ and the die roll results in $r \in \{1 \ldots 6\}$. The next state of the DTMC is

$$X_{n+1}(u) = \begin{cases} i-1, & \text{if } r < i \\ i, & \text{if } r = i \\ i+1, & \text{if } r > i \end{cases}$$

Provide a labeled state transition diagram for this DTMC. Also, find the stationary distribution and the probability of going from state 2 to state 4 in 4 steps.

5.6. Consider the full house game once more. Recall that this game is played by flipping a coin and noting how many flips are required to have at least 2 tails and 3 heads.

(a) Show that this game can be modeled as a 12 state DTMC and provide the corresponding state transition diagram.

(b) Use this DTMC to find the expected number of flips required to obtain a full house – *i.e.,* the mean of the full-house random variable considered in Problem 1.54.

(c) Plot the mean of the full-house random variable as a function of p. What value of p minimizes the expected time to a full house? Which value of p maximizes this mean value?

5.7. Consider the DTMC shown in Fig. 28.

(a) Is this chain irreducible? periodic? ergodic?

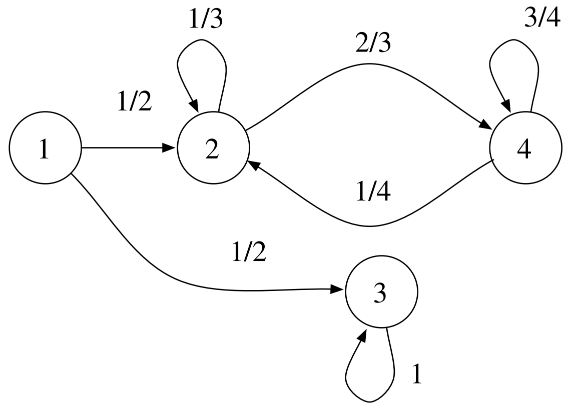

Figure 28: Four state DTMC considered in Problem 5.7.

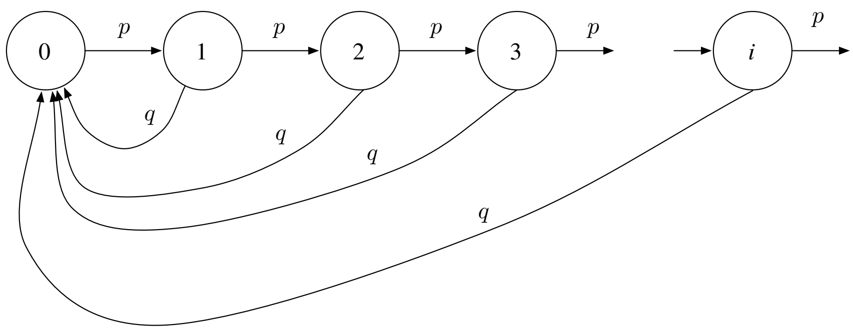

Figure 29: DTMC model for run-length of heads.

(b) Find the limiting value of $\boldsymbol{\pi}(n)$ for the following initial distributions:

i. $\boldsymbol{\pi}(0) = (\ 0\ p\ 0\ 1-p\ )$
ii. $\boldsymbol{\pi}(0) = (\ 0\ 0\ 1\ 0\ )$
iii. $\boldsymbol{\pi}(0) = (\ 1\ 0\ 0\ 0\ 0)$
iv. $\boldsymbol{\pi}(0) = (1/4)(\ 1\ 1\ 1\ 1\ )$

5.8. Consider a DTMC to model the run-length of heads in a sequence of independent coin flips. Specifically, $X_n(u)$ models the number of heads in a row at time n. When a tails is flipped, this count resets to zero so that $X_n(u)$ models the current run-length of heads. The state diagram for this DTMC is shown in Fig. 29.

(a) Find the steady state distribution for this DTMC:

(b) If $Y(u)$ has probability mass function $\Pr\{Y(u) = i\} = \pi_i$, what is the mean of $Y(u)$?

