Signals and Systems Problem Set

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1 Signals and Systems Preliminaries

1.1. Express $z = 4 + j2$ is magnitude-phase form. Show z, z^*, $1/z$ and $1/z^*$ in the complex plane.

1.2. Express $z = \frac{1}{2}e^{-j\pi/3}$ in Cartesian coordinates. Show z, z^*, $1/z$ and $1/z^*$ in the complex plane.

1.3. Simplify $\sqrt{j}$.

1.4. Consider the quantity $e^{j\frac{2\pi}{N}kn}$ for k and n both taking values in $\{0, 1, 2, \ldots N-1\}$.

(a) Show that $e^{j\frac{2\pi}{N}kn} = e^{j\frac{2\pi}{N}(kn)_{\mathrm{mod}\,N}}$ where $(kn)_{\mathrm{mod}\,N}$ is kn modulo N.

(b) For $N = 6$ how many distinct values of $e^{j\frac{2\pi}{N}kn}$ are there? Make a table of the 36 entries for all k and n and enter the simplified value of $e^{j\frac{2\pi}{N}kn}$.

1.5. Simplify and plot $\Re\left\{(2+j)e^{(-4+3j)t}/(1-j)\right\}$.

1.6. Verify that the following equations hold in general:

$$z(t) = \mathbb{EV}\{z(t)\} + \mathbb{ODD}\{z(t)\}$$
$$z(t) = \mathbb{HS}\{z(t)\} + \mathbb{HAS}\{z(t)\}$$

1.7. Show that if $z(t)$ is even, then

$$\int_{-T}^{+T} z(t)dt = 2\int_0^T z(t)dt$$

1.8. Show that for $z(t)$ odd,

$$\int_{-T}^{+T} z(t)dt = 0$$

1.9. Suppose $x(t)$ is even and $y(t)$ is odd. Find the even and odd parts of $z(t) = x(t)y(t)$ and simplify $\int_{-T}^{+T} z(t)dt$.

1.10. Show that $z(t)$ is HS if and only if $|z(t)|$ is even and $\angle z(t)$ is odd.

1.11. Show that $z(t)$ is HS if and only if $\Re\{z(t)\}$ is even and $\Im\{z(t)\}$ is odd.

1.12. If a signal is real and HS, what can be said about it?

1.13. If a signal is imaginary and HS, what can be said about it?

1.14. Consider the following function

$$g(\Omega) = \frac{1}{1 + (1/2)e^{-j\Omega}}$$

(a) Is this function HS? Is it HAS?
(b) Find $|g(\Omega)|^2$
(c) Find $\angle g(\Omega)$

1.15. Consider the following function

$$g(\Omega) = \frac{1}{1 + (1/2)(1+j)e^{-j\Omega}}$$

(a) Is this function HS? Is it HAS?
(b) Find $|g(\Omega)|^2$
(c) Find $\angle g(\Omega)$

1.16. Consider the following function

$$g(\omega) = \frac{1}{10 + j\omega}$$

(a) Is this function HS? Is it HAS?
(b) Find $|g(\omega)|^2$
(c) Find $\angle g(\omega)$

1.17. Consider the following function

$$g(\omega) = \frac{1}{(10 - 10j) + j\omega}$$

(a) Is this function HS? Is it HAS?
(b) Find $|g(\omega)|^2$
(c) Find $\angle g(\omega)$

1.18. Find the HS and HAS parts of $z(t) = e^{j\omega t}$.

1.19. Consider the complex number

$$z = \frac{1}{1 + e^{j\pi/4}}$$

Determine simplified numerical answers for $|z|$, $\angle z$, $\Re\{z\}$, and $\Im\{z\}$.

1.20. Evaluate the following sum

$$\sum_{n=0}^{9} 2^n$$

1.21. Let $y(t) = \cos(2\pi t) + \sin(t)$. Determine whether $y(t)$ is periodic and, if so, give the fundamental period of $y(t)$.

1.22. For arbitrary complex numbers z and w show that

$$\Re\{z\}\Re\{w\} = \frac{1}{2}\Re\{zw + zw^*\}$$

1.23. For continuous time, complex signals $z(t)$ and $w(t)$ show that

$$\Re\{z(t)\} * \Re\{w(t)\} = \frac{1}{2}\Re\{z(t) * w(t) + z(t) * w^*(t)\}$$

1.24. Let $x[n] = \delta_K[n] - 2\delta_K[n+1] + 2\delta_K[n-1] + \delta_K[n-4]$. Determine an expression for the even and odd parts of $x[n]$ and sketch these two signals. Clearly label your sketches.

1.25. A signal's power has been attenuated (reduced) by 6 dB by multiplying the signal by a constant. By what factor has the amplitude of the signal changed?

(a) $1/2$
(b) $1\sqrt{2}$
(c) $1\sqrt{6}$
(d) $1/4$

1.26. The equation below is an example of what property of the Kronecker delta function?

$$\sum_{k=-\infty}^{\infty} x[k]\delta[k-5] = x[5]$$

(a) Homogenity
(b) Scaling
(c) Sifting
(d) Superposition

1.27. Consider two periodic signals with periods T_1 and T_2, where $T_1 > T_2$. The sum of these two signals

(a) is periodic with period T_1
(b) is periodic with period T_2
(c) is periodic but with a period that may be different from either T_1 or T_2
(d) is not necessarily periodic

1.28. Which of the following systems is not a linear system?

(a) $y(t) = x(t-2) + x(t-1)$
(b) $y[n] = 3x[n] - 2x[n-1] + 1$
(c) $y[n] = 3n^2x[n-1]$
(d) $y(t) = \cos(2\pi t)x(t)$

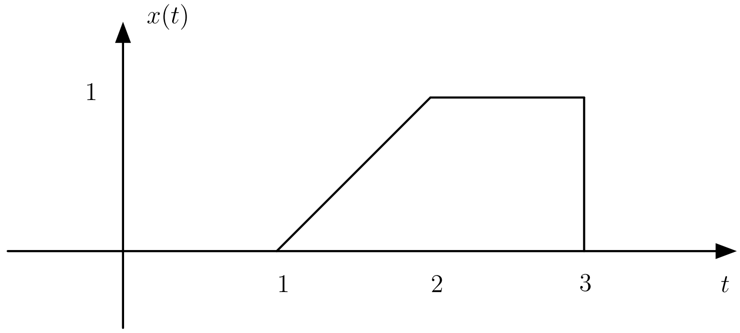

Figure 1: Signal considered in Problem 1.33.

1.29. Which of the following systems is not time-invariant?

(a) $y(t) = x(t+1) - x(t-1)$
(b) $y[n] = 4n^2 x[n-2]$
(c) $y[n] = 3x[n] - 2x[n-1] + 1$
(d) $y(t) = x(t+1) + x(t-1)$

1.30. Which of the following systems is not a causal system?

(a) $y[n] = x[n+2]$
(b) $y[n] = x[n] + 3$
(c) $y[n] = 3x[n] - 2n^2 x[n-1]$
(d) $y(t) = \cos(t/\pi)x(t-1)$

1.31. Is the continuous-time signal $x(t) = \cos(t)$ periodic? finite energy? finite power?

1.32. Is the discrete-time signal $x[n] = \cos[n]$ periodic? finite energy? finite power?

1.33. Consider the signal $x(t)$ as shown in Fig. 1; Sketch and label the signal $z(t) = 3x(4-2t)$.

1.34. Below is a list of several continuous-time systems, each described by the mapping of an arbitrary input $x(t)$ to the corresponding output $y(t)$. For each system, state whether the system has the listed properties by filling in each element of the table with 'YES' or 'NO'.

System	Linear	Time Invariant	Stable	Memoryless	Causal
$y(t) = \int_{t-3}^{t} x(\tau)d\tau$					
$y(t) = \int_{t-1}^{t+1} x(\tau)d\tau$					
$y(t) = \cos(10t + x(t))$					
$y(t) = 3x(t-7)$					
$y(t) = 3x(t+7)$					
$y(t) = x(t - x(t))$					
$y(t) = x(t)\cos(20t)$					

1.35. Below is a list of several continuous-time systems, each described by the mapping of an arbitrary input $x(t)$ to the corresponding output $y(t)$ ($x[n]$ and $y[n]$ are used for discrete time systems). For each system, state whether the system has the listed properties by filling in each element of the table with 'YES' or 'NO'. Explain your answers. Note: $\text{sgn}(v) = +1$ if $v \geq 0$ and -1 otherwise.

System	Linear	Time Invariant	Stable	Memoryless	Causal
$y(t) = 2x(t) + 4$					
$y[n] = 2^n y[n-1] + x[n+1]$					
$y(t) = \cos(2t)$					
$y(t) = \text{sgn}(x(t))$					

1.36. Below is a list of several systems, each described by the mapping of an arbitrary input $x(t)$ to the corresponding output $y(t)$ ($x[n]$ and $y[n]$ are used for discrete time systems). For each system, state whether the system has the listed properties by filling in each element of the table with 'YES' or 'NO'. Explain your answers.

System	Linear	Time Invariant	Stable	Memoryless	Causal
$y[n] = x[n-2]$					
$y[n] = x[3n-1]$					
$y(t) = x^2(t)$					
$y(t) = x(-t)$					
$y(t) = x(\cos(t))$					

1.37. Consider the signal $x(t)$ shown in Fig. 2. Sketch and carefully label the signal $y(t) = 6 - 3x(4 - t/2)$. If the mapping from $x(t)$ to $y(t)$ is considered as a system (*i.e.*, for arbitrary input $x(t)$), is this system stable? Casual? Time-invariant? Linear?

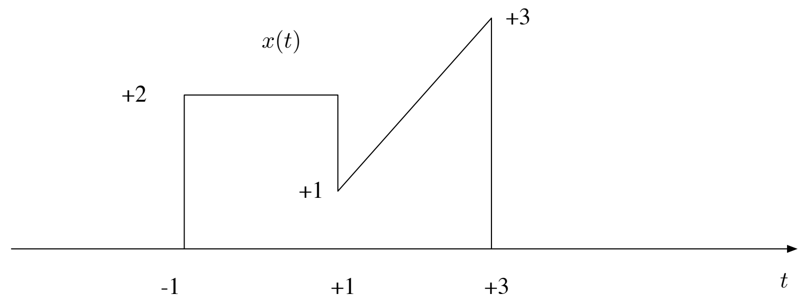

Figure 2: The signal $x(t)$ in Problem 1.37.

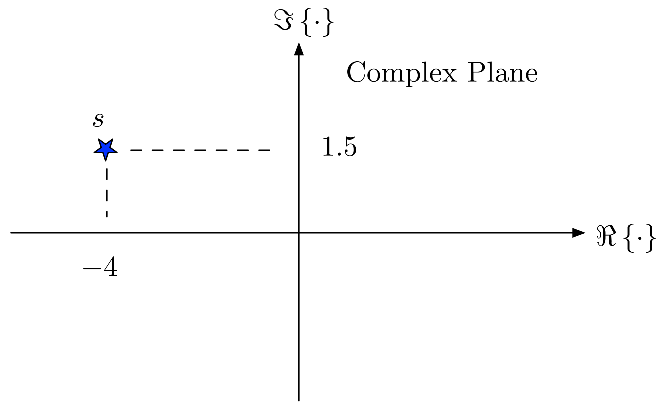

Figure 3: The location of the parameter s in the complex plane fore Problem 1.38.

1.38. Let $x(t) = \Re\left\{2e^{j\pi/3}e^{st}\right\}$ where s is the complex constant shown in Fig. 3.

(a) Determine a simplified expression for $x(t)$.

