EE364 - Prof. Chugg: Combinatorics Summary

1 Summary of Sampling Techniques

Many combinatorics problems are conceptually equivalent to pulling balls from an urn. Assume that there are n balls in the urn and that k draws will be made. There are four cases (2^2) to consider, each defined by whether the balls are replaced after being drawn and whether the order in which the balls are drawn is important. A summary of the results given in class is contained in Table 1

Ordering	Replacement	Restrictions on k	Number of possible draws
yes	yes	none	n^k
yes	no	$k \leq n$	$(n)_k = \frac{n!}{(n-k)!}$
no	no	$k \leq n$	$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$
no	yes	none	$\binom{n+k-1}{k}$

Table 1: Summary of sampling cases.

It is helpful to draw out these four cases for an example with relatively small n and k. Consider $n = 5$ and $k = 2$ (i.e. make two draws from an urn containing 5 labeled balls). The result of this experiment can be recorded by a 2-tuple - e.g. $(4, 2)$ represents the result that ball 4 was drawn first and ball 2 was drawn next. Tables 2-5 list the possible draws with each of the ordering/replacement possibilities.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

Table 2: Sampling w/ ordering and w/ replacement $\Longrightarrow$ $5^2 = 25$ possibilities.

·	(1,2)	(1,3)	(1,4)	(1,5)
(2,1)	·	(2,3)	(2,4)	(2,5)
(3,1)	(3,2)	·	(3,4)	(3,5)
(4,1)	(4,2)	(4,3)	·	(4,5)
(5,1)	(5,2)	(5,3)	(5,4)	·

Table 3: Sampling w/ ordering and w/o replacement $\implies$ $(5)_2 = (5)(4) = 20$ possibilities.

·	(1,2)	(1,3)	(1,4)	(1,5)
·	·	(2,3)	(2,4)	(2,5)
·	·	·	(3,4)	(3,5)
·	·	·	·	(4,5)
·	·	·	·	·

Table 4: Sampling w/o ordering and w/o replacement $\implies$ 5 pick 2, or 10 possibilities.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
·	(2,2)	(2,3)	(2,4)	(2,5)
·	·	(3,3)	(3,4)	(3,5)
·	·	·	(4,4)	(4,5)
·	·	·	·	(5,5)

Table 5: Sampling w/o ordering and w/ replacement $\implies$ $5 + 2 - 1 = 6$ pick 2, or 15 possibilities.

2 Typical Examples

The difficult part of applying this theory is determining which of the above models is proper, or if a combination or modification of these methods is necessary. We have discussed several examples of each sampling technique; here's a brief summary:

- **w/ order w/ replacement**
  - k-digit counting in base n.
  - Labeling each of k persons in a room with a number between 1 and n.
  - Place one of n types of balls into each of k bins.
- **w/ order w/o replacement** (k-permutations of n objects)
  - Picking order of first k finishes in a race with n competitors.
  - Permutations: How many different lists containing n different names ($k = n$).
  - Place one of n types of balls into each of k bins, so that no two bins contain the same type of ball.
- **w/o order w/o replacement** (combinations or subpopulations)
  - Partitioning a group of n people into 2 groups.
  - Extended Case: Partitioning a group of n people into m groups, so that group i has k_i members (multinomial coefficient).
  - Choosing k types of items from n possible.
  - Place one of 2 types of balls (e.g. k black and $(n - k)$ white) into each of n bins.
- **w/o order w/ replacement**(combinations or subpopulations w/ replacement)
  - Choose k toppings from n possible for a pizza with double, triple etc. toppings allowed.
  - Place k identical balls into n bins.