5.9. A famous question in probability is "will a monkey typing randomly at a keyboard eventually produce the works of William Shakespeare?" Consider a simplified version of this problem where a "binary monkey" produces a sequence of independent, identically distributed Bernoulli random variables (*e.g.,* coin flips) with success probability p.

This problem regards this binary monkey typing the specific length-3 binary string: 101 (this is analogous to the works of Shakespeare).

(a) Diagram and label the state transition diagram for a DTMC that models this problem.

(b) Use the model from part (a), or a suitably modified version, to find the mean time for the monkey produce the string 101. What is this mean time for $p = 1/2$?

(c) Return to the original question: will a monkey typing randomly at a keyboard eventually produce the works of William Shakespeare? Explain.

6 Computer-based Problems

6.1. Consider two digital clocks that display hours and minutes, but no seconds. Clock A leads Clock B by some fixed number of seconds (less than 60). So, if you check the clocks, sometimes they agree and sometimes Clock A reads one minute ahead of Clock B. Based on the intuition that we developed during lecture (coin flipping experiments), design a method for estimating the offset between the clocks via random measurements. Use a computer to characterize the accuracy of your approach.

6.2. Consider exploring problem 2.54 and approximating the solution numerically. Specifically, write a program/script that will generate sequences of coin flips (with probability of heads p) and count the first flip when the full house condition is met. Repeat this as many times as necessary to get a reasonable estimate of the probability that it takes k flips to get to a full house ($k = 5, 6, 7 \ldots$). Plot a these estimates vs k for $p = 0.1$ and $p = 0.5$.

6.3. Much effort has been put into the problem of generating a random number that is "uniformly distributed" on $[0, 1]$ – *i.e.,* when we call `v=random()` this returns a number between 0 and 1 and if you were to call this many times and plot a histogram, it will be flat on $[0, 1]$. The reason for the focus on developing such a random number generator, is that it can be converted into a random number generator for any probability distribution. Consider how to do this for a discrete random variable.

(a) If v is a randomly generated number on $[0, 1]$, find $g(v)$ so that $x = g(v)$ is a realization of a Bernoulli random variable with success probability p.

(b) Repeat (a) to produce a realization for a binomial($n = 10$, $p = 0.3$).

(c) Describe your method for any integer-valued random variable $X(u)$ with probability mass function $p_{X(u)}(k) = \textsc{Pr}\{X(u) = k\}$.

6.4. For the full-house random variable (problem 2.54), estimate the mean and variance of X(u). To do this, use the sample mean and sample variance. Specifically, assuming that you have

n independent realizations of $X(u) : x_1, x_2, \ldots x_n$

$$\text{Sample mean:}\hat{m}_n = \frac{1}{n}\sum_{i=1}^{n} x_n$$

$$\text{Sample variance:}s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(z_i - \hat{m}_n)^2 = \left(\frac{1}{n-1}\sum_{i=1}^{n} x_i^2\right) - \left(\frac{n}{n-1}\right)\hat{m}_n^2$$

These functions are built into Matlab, Excel, Python, etc. Also, if you have already worked Problem 6.2, you have generated the realizations $\{x_i\}$ for for estimating the pmf.

Plot the sample mean and sample variance of $X(u)$ as a function p for $n = 10, 100, 1000$.

6.5. Simulate problem 1.47 (parking lot problem). Specifically, consider a parking lot with:

- N_{spots} = the total number of parking spots
- N_{empty} = total number of empty spots
- N_{sought} = number of contiguous empty spots sought

(a) Write a program/script to estimate the probability of finding a run of N_{sought} empty spots. Run your program for several cases, including the case of 1.47 and

$$N_{\text{spots}} = 30$$
$$N_{\text{empty}} = 9$$
$$N_{\text{sought}} = 4$$

(b) Let the random variable $X(u)$ model the length of the largest run of contiguous empty spots. Use your simulation to estimate the mean, variance, and pmf of $X(u)$.