(b) Provide a description of the behavior of $x(t)$ and a labeled sketch describing this behavior (this does not have to be a detailed plot, just show that you understand the behavior of this signal).

1.39. Let $x[n] = (z_1^n + z_2^n)$ where z_1 and z_2 are the complex constants shown in Fig. 4.

(a) Determine a simplified expression for $x[n]$.

(b) Provide a description of the behavior of $x[n]$ and a labeled sketch describing this behavior (this does not have to be a detailed plot, just show that you understand the behavior of this signal).

1.40. Consider the frequency domain signal $X(f) = \mathbb{HS}\left\{e^{j6\pi f}\right\}$, where $\mathbb{HS}\left\{\cdot\right\}$ takes the Hermitian symmetric part. Find $x(t)$.

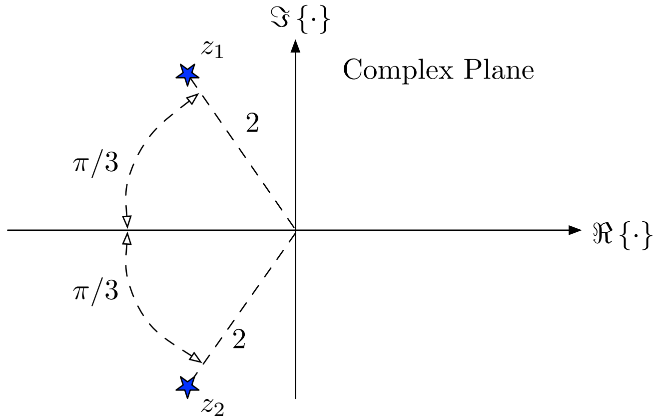

Figure 4: The parameters z_1 and z_2 for Problem 1.39.

1.41. Consider the system with input $x(t)$ and output $y(t)$ governed by

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Is this system

(a) Stable?

(b) Causal?

(c) Time invariant?

(d) Linear?

If the input is $x(t) = \delta_D(t)$ what is the output?

2 LTI Systems (time domain)

2.1. The output of a specific LTI system to the unit step input is given by the signal shown in Fig. 5. Determine the output of the system when the input is $x(t) = \mathrm{rect}(t)$ – recall $\mathrm{rect}(t)$ is 1 for $|t| < 1/2$ and zero elsewhere. Sketch and label this output signal when the input is a rect-function.

2.2. Consider $x[n] = \mathrm{u}[n-4] - \mathrm{u}[n-7]$. Sketch $x[n]$ and find and sketch $r[n] = x[n] * x[-n]$.

2.3. Consider a discrete time LTI system with impulse response $h[n] = 2^n\mathrm{u}[n]$. Is this system stable? Causal? Memoryless? Invertible? If the system is invertible, provide the impulse response of the inverse system – *i.e.*, find $g[n]$ so that $g[n] * h[n] = \delta[n]$. If the system is not invertible, provide an input $x[n]$ that cannot be recovered from $y[n] = h[n] * x[n]$.

2.4. Consider a LTI system with input $x[n]$ and output $y[n]$ governed by

$$y[n] = 2x[n] - 2x[n-1] + 4x[n-6]$$

Sketch and label the output of this LTI system when the input is the signal shown in Fig. 6.

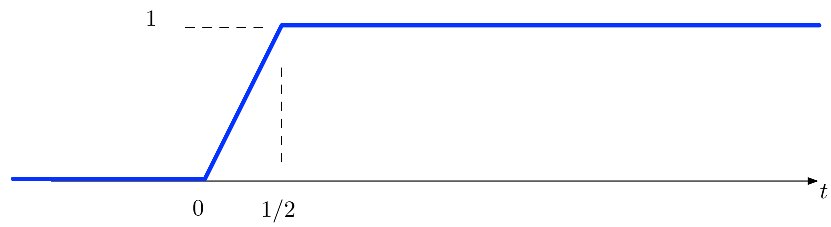

Figure 5: The step response of the system considered in Problem 2.1.

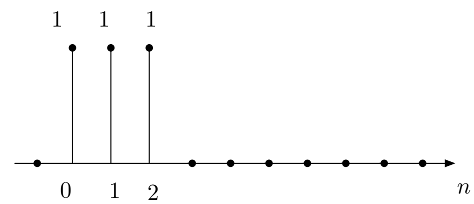

Figure 6: Input to the LTI system considered in Problem 2.4 .

2.5. Let $x(t) = \text{trian}(t)$ – recall this is $1 - |t|$ for $|t| < 1$ and 0 otherwise. Also, let $v(t) = \delta_D(2t-1)$ where $\delta_D(t)$ is the Dirac delta function. Find and sketch/label the following two signals:

(a) $w(t) = x(t) * v(t)$

(b) $z(t) = x(t)v(t)$

2.6. Consider a discrete time LTI system with impulse response

$$h[n] = \sin\left(\frac{\pi}{3}n\right) \mathrm{u}[n]$$

(a) Answer the following about the system and $h[n]$, providing a short justification for each response:

i. Is this system causal?

ii. Is this system stable?

iii. Is $h[n]$ periodic?

(b) Sketch $h[n]$ vs. n (label your sketch and include points corresponding to $-10 \leq n \leq 10$).

(c) Find a simplified expression for the output of this system $y[n]$ when the input is given by

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

(d) Provide an accurate, labeled sketch of $y[n]$.

2.7. The discrete time signal $x[n]$ is the input to an LTI system with impulse response $h[n]$ with both signals shown in Fig. 7. Sketch and fully label the output of the system $y[n]$.

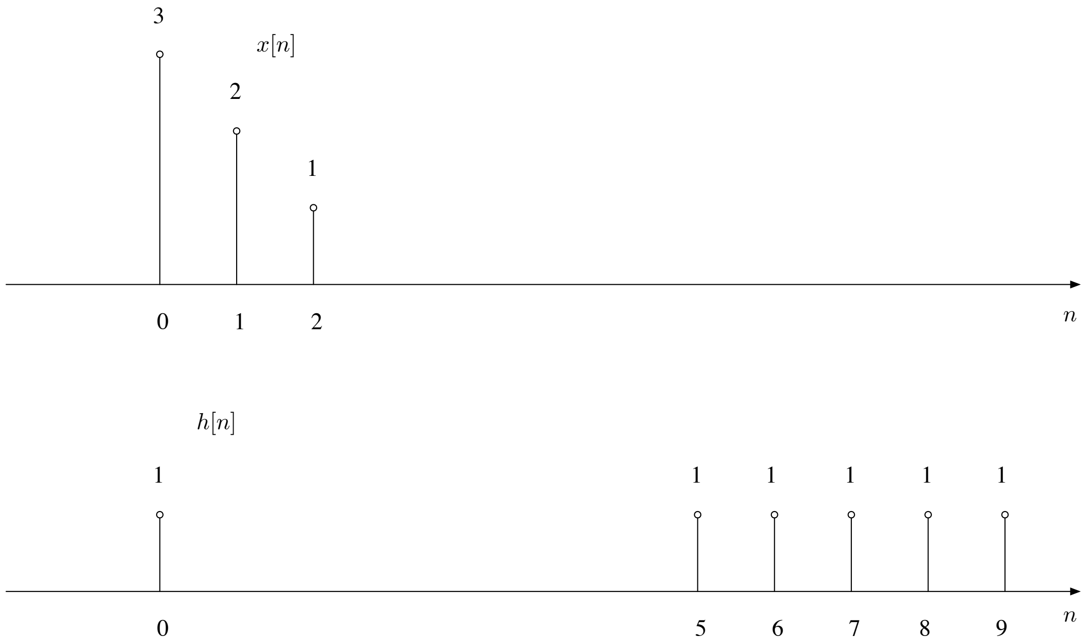

Figure 7: The input and impulse response of the LTI system in Problem 2.7.

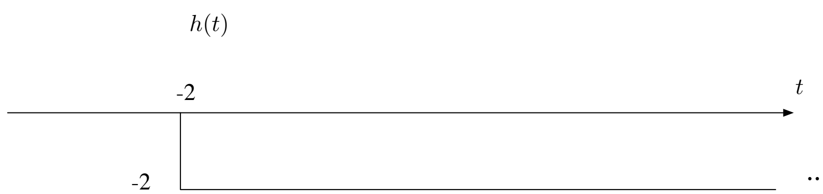

Figure 8: The impulse response $h(t)$ in Problem 2.8.

2.8. Find $y(t) = x(t) * h(t)$ where

$$x(t) = e^{-3(t-4)}\mathrm{u}(t-3)$$

where $\mathrm{u}(t)$ is the unit step function and $h(t)$ is sketched in Fig. 8. Determine a simplified expression for $y(t)$. Also, sketch $y(t)$ (provide labeling on your sketch).

2.9. Consider an LTI system with impulse response

$$h(t) = e^{-2t}\cos(\sqrt{12}\,t + \pi/3)\mathrm{u}(t)$$

(a) Sketch $h(t)$. Is this system stable? Causal? Memoryless?

(b) Find and sketch the unit step response to this system $y(t) = h(t) * \mathrm{u}(t)$.

2.10. A lab. technician at a recording studio is told to process audio recordings using two boxes. The boxes are labeled "LTI" and "TS", for Linear Time Invariant (LTI) and Time-Scaling

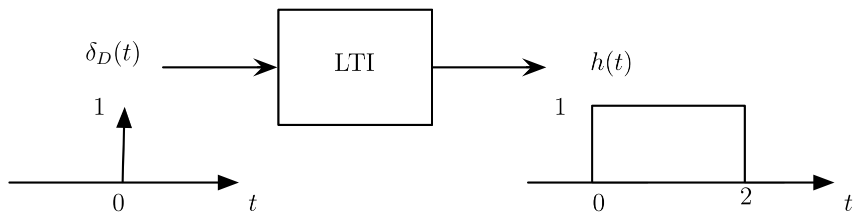

Figure 9: The LTI box with impulse response shown for Problem 2.10.

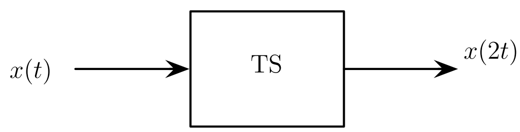

Figure 10: The input-output characteristics of the TS box in Problem 2.10.

(TS). The LTI box is an LTI system and its response to a unit delta function is known and shown in Fig. 9. The behavior of the TS box is shown in Fig. 10.

The technician is not sure exactly how to use these boxes, so he decides to run some tests. He runs two tests, in each case he uses an input test signal given by $x(t)$ as shown in Fig. 11

(a) The first test he conducts is to put the above signal $x(t)$ through the test system shown in Fig. 12. Determine and sketch (with labeling) the signals $v(t)$ and $w(t)$.

(b) The second test he conducts is to put the same signal $x(t)$ (given above) through the test system shown in Fig. 13. Determine and sketch (with labeling) the signals $y(t)$ and $z(t)$.

(c) Are test systems A and B equivalent models of the same system? Fill-in each element of the table below with 'YES' or 'NO' and explain your answers.

System	Linear	Time Invariant	Stable	Memoryless	Causal
Test System A					
Test System B					

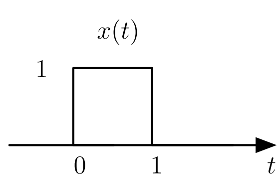

Figure 11: The input signal for the audio processing system in Problem 2.10.