6.6. Consider the experiment discussed in lecture: take a coin and repeatedly flip it. After n flips, denote the actual (possibly unknown) probability of heads as p and the estimate for the probability of heads after n flips as:

$$\hat{p}(n) = \frac{k_n}{n}$$

where k_n is the number of heads observed in n flips.

(a) Consider n as a variable and plot $\hat{p}(n)$ vs. n for one sequence of flips. Plot at least out to 500 flips. Use $p = 1/2$, but feel free to explore.

(b) Repeat the experiment from (a) many times and plot the trajectory of $\hat{p}(n)$ vs. n for these trials all on the same plot. Plot at least 400 trials on the plot.

(c) Plot histograms of $\hat{p}(n)$ at fixed n. For example, for $n = 2$, plot a histogram of $\hat{p}(2)$ over all the trials from part (b). Plot these histograms for several values of n – e.g., $n = 2, 5, 50, 500$. Explore and experiment.

(d) Based on the plots in (b)-(c), describe the behavior of $\hat{p}(n)$ as n increases. How do the different trials vary around the actual value of p?

(e) Use the np.std() command from numpy to compute the sample standard deviation of $\hat{p}(n)$ at fixed n – i.e., call this $\hat{\sigma}(n)$. Specifically, you are computing a measure of the variation from the actual valaue of p over all of the trials from (b)-(c) which is also a measure of the "spread" of the histograms from (c). Plot $\hat{\sigma}(n)$ vs. n. Compare this plot of $1/(2\sqrt{n})$.

6.7. Numerically demonstrate the method for Problem 2.58. Specifically, generate random realizations from a uniform distribution in $[-\pi, +\pi]$ and plot the histograms. Compare these to the analytical expression for a Rayleigh pdf. Do this for various values of α^2 – *e.g.,* $\alpha^2 \in \{1, 2, 4, 8\}$.

7 Assorted Short Problems

7.1. State whether each of the following is true ALWAYS, SOMETIMES, or NEVER.

(a) $P(A \cup B) = P(A) + P(B)$

(b) An arbitrary subset of the sample space ($\mathcal{U}$) is an event.

(c) If A and B are events, then $A \cap B$ is an event.

(d) If B is an Borel subset of the real line, $\text{Pr}\{X(u) \in B\}$ can be determined from the cdf of $X(u)$.

7.2. Determine whether the following are TRUE or FALSE:

(a) If $\mathbb{E}\{X(u)Y(u)\} = \mathbb{E}\{X(u)\}\mathbb{E}\{Y(u)\}$ then $X(u)$ and $Y(u)$ are independent.

(b) If $X(u)$ and $Y(u)$ are independent, then $\mathbb{E}\left\{[X(u)]^3 \cos(2\pi Y(u))\right\} = \mathbb{E}\left\{[X(u)]^3\right\}\mathbb{E}\{\cos(2\pi Y(u))\}$

(c) If $m_X = m_Y$ and $\sigma_X^2 = \sigma_Y^2$, then $\text{Pr}\{X(u) \in B\} = \text{Pr}\{Y(u) \in B\}$ where B is any subset of the real line.

(d) If $f_{X(u)}(z) = f_{Y(u)}(z)$ for all $z \in (-\infty, \infty)$, then $\text{Pr}\{X(u) = Y(u)\} = 1$

(e) $\mathbb{E}\left\{[X(u)]^2\right\} \leq [\mathbb{E}\{X(u)\}]^2$

7.3. Let $X(u)$ and $Y(u)$ be uncorrelated random variables with means and variances $m_X = 2$, $\sigma_X^2 = 4$, $m_Y = -2$, $\sigma_Y^2 = 2$, respectively. Let $Z(u) = X(u) + Y(u)$ and determine the following.

(a) The mean and variance of $Z(u)$

(b) Ture or False: $Z(u)$ is Gaussian

7.4. Let $Y(u)$ be an exponential random variable with parameter $\lambda = 1$. Find the probability that $Y(u)$ is between 0 and 1.

7.5. The pdf of $X(u)$ is $f_{X(u)}(z) = \mathcal{N}(z; 1; 4)$. What is the probability, p, that $X(u)$ is between 0 and 4 or less than -3? Express your answer in terms of Q-functins with non-negative arguments.

7.6. Let $F_{X(u)}(z|A)$, $P(A) \in (0, 1)$ and $F_{X(u)}(z)$ all be known. Express $F_{X(u)}(z|A^c)$ in terms of these known quantities.

7.7. The binomial distribution takes its maximum value at its mean - TRUE or FALSE?

7.8. The discrete random variable $Y(u)$ takes on the values -2, 0, $+2$ with probability 1/4, 1/2, and 1/4, respectively. Determine the characteristic function of $Y(u)$.

7.9. Consider the pdf

$$f_{X(u)}(x) = \begin{cases} Cx^2(1-x)^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the constant C, the mean and the variance.

7.10. $X(u)$ has mean 2 and variance 4. If $V(u) = -3X(u) + 2$, determine the mean and variance of $V(u)$.

7.11. Let $X(u)$ be a continuous random variable, uniformly distributed between 0 and 1. Given a realization $X(u) = x$, the random variable $Y(u)$ is Gaussian with mean x and variance 1. Determine $f_{Y(u)}(y)$ and $f_{X(u)|Y(u)}(x|y)$.

7.12. The random variables $X(u)$ and $Y(u)$ are jointly Gaussian with parameters $m_X = 1$, $\sigma_X^2 = 3/4$, $m_Y = 3$, $\sigma_Y^2 = 3$ and $\rho = 1/2$. Determine $\Pr\{Y(u) > 3|X(u) = 4\}$

7.13. The pdf of $X(u)$ is given by

$$f_{X(u)}(x) = \frac{1}{2}e^{-|x|}$$

Determine the following: $\mathbb{E}\left\{[X(u)]^4\right\}$, $\mathbb{E}\left\{[X(u)]^9\right\}$.

7.14. The time to process a print job (print time) is modeled as an exponential random variable. Given a print time of τ, the number of print jobs in the print queue is modeled as Poisson with mean τ. What type of unconditional distribution models the number of print jobs (*i.e.,* this is a well-know distribution)?

7.15. The joint pdf of $X(u)$ and $Y(u)$ is

$$f_{X(u)Y(u)}(x, y) = xe^{-x(1+y)}\mathrm{U}(x)\mathrm{U}(y).$$

These two random variables are independent. TRUE or FALSE?

7.16. Let $\{X_i(u)\}_{i=1}^5$ be i.i.d., each exponentially distributed with variance 1/4 and let $Y(u) = \sum_{i=1}^5 X_i(u)$. Determine the characteristic function of $Y(u)$

7.17. Let $X(u)$ and $Y(u)$ be uncorrelated, each with unit variance and mean zero. Find a good upperbound for the following probability: $\Pr\{|2X(u) + 3Y(u)| \geq 10\}$.

7.18. A large city has a population with 10% left-handed people. If 10 people are drawn randomly from this population, what is the probability that 2 or more will be left-handed?

7.19. The Central Limit Theorem (CLT) states that

(a) A binomial random variable with large N and small p can be approximated by a Poisson random variable.

(b) The sum of two Gaussian random variables is Gaussian

(c) The sample variance of an independent, identically distributed (i.i.d.) sequence goes to zero in the limit

(d) None of the above.

7.20. The random variable $X(u)$ has pdf given by

$$f_{X(u)}(z) = \begin{cases} cz(1-z) & 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the following probability $\Pr\{1/2 \leq X(u) \leq 3/4\}$.

7.21. Let $X(u)$ and $Y(u)$ be independent, both uniformly distributed on $[-0.5, 0.5)$ and let $Z(u) = X(u) + Y(u)$. Determine the following: m_Z, σ_Z^2, $\Pr\{Z(u) > 0.5\}$.