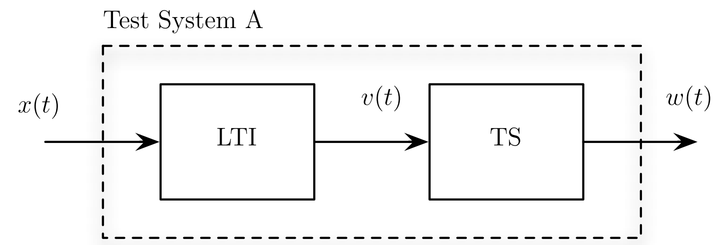

Figure 12: Audio test system A in Problem 2.10.

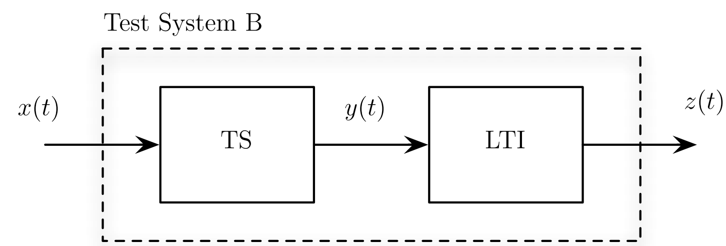

Figure 13: Audio test system B in Problem 2.10.

2.11. Consider the composite system shown in Fig. 14 where System 1 is governed by

$$y(t) = x\,(16 - 4t)$$

and System 2 is an LTI system with impulse response given by

$$h(t) = 2e^{-2(t+4)}\mathrm{u}(t+4)$$

This composite system is to be considered with the input signal shown in Fig. 15.

(a) Express the input signal $x(t)$ functionally using unit step functions.

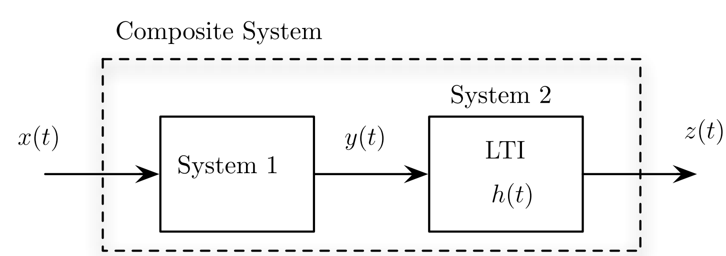

Figure 14: Concatenated system considered in Problem 2.11.

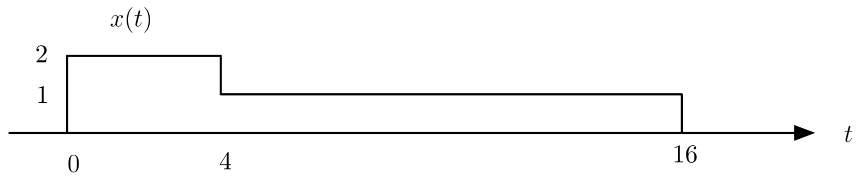

Figure 15: Input signal considered in problem Problem 2.11.

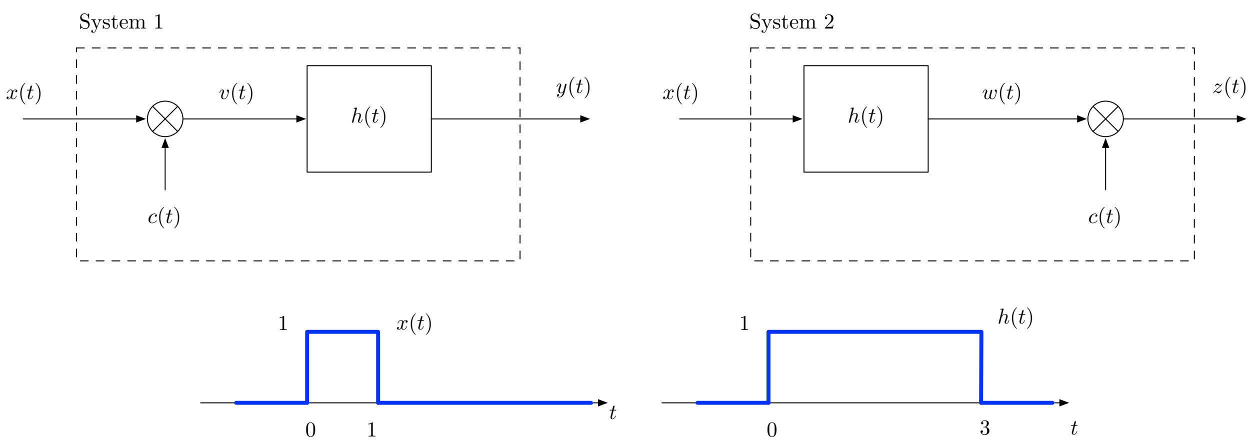

Figure 16: Systems considered in Problem 2.12.

(b) Sketch and $y(t)$, the output of the System 1, when $x(t)$ is the input. Clearly label your sketch.

(c) Find the output of the composite system $z(t)$ when $x(t)$ is the input. Specifically, find a simplified expression for $z(t)$ and sketch $z(t)$.

(d) Fill-in each element of the table with 'YES' or 'NO' and explain your answers.

System	Linear	Time Invariant	Stable	Memoryless	Causal
System 1					
System 2					
Composite System					

2.12. Consider the composite systems shown in Fig. 16 where $c(t) = \cos(4\pi t)$.

(a) Determine the fundamental period of $c(t)$ and plot this signal with labels.

(b) Consider System 1: find a simplified expression for $v(t)$ and $y(t)$ and provided a labeled sketch for each.

(c) Consider System 2: find a simplified expression for $w(t)$ and $z(t)$ and provided a labeled sketch for each.

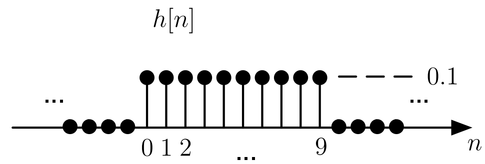

Figure 17: The impulse response for the system considered in Problem 2.13.

(d) Fill-in each element of the table with 'YES' or 'NO' and explain your answers.

System	Linear	Time Invariant	Stable	Memoryless	Causal
System 1					
System 2					

2.13. You just started a new job working for a wireless communications company. A digital communication receiver has a (discrete time) LTI system with impulse response $h[n]$ as shown in Fig. 17. Your boss wants to you to investigate the response of this system to an interference signal $x[n]$ defined by

$$x[n] = e^{j\frac{\pi}{8}n}$$

(a) To gain some insight, you first consider the signal $x[n]$. Determine the following about $x[n]$:

i. Is $x[n]$ periodic? Explain your answer by either determining the fundamental period (if YES) or showing that no period exists (if NO).

ii. Sketch and label the signal $x_R[n] = \Re\{x[n]\}$.

iii. Sketch and label the signal $x_I[n] = \Im\{x[n]\}$.

(b) Your boss claims that the output of this LTI system, $y[n] = x[n] * h[n]$, is of the form

$$y[n] = Cx[n]$$

where C is a complex constant. Show that this is true and determine this constant C.

2.14. Consider a discrete time LTI system with impulse response

$$h[n] = 2^{-n/2}\cos\left(\frac{\pi}{4}n\right)\mathrm{u}[n]$$

In this problem you will find the unit step response for this system – *i.e.*, find the output $y[n]$ when the input is $x[n] = \mathrm{u}[n]$.

(a) Answer the following about the system and $h[n]$, providing a short justification for each response:

i. Is this system causal?

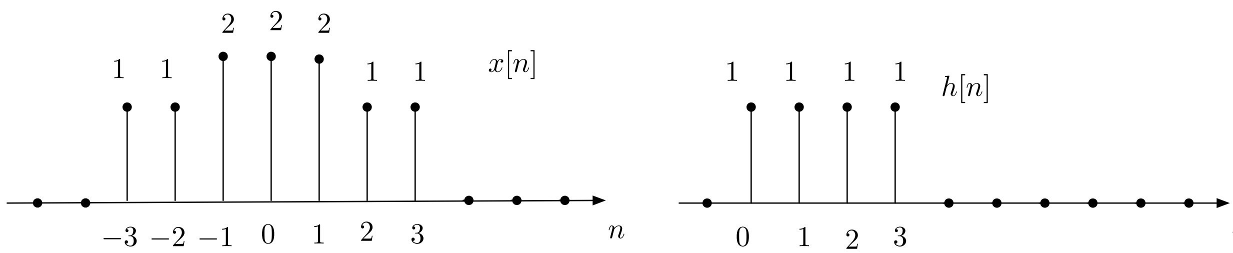

Figure 18: The signal $x[n]$ and impulse response $h[n]$ in Problem 2.15.

ii. Is this system stable?

iii. Is $h[n]$ periodic?

(b) Sketch $h[n]$ vs. n (label your sketch and include points corresponding to $-10 \leq n \leq 10$).

(c) Find the unit step response for this system and give a simplified functional description of this signal.

(d) Provide an accurate, labeled sketch of the unit step response (including the points corresponding to $-10 \leq n \leq 10$).

2.15. The input $x[n]$ and impulse response $h[n]$ for an LTI systems are shown in Fig. 18.

(a) This system is stable (true/false)?

(b) This system is causal (true/false)?

(c) This system is memoryless (true/false)?

(d) Determine the output of the system $y[n]$ and provide a labeled sketch of this signal.

2.16. Let $x(t) = \text{trian}(t)$ – recall this is $1 - |t|$ for $|t| < 1$ and 0 otherwise. Also, let $c(t) = \sum_{k=-\infty}^{\infty} \delta_D(t-3k)$ where $\delta_D(t)$ is the Dirac delta function. Find and provide a labeled sketch the following two signals:

(a) $w(t) = x(t) * c(t)$

(b) $z(t) = x(t)c(t)$

2.17. Determine the following

$$\int_{-\infty}^{\infty} \angle G(\omega) d\omega$$

where ω is a real variable of integration and

$$G(\omega) = \frac{1 + j\omega}{(3 + j + j\omega)(3 - j + j\omega)}$$

2.18. Consider the system shown in Fig. 19. Here the input $x(t)$ is passed through an LTI system with impulse response $h(t) = x^*(-t)$ and the number V is the system output at time $t = 0$.

For this problem, consider the input signal shown in Fig. 20.

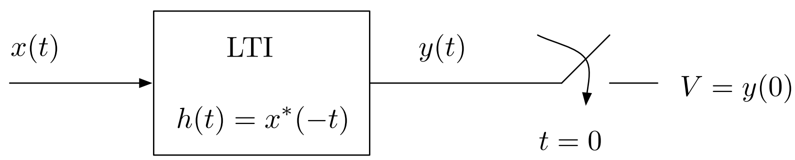

Figure 19: The sampled, matched filter for Problem 2.18.

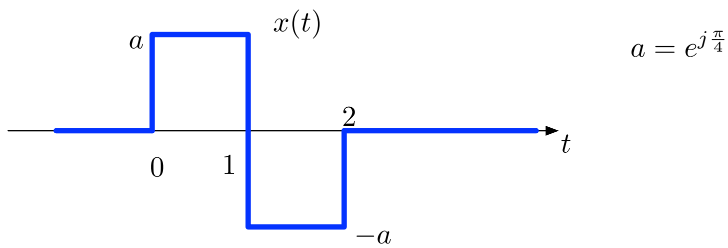

Figure 20: The input to the matched filter for Problem 2.18.

(a) Answer TRUE/FALSE questions below and provide the requested sketch for the last part.

i. This system is stable.

ii. This system is causal.

iii. This system is memoryless.

iv. Provide a labeled sketch of $h(t)$.

(b) Express $x(t)$ and $h(t)$ in terms of the rect-function – *i.e.*, $\text{rect}(t) = 1$ for $|t| \leq 1/2$ and 0 for $|t| > 1/2$.

(c) Find and provide a labeled sketch of $y(t)$.

(d) What is the numerical value of V? What is the physical interpretation of V?

2.19. Consider a continuous time LTI system with unit step response $s(t)$ given by

$$s(t) = \left(1 - e^{-at}\right) \text{u}(t)$$

where a is a real number with $a > 0$. In other words when the input to the LTI system is $\text{u}(t)$, the output is $s(t)$.

(a) Find the output of this system when the input is $x_1(t)$ as shown in Fig. 21. Sketch this output (this is a function of t and a).

(b) Find the output of this system when the input is $x_2(t)$ as shown in Fig. 22. Sketch this output (this is a function of t and a).

(c) Find the output of this system when the input is $x(t)$ as shown Fig. 23. Note that this is a square wave that starts at $t = 0$ continuous forever. The output signal can be completely specified by specifying it for $t \in [2m, 2m + 2]$ where m is an integer. Sketch the output on $t \in [2m, 2m + 2]$ (this is a function of t, a, and m).

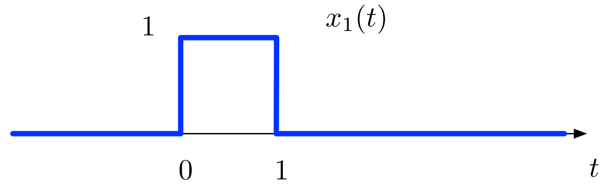

Figure 21: The input $x_1(t)$ for Problem 2.19.

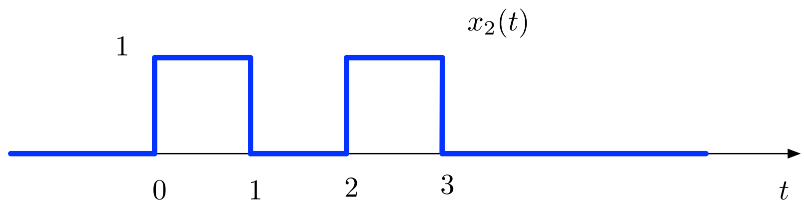

Figure 22: The input $x_2(t)$ for Problem 2.19.

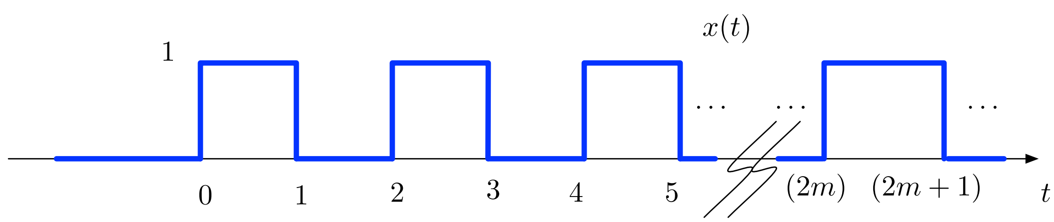

Figure 23: The input $x(t)$ for Problem 2.19.

(d) For the previous part (with the square wave that continues forever), the output will reach a steady state behavior for large t. Specify and sketch the output on $t \in [2m, 2m+2]$ as $m \to \infty$ (this is a function of t and a).

2.20. If $x[n] = \left(\frac{1}{2}\right)^n \mathrm{u}[n]$ is the input to an LTI system with impulse response $h[n] = \left(\frac{1}{4}\right)^n \mathrm{u}[n]$, what is the output $y[n]$?

3 Continuous Time Fourier Analysis

3.1. Which one of the following is not true about the relation between the frequency bandwidth and time duration of a signal?

(a) A signal can not be both bandlimited and have a finite time duration.

(b) Signals that are short in time duration have wider bandwidth.

(c) Signals that are narrow in bandwidth have longer time duration.

(d) A band limited signal is always causal.

3.2. If $x(t)$ is a real function, then the Fourier transform, $X(j\omega)$ has which of the following properties?

(a) Hermitian symmetry

(b) $\Im\{X(j\omega)\} = 0$

(c) $\Re\{X(j\omega)\}$ is an odd function

(d) Phase angles are all zero

3.3. Let $x(t)$ and $y(t)$ be the periodic signals shown in Fig. 24 with Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{j\frac{2\pi}{T}kt}$$
$$y(t) = \sum_{k=-\infty}^{+\infty} Y_k e^{j\frac{2\pi}{T}kt}$$

Find the coefficients Y_k in terms of the coefficients X_k.

3.4. For each of the cases below, plot the truncated Fourier Series approximation

$$\hat{x}_K(t) = \sum_{k=-K}^{+K} X_k e^{j\frac{2\pi}{T}kt}$$

for various values of K. Specifically, include at least $K = 1, 3, 7, 19$ in your plots. In each case below X_k is Hermitian Symmetric – *i.e.,* $X_k = X_{-k}^*$ – so X_k is specified for positive k. Also, unless otherwise indicated, you can assume that $X_k = 0$.

(a) $X_k = 0$ for even k. For odd and positive k, let $k = 2m+1$, then $X_k = \frac{(-1)^m}{k}$.

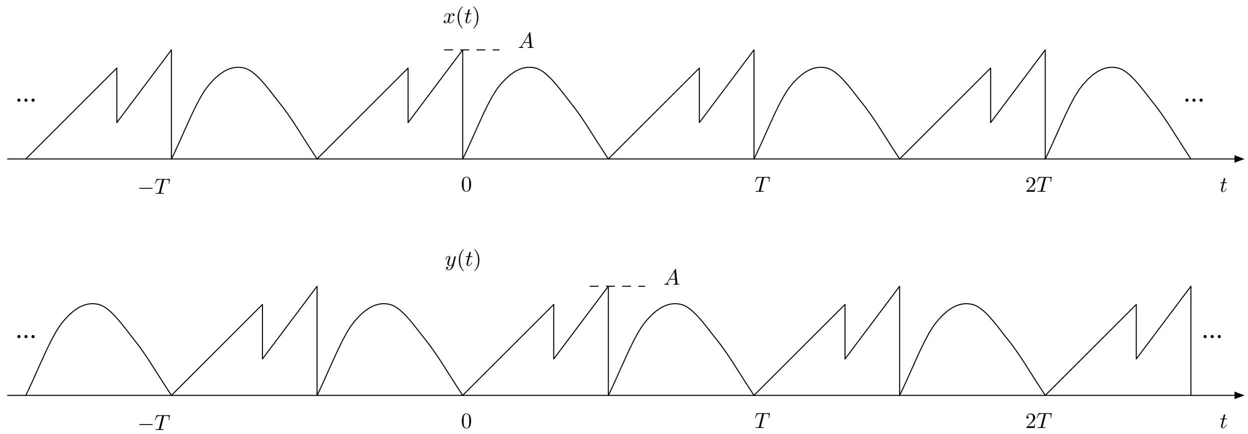

Figure 24: The periodic signals considered in Problem 3.3.

(b) $X_k = 0$ for even k. For odd and positive k, $X_k = \frac{(-j)^k}{k^2}$.

(c) $X_k = 1/k^2$.

(d) $X_k = 1$ for all k, including $X_0 = 1$.

(e) Make up one sequence of number with some structure, assign this to be X_k.

3.5. Which of the following is not true of an LTI system?

(a) If the magnitude of the input signal is doubled, the magnitude of the output signal will double.

(b) Frequencies can be present in the input that are not in the output.

(c) Frequencies can be present in the output that are not in the input.

(d) The phase of the frequency components can shift by any amount.

3.6. For the cascaded LTI systems show below with frequency response functions $H_1(j\omega)$ and $H_2(j\omega)$, what is the frequency response of the overall cascade system?

(a) $H_1(j\omega) \times H_2(j\omega)$

(b) $H_1(j\omega) * H_2(j\omega)$

(c) $H_1(j\omega) + H_2(j\omega)$

(d) $\frac{H_1(j\omega)}{H_2(j\omega)}$

3.7. Use Fourier relationships to find the following integral: $\int_{-\infty}^{\infty} \text{sinc}(x)dx$.

3.8. If $x(t)$ is a signal with the property

$$\int_{-\infty}^{\infty} x(t)dt = 0$$

then what can be said about $X(j\omega)$ at $\omega = 0$?

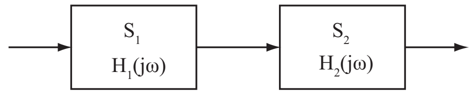

Figure 25: The system considered in Problem 3.6.

3.9. The following relation is true:

$$4\cos(t) + 2\cos(t-1) + \cos(t-10) = A\cos(t+\theta)$$

Determine the real numbers A and θ.

3.10. Use Fourier relationships to find the following integral:

$$\int_{-\infty}^{\infty} \text{sinc}(t)\text{sinc}(t-5)dt$$

3.11. Use Fourier properties to evaluate the following integral:

$$\int_{0}^{\infty} \text{sinc}^4(x)dx$$

3.12. Use Fourier theory to evaluate the following integral

$$\int_{0}^{\infty} \frac{dx}{4+x^2}$$

3.13. Find the Fourier Transform of

$$x(t) = \frac{1}{3}e^{-4(t-3)}\text{u}(t-2)$$

3.14. Find the inverse Fourier Transform of

$$X(j\omega) = \frac{j\omega}{3+j\omega}$$

3.15. Find the Fourier Transform of

$$x(t) = \frac{1}{2}e^{-2(t-3)}\text{u}(t-1)$$

3.16. Use Fourier relationships to evaluate the following integral:

$$\int_{0}^{\infty} \text{sinc}^3(z)dz$$

3.17. Determine the inverse Fourier Transform of $X(j\omega) = \frac{\sin(3\omega)}{6\omega}$.

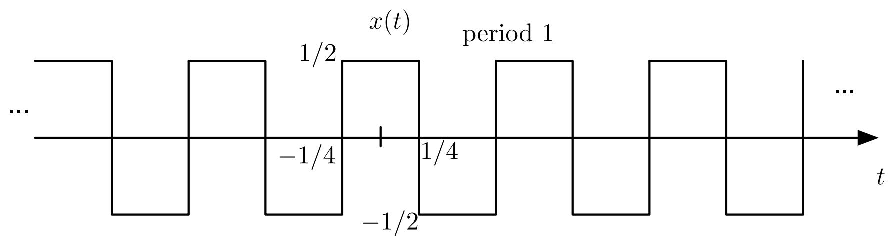

Figure 26: Input to the LTI system considered in Problem 3.27 .

3.18. Let $h(t) = \delta(t) - \frac{1}{2}\delta(t-1)$, determine $|H(j\omega)|^2$.

3.19. Determine the Fourier Transform of $\cos(\omega_0 t + \theta)$ where ω_0 and θ are real constants.

3.20. Determine the Fourier Transform of $\frac{2a}{a^2+t^2}$ where a is a real, positive constant.

3.21. Consider a continuous time LTI system with frequency response $H(j\omega)$. Determine the output of this system, $y(t)$, when the input is $x(t) = e^{j20t}$.

3.22. Consider a continuous time LTI system with input $x(t)$ and output $y(t)$ governed by

$$y(t) + RC\frac{d}{dt}y(t) = x(t)$$

If the input to this system is $x(t) = e^{j100t}$ what will be the output $y(t)$?

3.23. Consider an RC circuit with input signal $x(t) = \sin(20t)$ (source voltage) and output $y(t)$ (voltage across capacitor). Find the output signal $y(t)$ as a function of R and C.

3.24. A student measures and records an output voltage, $v(t)$, of a linear circuit in the laboratory. She then computes the Fourier transform of this signal, $V(j\omega)$. She tells you that $V(j20) = 0.4(1+2j)$. What is $V(-j20)$?