7.22. Let $X_1(u)$ and $X_2(u)$ be jointly-Gaussian, each with mean zero and unit variance. Also, $\mathbb{E}\{X_1(u)X_2(u)\} = -1/2$. Define $Y_1(u) = X_1(u) + X_2(u)$ and $Y_2(u) = X_1(u) - X_2(u)$ and determine $\Pr\{Y_1(u) > 1, Y_2(u) < 1\}$.

7.23. **TRUE/FALSE:** The cumulative distribution function exists for any random variable.

7.24. **TRUE/FALSE:** The variance exists for any random variable.

7.25. Let $X(u)$ be a Gaussian random variable with zero mean and unit variance. If $Y(u) = 3X(u) - 2$, determine $\Pr\{Y(u) > 4\}$.

7.26. A random variable $X(u)$ has known mean and variance, m_X and σ_X^2, respectively. Also, it is known that the pdf of $X(u)$ is symmetric around its mean – *i.e.*, $f_{X(u)}(x - m_X) = f_{X(u)}(x + m_X)$ for all real x. Use this information to obtain a good upper bound for $\Pr\{X(u) \geq m_X + \alpha\sigma_X\}$ where α is a positive constant. Evaluate this for $\alpha = 3$.

7.27. Consider the pdf

$$f_{X(u)}(x) = K\beta^2 e^{-\beta^2|x|}$$

where β is a parameter of the pdf and K is a constant.

Determine the following:

(a) K

(b) m_X

(c) σ_X^2

(d) $\Pr\{|X(u)| < 10\}$

(e) $\mathbb{E}\{[X(u)]^k\}$

7.28. Let $S(u) = V(u) + X(u)$ where $X(u)$ is uniformly distributed on $(-1, +1)$ and $V(u)$ is uniformly distributed on $(+5, +10)$. Also, $X(u)$ and $V(u)$ are statistically independent. Determine the mean of $S(u)$. Determine and sketch the pdf of $S(u)$.

7.29. What is the main purpose of the characteristic function (as presented thus far)? (circle one)

(a) To enable efficient transformation of the pdf into the frequency domain.
(b) To simplify the computation of tail probabilities for Gaussian random variables.
(c) To simplify computation of the moments of some random variables.
(d) To express point masses in continuous distributions.

7.30. Which of the following functions is guaranteed to exist for any random variable? (circle one)

(a) The probability mass function.
(b) The probability density function.
(c) The moment generating function.
(d) The cumulative distribution function.
(e) All of the above.

7.31. $\mathbb{E}\left\{[X(u)]^2\right\} \geq (\mathbb{E}\{X(u)\})^2$ (circle one)

(a) Always
(b) Sometimes
(c) Never

7.32. A random variable $X(u)$ has pdf

$$f_{X(u)}(x) = C|x|^3 \qquad x \in [-1, +1]$$

and zero for $|x| > 1$.

Determine the following: C, m_X, σ_X^2, and $\Pr\left\{|X(u) - \frac{1}{4}| > \frac{1}{4}\right\}$.

7.33. List three important properties of Gaussian random variables and/or pairs jointly-Gaussian random variables.

7.34. A fair die is rolled and $X(u)$ models the square of the number that is rolled. Determine the probability mass function for $X(u)$ and $\Pr\{X(u) < 7\}$.

7.35. Circle the best answer: A random variable is

(a) a function that associates a real number with each element of the sample space
(b) an element of the sample space of an experiment
(c) an event that is quantified for each element of the sample space

7.36. $X(u)$ is a continuous random variable that is uniformly distributed on $[a, b]$ and has $m_X = 0$, $\sigma_X^2 = 1$. Determine a and b.

7.37. **TRUE/FALSE:** If $A \subset B$, then $P(B|A) = 1$.

7.38. **TRUE/FALSE:** If $A \subset B$, then $P(A|B) = 1$.