3.25. A causal, stable LTI system with input $x(t)$ and output $y(t)$ is governed by

$$13y(t) - 6y'(t) + y''(t) = 2x(t)$$

Determine the frequency response and the impulse response of this system.

3.26. Consider an LTI system with real-valued impulse response $h(t)$.

(a) Show that when the input is $x(t) = \cos(\omega_0 t)$ the output of the systems is $y(t) = |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$

(b) Find the output when the input is $x(t) = \sin(\omega_0 t)$.

(c) Would these results hold if $h(t)$ were not real-valued (*i.e.,* if it was complex)? Explain.

3.27. In this problem, you will consider the output of various LTI systems when the input is the square wave shown in Fig. 26.

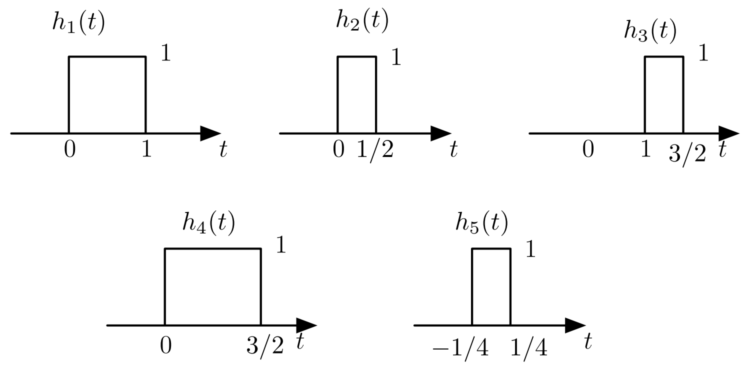

Figure 27: The impulse responses for LTI systems considered in Problem 3.27 .

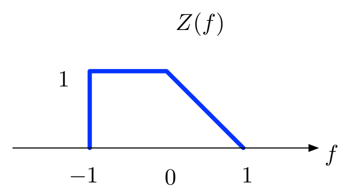

Figure 28: The signal considered in Problem 3.29.

(a) Find the output $y_i(t)$ when $x(t)$ is passed through an LTI system with impulse response $h_i(t)$ as shown in Fig. 27.

(b) Consider the system with impulse response $h_5(t)$. What is the frequency response $H_5(j\omega)$ for this system (*i.e.,* the Fourier transform of $h_5(t)$). Using the properties of LTI systems along with your knowledge of the Fourier series for $x(t)$, determine the Fourier series for $y_5(t)$.

(c) Using a similar method as in the previous part, determine $H_1(j\omega)$ and explain your result for $y_1(t)$ based on $H_1(j\omega)$.

3.28. Consider the causal, stable, LTI system with system response

$$H(s) = \frac{10^5}{(s+10)(s+10^4)}$$

Provide approximate Bode plots (with labels!) for the $|H(j\omega)|$ and $\angle H(j\omega)$.

3.29. Let $x(t) = \Re\{z(t)\}$ with $Z(f)$ as shown in Fig. 28. Find $x(t)$.

3.30. Find the Fourier transform of $x(t)$, the signal shown in Fig 29.

3.31. Find the inverse Fourier transform of $X(f) = e^{-2f}\mathrm{u}(f)$

3.32. Consider approximating a signal using Hadamard functions. Specifically, consider $x_1(t) = \cos(2\pi t)$ and $x_2(t) = \sin(20\pi t)$ where each signal is defined on $t \in [-1/2, 1/2]$. Use the

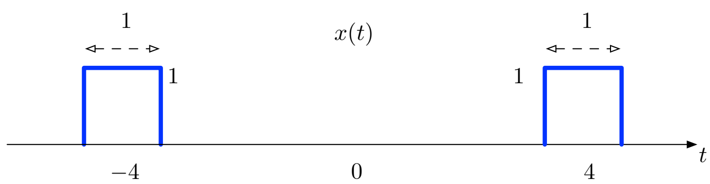

Figure 29: The signal considered in Problem 3.30.

Hadamard sequences to define a set of orthonormal signals on the interval $[-1/2, +1/2]$ and find the linear combination of these signals that best approximates the signals $x_1(t)$ and $x_2(t)$. Do this for length 4 Hadamard sequences and repeat with length 16 Hadamard sequences. Compute the expansion coefficients in each case and plot the approximation and $x_i(t)$ on the same plot.

3.33. Consider the orthonormal signals defined for $-1 \leq t \leq +1$

$$\phi_0(t) = \frac{1}{\sqrt{2}}$$
$$\phi_1(t) = \sqrt{\frac{3}{2}}t$$
$$\phi_2(t) = \frac{1}{2}\sqrt{\frac{5}{2}}(3t^2 - 1)$$

Consider approximating $x(t) = t^3$ for $t \in [-1, +1]$ by

$$\hat{x}(t) = X_0\phi_0(t) + X_1\phi_1(t) + X_2\phi_2(t)$$

Find the numerical values for the coefficients $\{X_0, X_1, X_2\}$ that minimize the energy in the approximation error. For these best coefficients, what is the fraction of approximation error signal energy to the energy in $x(t)$?

3.34. Consider the orthogonal signals defined for $-1 \leq t \leq +1$ as shown in Fig. 30. Consider approximating $x(t) = t(t-1)$ for $t \in [-1, +1]$ by

$$\hat{x}(t) = A_1 v_1(t) + A_2 v_2(t)$$

Find the numerical values for the coefficients that minimize the energy in the approximation error. Sketch and label the signal $x(t)$ and the approximation $\hat{x}(t)$.

3.35. Consider the signal $x(t)$ given by

$$x(t) = \text{trian}(t/2) * \sum_{n=-\infty}^{\infty} \delta(t - 3n)$$

(a) Determine the fundamental period of $x(t)$ and sketch $x(t)$.

(b) Determine the Fourier Series coefficients X_k for $x(t)$ with respect to period T and provide numerical answers for X_k for $|k| \leq 6$.

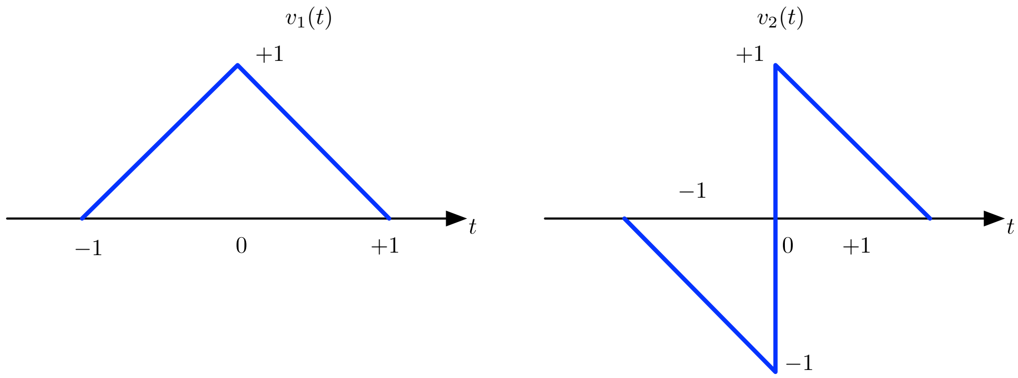

Figure 30: Two orthogonal signals in Problem 3.34.

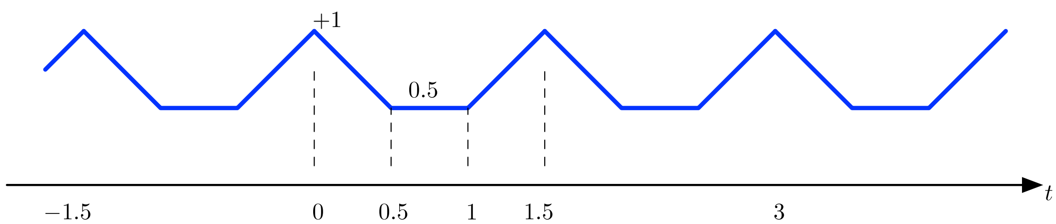

Figure 31: The periodic signal in Problem 3.36.

(c) What is the Fourier transform of $x(t)$?

3.36. Find the Fourier Series coefficients for the periodic signal $x(t)$ shown in Fig. 31. Evaluate X_k numerically for $|k| \leq 4$.

3.37. A causal, stable LTI system with input $x(t)$ and output $y(t)$ is governed by

$$y(t) + 0.1y'(t) = x'(t)$$

(a) Determine the frequency response and the impulse response for this system.

(b) Let the input to this system be $x(t) = \cos(10t)$, determine the output.

(c) What type of filtering does this system perform?

3.38. Consider $x(t)$, periodic with period T, as shown in Fig. 32.

(a) This signal can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi \frac{k}{T} t}$$

Find the coefficients X_k in a simple form as a function of k.

(b) Evaluate the expression in the previous part to provide numerical values for X_k, $|k| \leq 5$.

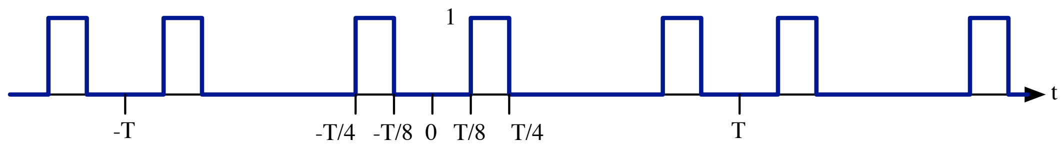

Figure 32: The signal considered in Problem 3.38.

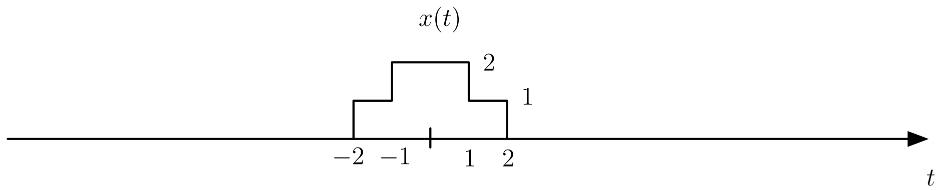

Figure 33: The signal considered in Problem 3.39.

(c) It is desired to find an approximation of $x(t)$ of the form

$$\hat{x}(t) = A_0 + A_1 \cos\left(\frac{2\pi}{T}t\right) + B_1 \sin\left(\frac{2\pi}{T}t\right) + A_2 \cos\left(\frac{4\pi}{T}t\right) + B_2 \sin\left(\frac{4\pi}{T}t\right) + A_3 \cos\left(\frac{6\pi}{T}t\right) + B_3 \sin\left(\frac{6\pi}{T}t\right)$$

Find the coefficients that minimize the energy in the approximation error.

3.39. This problem deals with the signal $x(t)$ shown in Fig. 33.

(a) Find the Fourier transform of $x(t)$.

(b) Consider the periodic signal $x_4(t)$ as shown in Fig. 34. Find the Fourier series for this signal. Specifically, for the expansion

$$x_4(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

determine ω_0 for this expansion and the coefficients. To demonstrate your understanding of the above answer, specify the numerical values for the coefficients a_k, $|k| \leq 7$.

(c) Consider the periodic signal $z(t)$ shown in Fig. 35. This signal can be expressed as a Fourier series with the same value of ω_0 used in part (b) and coefficients c_k. Specify how the coefficients c_k are related to the coefficients a_k found in part (b).

(d) Consider the periodic signal $x_8(t)$ as shown in Fig. 36. Find the Fourier series for this signal. Specifically, for the expansion

$$x_8(t) = \sum_{k=-\infty}^{\infty} d_k e^{j\omega_0 k t}$$

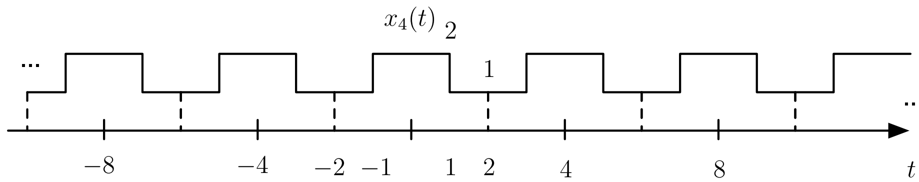

Figure 34: The period 4 signal considered in Problem 3.39.

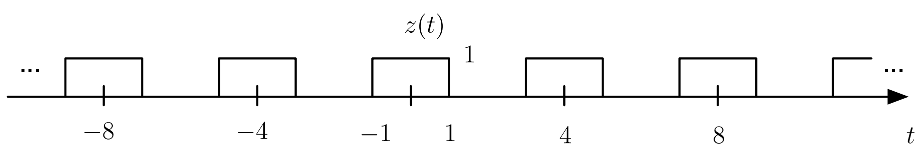

Figure 35: Another period 4 signal considered in Problem 3.39.

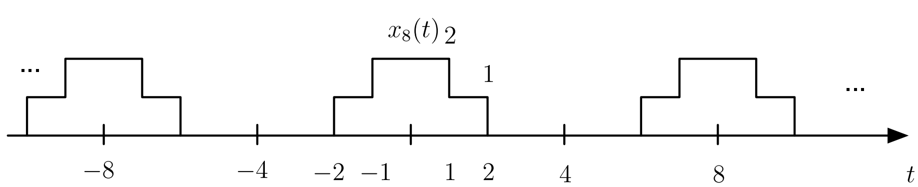

Figure 36: The period 8 signal considered in Problem 3.39.

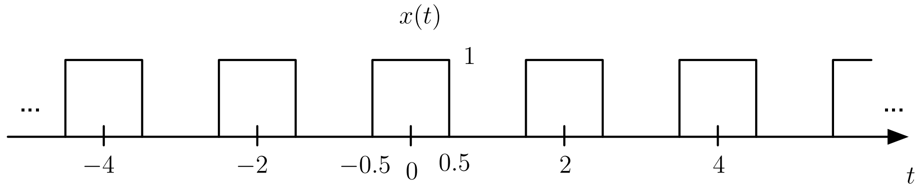

Figure 37: The input signal considered in Problem 3.40 and Problem 3.41.

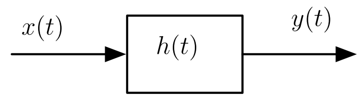

Figure 38: The LTI systems considered Problem 3.40.

determine ω_0 for this expansion and the coefficients. To demonstrate your understanding of the above answer, specify the numerical values for the coefficients d_k, $|k| \leq 7$.

3.40. This problem deals with the signal $x(t)$ shown in Fig. 37. Note that $x(t)$ is periodic with period 2.

(a) Consider passing $x(t)$ through a linear time invariant system with impulse response given by

$$h(t) = 2\text{sinc}(2t)$$

to produce $y(t)$ as shown in Fig. 38. Determine the frequency response $H(j\omega)$ for this LTI system. Determine and sketch the output signal $y(t)$.

(b) Now consider the system comprising a serial cascade of and RC circuit and the above LTI system as shown in Fig. 39. The RC circuit is an LTI system with input output relation governed by

$$r(t) + RC\frac{dr(t)}{dt} = x(t)$$

Determine the frequency response $H_{RC}(j\omega)$ for this RC system – *i.e.*, $H_{RC}(j\omega)$. Determine and sketch the output signal for this cascaded system $z(t)$.

3.41. Consider the periodic signal $x(t)$ shown in Fig. 37. This signal is processed by an LTI filter with frequency response $H(j\omega)$. It is desired that the output $y(t)$ be $\cos(9\pi t)$. Sketch an $H(j\omega)$ to achieve this.

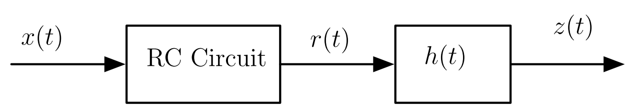

Figure 39: The concatenated system considered in Problem 3.40.

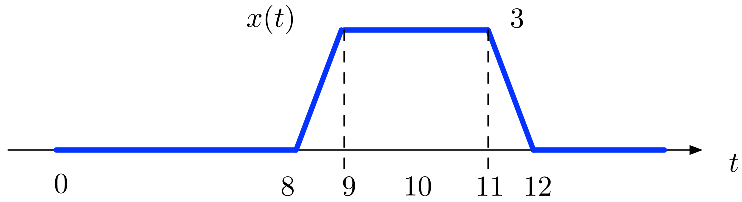

Figure 40: The signal for Problem 3.44.

3.42. An LTI system governed by the following equation

$$y(t) + 0.01\frac{d}{dt}y(t) = x(t)$$

(a) Determine the frequency response $H(j\omega)$ and impulse response $h(t)$ for this system.

(b) A signal of the form $x(t) = \cos(\omega_0 t)$ is the input to this system. What value of ω_0 should be selected in order to have the output be another cosine with the same frequency, but with an amplitude of 0.1?

(c) Consider the causal LTI system with input $x(t)$ and output $y(t)$ governed by

$$5y(t) - 2\frac{d}{dt}y(t) + \frac{d^2}{dt^2}y(t) = x(t)$$

i. Find the system function $H(s)$ include the corresponding ROC. Sketch the ROC and indicate the poles and zeros.

ii. Using only integrators, real-coefficient multipliers, and adders, draw a block diagram that implements this system.

iii. Find and sketch the impulse response of the system $h(t)$.

3.43. What is the Fourier transform of each of the following signals?

(a) $x(t) = \sum_{m=-\infty}^{\infty} 2\delta(t - m/4)$.

(b) $h(t) = e^{-3t-2}\mathrm{u}(t-5)$

(c) $v(t) = \mathrm{rect}(4t - 12)$

3.44. Determine the Fourier transform of the signal $x(t)$ sketched in Fig. 40.

3.45. An LTI system with input $x(t)$ and output $y(t)$ is governed by

$$\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t) + 100x(t)$$

(a) Determine the frequency response of this system.

(b) Provide an approximate Bode plot (magnitude and phase). Make sure to label your Bode plot to show that you understand the approximation.

(c) If the input to this system is $x(t) = \cos(10t + \pi/3)$, determine the output $y(t)$.

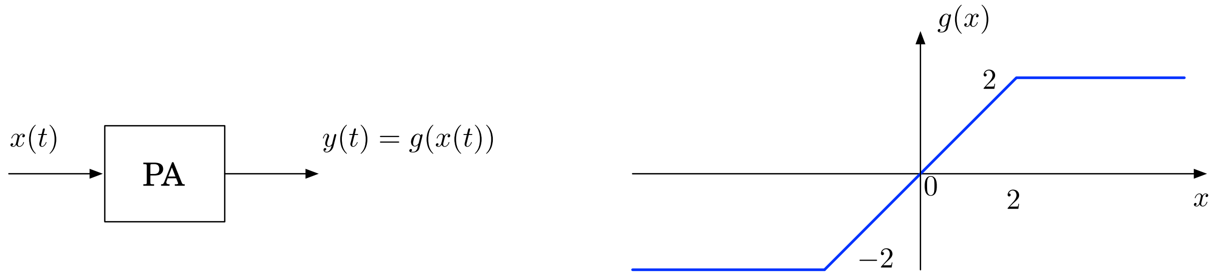

Figure 41: The input-output characteristic for the power amplifier in Problem 3.46.

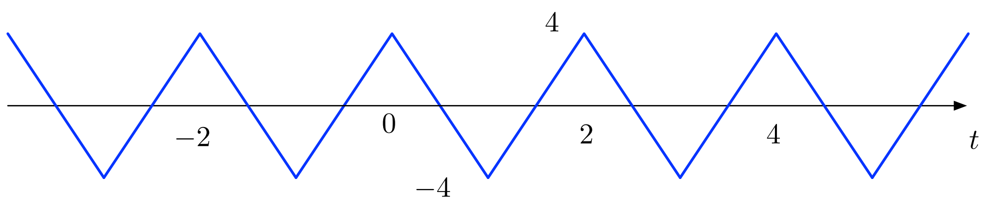

Figure 42: The input to consider for Problem 3.46.

3.46. A wireless communication system using a power amplifier (PA) to transmit the signal. Many power amplifiers exhibit "saturation" which can be modeled as a simple memoryless system. Specifically, if the input to the system at time t is x, then the output at time t is $y = g(x)$, where $g(x)$ is shown in Fig. 41.

(a) Consider the case when the input to the PA is $x(t) = \cos(4\pi t)$.

- Determine $y(t)$.
- Provide labeled sketches of $x(t)$ and $y(t)$.
- Determine the Fourier transform of $x(t)$ and $y(t)$.
- Provide labeled sketches for the Fourier transform of $x(t)$ and $y(t)$.

(b) Consider the case when the input to the PA is $x(t)$ is the triangle wave shown in Fig. 42.

- Determine $y(t)$.
- Provide a labeled sketch of $y(t)$.
- Determine the Fourier transform of $x(t)$ and $y(t)$.
- Provide labeled sketches for the Fourier transform of $x(t)$ and $y(t)$.

(c) Is this PA system linear? Is it stable? Time invariant?

3.47. Consider an LTI system with real impulse response $h(t)$. When the input is $x(t) = \cos(\omega_0 t)$ the output is $y(t) = |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$. Show that this is equivalent to a delay for this cosine input and identify the delay $\tau(\omega_0)$ – *i.e.*, the delay may depend on ω_0. For what type of function $\angle H(j\omega_0))$ will all frequencies be delayed by the same amount? What is this delay? This delay is called the group-delay of a filter.

4 Sampling, Reconstruction and Modulation

4.1. The temperature at USC is measured continuously and it is desired to store and distribute this signal using a discrete time version via sampling. The frequency spectrum of the temperature signal has been estimated and it has been concluded that this signal has frequency content from 0 up to 0.0833 Hz (1 Hz = 1/sec). This means that the maximum rate of change is every 120 seconds (120=1/0.0833). What is the minimal number of samples that should be collected every hour to ensure that temperature variations can be perfectly reconstructed?

4.2. A continuous time speech signal has been filtered so that it has no frequency content above 10 KHz (1 KHZ = 1000 Hz). This signal is sampled at the Nyquist rate and the resulting samples are stored in "frames" where each frame is a segment of speech lasting 20 msec (1 msec = 0.001 sec). How many samples are in each frame?

4.3. A continuous-time signal sampled at 6kHz has a component at a normalized radian frequency of $\Omega_0 = \pi/3$. Which of the following could correspond to the same component in the continuous time signal?

   (a) 500 Hz
   (b) 1 kHz
   (c) 2 kHz
   (d) 4 kHz

4.4. A continuous-time signal has frequency components at 15kHz, 17kHz, 21kHz and 23kHz. In order to later reconstruct the signal without aliasing, what is the minimum sampling rate that should be used?

   (a) The highest frequency in the signal: 23 kHz
   (b) Twice the highest frequency in the signal: 46 kHz
   (c) Sum of the frequencies: 15 kHz + 17 kHz + 21 kHz + 23 kHz = 76 kHz
   (d) Twice the midpoint of the frequency spectrum: 2× (15 kHz + 23 kHz)/2 = 38 kHz

4.5. CD quality audio uses a 20 KHz bandwidth with a sample rate that is 10% faster than the Nyquist rate. It separately samples a left and right audio channels. How many samples are there in a 3 minute song? If each sample is quantized to 2 bytes (16 bits), how many bytes are required to represent 3 minutes of raw CD audio?