7.39. **TRUE/FALSE:** If $\mathbb{E}\{X(u)Y(u)\} = m_x m_y$, then $\rho_{xy} = 0$.

7.40. **TRUE/FALSE:** The Central Limit Theorem says that the Law of Large Numbers applies to Gaussians.

7.41. In the context of probability and random variables, state the Cauchy-Schwartz inequality.

7.42. **(10 points)** Consider the random variable $Z(u)$

$$Z(u) = 3X(u) + 2Y(u) + 2$$

where $X(u)$ and $Y(u)$ are uncorrelated random variables with $m_X = 2$, $\sigma_X^2 = 1$, $m_Y = 1$, $\sigma_Y^2 = 4$. Determine:

(a) m_Z, σ_Z^2 and $\text{cov}\,[Z(u), Y(u)]$

(b) A good upperbound for $\text{Pr}\{|Z(u) - m_Z| > 10\}$

7.43. Let $X(u)$ be an exponential random variable with mean 1/3 and $Y(u) = 10 - 2X(u)$. Find and sketch the pdf of $Y(u)$. What is the mean and variance of $Y(u)$?

7.44. $X(u)$ is Gaussian with mean $m_X = 1$ and variance $\sigma_X^2 = 9$. Let $Y(u) = 3X(u) + 2$ and determine the following:

(a) m_Y

(b) σ_Y^2

(c) $\text{Pr}\{Y(u) > 32\}$

(d) $\text{Pr}\{Y(u) > -4\}$

7.45. Which of the following exists for every random variable?

- The cumulative distribution function (cdf).
- The probability mass function (pmf).
- The probability density function (pdf).

7.46. $X(u)$ and $Y(u)$ have mean and variance: $m_X = 0$, $m_Y = 2$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$. The normalized correlation coefficient is $\rho = -1/2$. Let $Z(u) = 4X(u) + 2Y(u) + 4$. Find the mean and variance of $Z(u)$.

7.47. A random variable $X(u)$ has mean 0 and variance 1. Give an upper bound on the $\text{Pr}\{|X(u)| > 4\}$.

7.48. The Law of Large Number states that (circle best choice):

(a) The variance of a random variable goes to zero over time.

(b) The sample mean tends towards zero with large sample size.

(c) The sum of many independent random variables is approximately Gaussian.

(d) The sample mean converges to the mean for a sequence of iid random variables.

7.49. Which of the following distributions is the best model for the time to failure of a computer power supply, measured from the date of purchase?

(a) Gaussian
(b) uniform
(c) exponential
(d) binomial

7.50. **True or False:** The following is a valid covariance matrix for two random variables. Explain your answer briefly.

$$\begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$$

(a) TRUE
(b) FALSE

7.51. Consider the random variable $Z(u)$

$$Z(u) = 4X(u) - 3Y(u) + 1$$

where $X(u)$ and $Y(u)$ are uncorrelated random variables with $m_X = 0$, $\sigma_X^2 = 2$, $m_Y = 1$, $\sigma_Y^2 = 3$. Determine mean and variance of $Z(u)$ and $\mathrm{cov}\left[Z(u), Y(u)\right]$.

7.52. For a continuous random variable $X(u)$, the probability $\mathrm{Pr}\left\{X(u) = x\right\}$ is

(a) 1
(b) 0
(c) 1/2

7.53. You are in a class with a total of 50 students. The teacher has decided to randomly assign grades with 15 As, 18 Bs, and 17 Cs. What is the probability that you receive an A?

7.54. State the law of large numbers and the central limit theorem in words.

7.55. The USC football team is scheduled to play 10 games in a given season. Before the season, ESPN estimates the probability that USC wins each game at 0.7. What is the probability that USC wins 8 or more games in 2018?

7.56. If A and B are independent events with $P(A) = 1/2$ and $P(B) = 1/4$, what is $P(A \cup B)$?

7.57. What is the allowable set of values for the correlation coefficient ρ?