4.6. A continuous time audio signal $x(t)$ is sampled at 44 kHz to produce a discrete time signal $y[n]$. It is desired to perform a bandpass filter (BPF) operation centered around 11 kHz in the original signal. What should be the center frequency in a discrete-time BPF to accomplish this?

   (a) $\Omega_0 = \pi/4$ rad/sample
   (b) $\nu_0 = \pi/2$ sample^{-1}
   (c) $\nu_0 = 1/4$ sample^{-1}
   (d) $\nu_0 = 4$ sample^{-1}

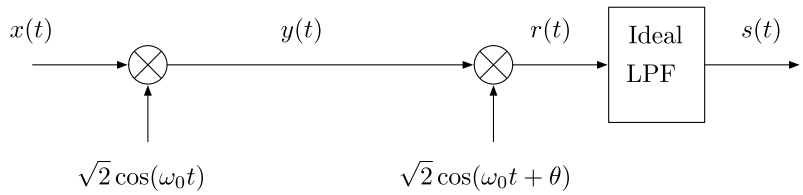

Figure 43: The system considered in Problem 4.9.

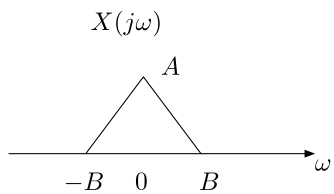

Figure 44: The spectrum of the input signal considered in Problems 4.9, 4.18 and 4.14.

4.7. Consider an ideal impulse sampler, mapping input $x(t)$ to output $y(t)$ via

$$y(t) = x(t)\left[\sum_{-\infty}^{\infty} \delta_D(t - nT_s)\right]$$

Is this system causal? Time-invariant? Linear? Memoryless?

4.8. Find the Fourier transform of $x(t) = \text{sinc}(t/200)\sin(20000\pi t)$

4.9. Consider the system shown in Fig. 43 with the real-valued message signal $x(t)$ has Fourier transform as sketched in Fig. 44 where $\omega_0 \gg B$. Note that in this system, there is a relative phase difference θ between the sinusoidal carrier used at the transmitter and that used as the receiver. This problem deals with the effect of this phase difference.

(a) Determine the Fourier transform of $\cos(\omega_0 t + \theta)$.

(b) Determine an expression for $Y(j\omega)$. Provide a labeled sketch of $Y(j\omega)$.

(c) Determine an expression for $R(j\omega)$. Provide a labeled sketch of $R(j\omega)$.

(d) Assuming that the ideal low pass filter passes all frequencies below B with unit gain and perfectly rejects all higher frequencies, give an expression for $S(j\omega)$ and $s(t)$ in terms of $X(j\omega)$ and $x(t)$. Define the loss in signal power for a given value of θ as

$$L(\theta) = \frac{\text{Energy in } s(t)}{\text{Energy in } x(t)}$$

(note that the energy in $s(t)$ is a function of θ). Determine an expression for $L(\theta)$ and plot this function for $0 < \theta < 2\pi$.

4.10. An ideal band-pass filter, centered at ω_0, has frequency response

$$H(j\omega) = \begin{cases} 1 & \omega_0 - B < \omega < \omega_0 + B \\ 1 & -\omega_0 - B < \omega < -\omega_0 + B \\ 0 & \text{otherwise} \end{cases}$$

Typically, $\omega_0 \gg B$. Determine and sketch the impulse response of this ideal BPF.

4.11. Consider the signal

$$r(t) = x(t)\sqrt{2}\cos(\omega_0 t) - y(t)\sqrt{2}\sin(\omega_0 t)$$

where $X(j\omega)$ and $Y(j\omega)$ are both zero for $|\omega| > B$ and $\omega_0 \gg B$. It was shown in lecture, using the multiplication property, that if we form two signals

$$z_I(t) = \sqrt{2}\cos(\omega_0 t) r(t) \tag{1}$$
$$z_Q(t) = -\sqrt{2}\sin(\omega_0 t) r(t) \tag{2}$$

then we have that $z_I(t) = x(t) +$ terms at frequency $2\omega_0$; similarly $z_Q(t)$ is $y(t)$ plus double frequency terms.

(a) Show that the same conclusion is arrived at using basic trigonometric identities by substituting the expression for $r(t)$ into the expressions for $z_i(t)$ and $z_Q(t)$.

(b) Verify that $r(t)$ can be expressed as the real part of the complex signal $w(t) = s(t)\sqrt{2}e^{j\omega_0 t}$, where $s(t) = (x(t) + jy(t))$.

(c) Find $W(j\omega)$ and $R(j\omega)$ in terms of $S(j\omega)$. **Hint:** use the multiplication property (or frequency shift property) to find $W(j\omega)$ and then use the relationship between the real-part and Hermitian-symmetric-part operators in the time and frequency domains.

4.12. Let $x(t) = \cos(2\pi(100)t) + \frac{1}{3}\cos(2\pi(300)t)$. This signal is sampled with sample time T_s as described in lecture to form

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta_D(t - nT_s)$$

This sampled signal is input to a reconstruction filter with impulse response $h_r(t) = \text{sinc}(t/T_s)$, which is the ideal reconstruction filter to produce the reconstructed signal $x_r(t)$.

(a) Consider $T_s = 1/700$ seconds. What is the reconstructed signal? Sketch $x_r(t)$ and discuss its relation to $x(t)$.

(b) Consider $T_s = 1/250$ seconds. What is the reconstructed signal? Sketch $x_r(t)$ and discuss its relation to $x(t)$.

(c) For one of the above examples, one of the cosines is aliased into a different frequency. Identify which case this is and state the original frequency and the frequency it is aliased into.

4.13. Consider $x_1(t)$ and $x_2(t)$ with $X_1(j\omega) = 0$ for $|\omega| > 1000\pi$ and $X_2(j\omega) = 0$ for $|\omega| > 2000\pi$.

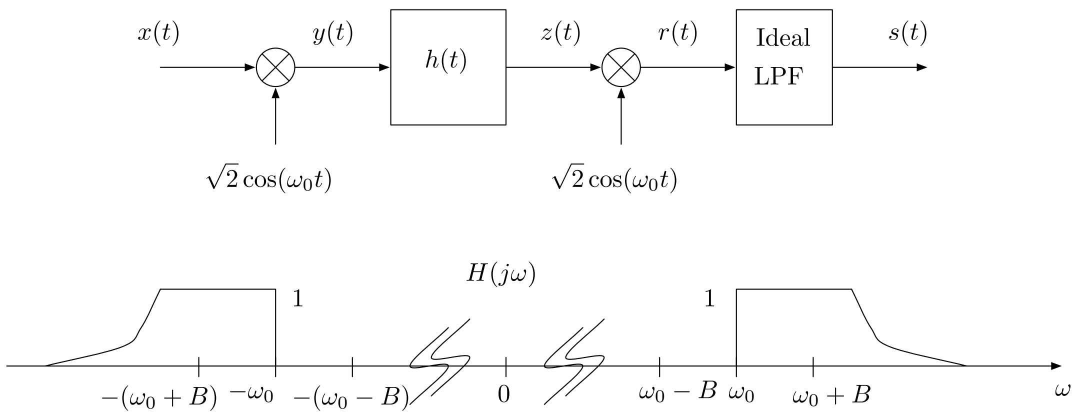

Figure 45: The modulation system and filter frequency response. in Problem 4.14.

(a) Let $y(t) = x_1(t) * x_2(t)$. What is the largest sample period T_s that can be used such that $y(t)$ can be perfectly reconstructed from its samples?

(b) Let $y(t) = x_1(t)x_2(t)$. What is the largest sample period T_s that can be used such that $y(t)$ can be perfectly reconstructed from its samples?

4.14. Consider the real signal $x(t)$ with Fourier transform $X(j\omega)$ as shown in Figure 44. This signal is processed as shown in Figure 45. This signal is modulated to create $y(t) = x(t)\sqrt{2}\cos(\omega_0 t)$ where $\omega_0 \gg B$. Then, $y(t)$ is filtered by the filter $h(t)$ with frequency response $H(j\omega)$ and down converted as shown in Figure 45. The ideal low-pass filter in Figure 45 passes frequencies $|\omega| \leq B$ and rejects all higher frequencies.

Provide a labeled sketch of $Y(j\omega)$, $Z(j\omega)$, $R(j\omega)$, and $S(j\omega)$. This method of amplitude modulation is called “single sideband (SSB)” modulation.

4.15. Consider $x_1(t)$ and $x_2(t)$ with $X_1(j\omega) = 0$ for $|\omega| > 1000\pi$ and $X_2(j\omega) = 0$ for $|\omega| > 2000\pi$.

(a) Let $y(t) = x_1(t) * x_2(t)$. What is the largest sample period T_s that can be used such that $y(t)$ can be perfectly reconstructed from its samples?

(b) Let $y(t) = x_1(t)x_2(t)$. What is the largest sample period T_s that can be used such that $y(t)$ can be perfectly reconstructed from its samples?

4.16. The sampling theorem was developed assuming multiplication of the continuous time signal $x(t)$ by a train of Dirac impulses – *i.e.*, the delta functions are spaced T seconds apart where T is the sampling interval. This problem considers the case where a narrow pulse is used in place of the Dirac delta function. Specifically, the sampled signal is

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)p(t - nT)$$

where $p(t)$ is a pulse of unit area and width less than T.

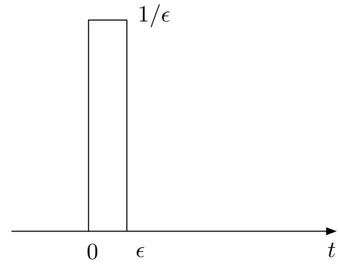

Figure 46: The sampling pulse considered in Problem 4.16.

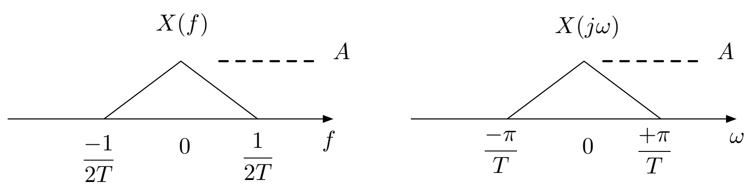

Figure 47: The spectrum of the input signal considered in Problem 4.16.

(a) As a concrete example, consider the special case of $p(t)$ shown in Fig. 46 ($\epsilon < T$). Find the Fourier Transform of this $p(t)$ and sketch the magnitude of this FT.

(b) Find a simplified expression for the Fourier transform of the sampled signal $x_s(t)$. Sketch the magnitude of this Fourier transform using the following assumptions: the pulse of part (a) with $\epsilon = T/4$ and the Fourier Transform of $x(t)$ as shown in Fig. 47. Note: the spectrum is shown in terms of both linear and angular frequency – you may sketch the spectrum of the sampled signal using either convention.

(c) It is still possible to reconstruct the original signal from the sampled signal $x_s(t)$ using an LTI filter – *i.e.,* there is a filter with impulse response $g(t)$ such that $x(t) = g(t) * x_s(t)$. Specify the Fourier transform of $g(t)$ (assuming a general $p(t)$). Assuming the $p(t)$ used in the previous part, sketch the magnitude of the Fourier transform of $g(t)$.

4.17. A signal has been modulated on a sinusoidal carrier with frequency $\omega_0 = 2\pi \times 10^8$ (i.e., 100 MHz). The modulated signal has Fourier transform given by

$$R(j\omega) = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(-j(\omega + \omega_0))$$

where $S(j\omega)$ is the bandlimited signal sketched in Fig. 48 where $B = 2\pi \times 500$ (i.e., 500 Hz).

This problem deals with a way of moving this signal to a lower frequency using sampling instead of mixing by sinusoids. Specifically, the system in Fig. 49 is to be considered where the sampling time is $T_s = \frac{2\pi}{\omega_s}$ with $\omega_s = \frac{3}{10}\omega_0 = 6\pi \times 10^7$ (i.e., 30 MHz). It is your task to show that this is possible and to design the low-pass filter (select W and A) so that the Fourier transform of $v(t)$ is

$$V(j\omega) = \frac{1}{2}S(j(\omega - \omega_{IF})) + \frac{1}{2}S(-j(\omega + \omega_{IF}))$$

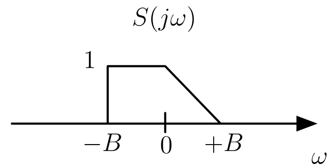

Figure 48: The spectrum of the information signal in Problem 4.17.

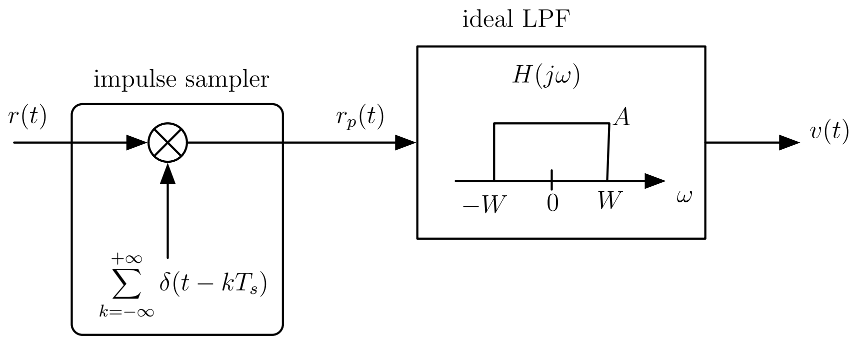

Figure 49: The IF sampler analyzed in Problem 4.17.

where ω_{IF} is an "intermediate frequency" less than ω_0.

(a) Sketch $R(j\omega)$

(b) Determine and sketch the Fourier transform of the sampled signal $r_p(t)$.

(c) Determine values for A and W (the parameters of the low-pass filter) such that

$$V(j\omega) = \frac{1}{2}S(j(\omega - \omega_{IF})) + \frac{1}{2}S(-j(\omega + \omega_{IF}))$$

and specify the value of ω_{IF}.

4.18. Consider the signal

$$y(t) = x(t)\cos(4Bt)$$

where $x(t)$ is a real-valued signal with Fourier transform shown in Fig. 44. The signal $y(t)$ is to be impulse sampled in an effort to represent the signal $x(t)$ from these samples as shown in Fig. 50. This problem deals with the valid sample rates to allow for perfect reconstruction of $x(t)$ from the samples of $y(t)$ – *i.e.*, so that $x_r(t) = x(t)$.

(a) Sketch the Fourier transform of $y(t)$.

(b) Let the sampling frequency $\omega_s = 2\pi/T_s$ be $\omega_s = 4B$. Sketch the Fourier transform of $y_s(t)$. For this value of ω_s, can $x(t)$ be perfectly reconstructed? If so, give the impulse response of the ideal LPF to achieve this.

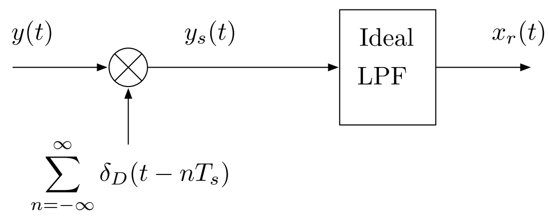

Figure 50: The sampling system considered in Problem 4.18.

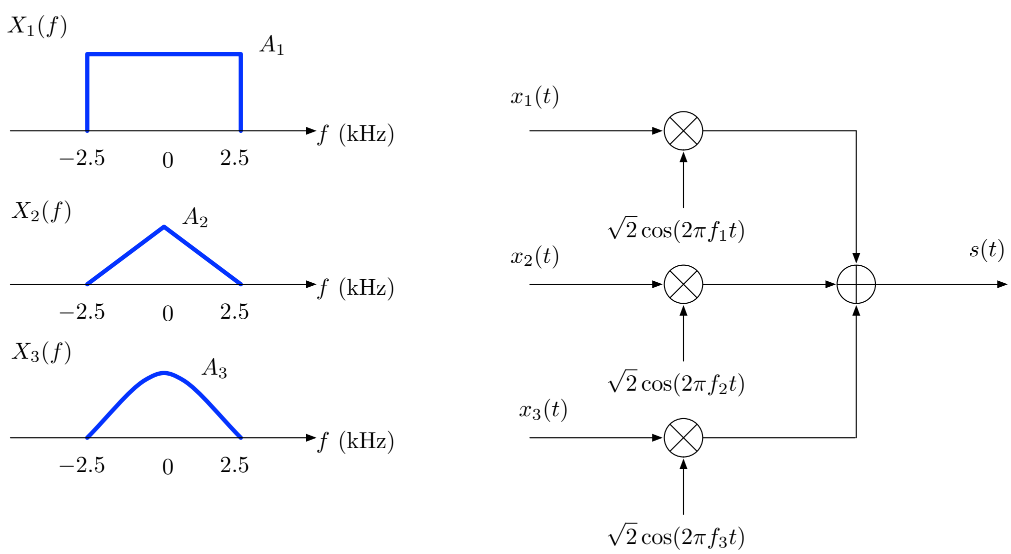

Figure 51: The 3-channel radio transmission system in Problem 4.19.

(c) Let the sampling frequency $\omega_s = 2\pi/T_s$ be $\omega_s = 2B$. Sketch the Fourier transform of $y_s(t)$. For this value of ω_s, can $x(t)$ be perfectly reconstructed? If so, give the impulse response of the ideal LPF to achieve this.

4.19. Consider the 3-channel radio transmission system shown in Fig. 51 where the spectrum of 3 message signals is shown. Note each has the same bandwidth. The modulating frequencies used are $f_1 = 960$ kHz, $f_2 = 970$ kHz, and $f_3 = 980$ kHz.

(a) Given an expression for the Fourier transform of $s(t)$ in terms of $X_1(f)$, $X_2(f)$ and $X_3(f)$. Provide a labeled sketch of $S(f)$.

(b) Consider the "super-heterodyne" receiver shown in Fig. 52 where the low pass filter passes frequencies in the band $|f| < 2.5$ kHz with unit gain and rejects other frequencies. This receiver can be tuned to any of the three channels by altering the *local oscillator (LO)* frequency, which frequency translates the desired channel to the *intermediate frequency (IF)* – which is $f_{IF} = 450$ kHz in this problem.

Determine the value of f_{LO} to tune the receiver to channel 2 – *i.e.,* $f_2 = 970$ kHz. For this choice of f_{LO}, provide a labeled sketch of $V(f)$, $Z(f)$, and $Y(f)$.

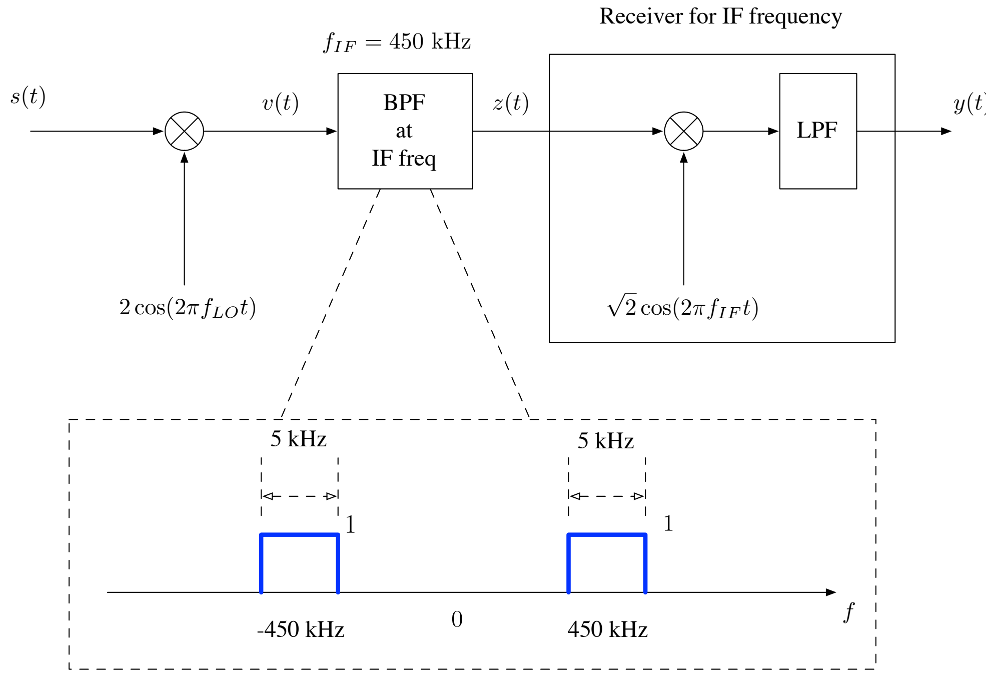

Figure 52: The "super-heterodyne" receiver considered in Problem 4.19.

(c) Determine the value of f_{LO} to be used to tune the receiver to channel 1 and channel 3. Are the values for f_{LO} provided above and in the previous section unique? Explain.

4.20. Consider the system in Fig. 53 where a continuous time signal $x(t) = \cos(10\pi t)$ is the input to "composite system" comprising a hard-limiter and an LTI system with impulse response $h(t)$. The hard-limiter is defined by the input-output relation

$$s(t) = \begin{cases} +1 & x(t) \geq 0 \\ -1 & x(t) < 0 \end{cases}$$

(a) Sketch $s(t)$, the output of the hard-limiter. Is the hard-limiter linear? time-invariant? memoryless? stable?

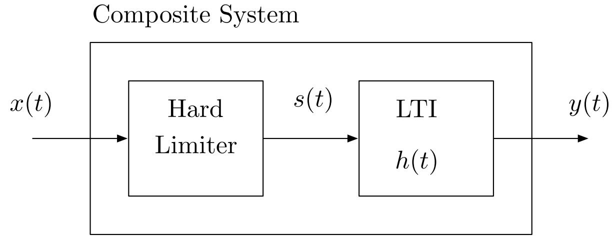

Figure 53: The frequency translation system considered in Problem 4.20.

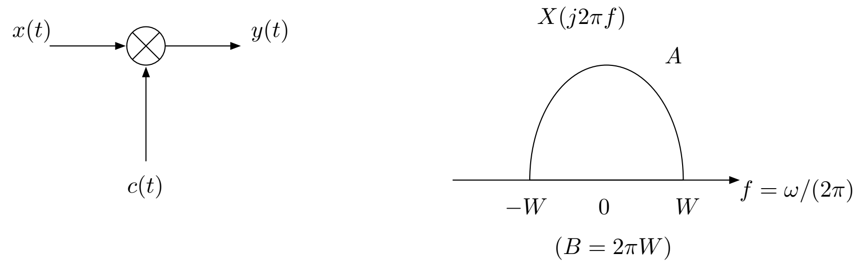

Figure 54: The sampling system considered in Problem 4.20.

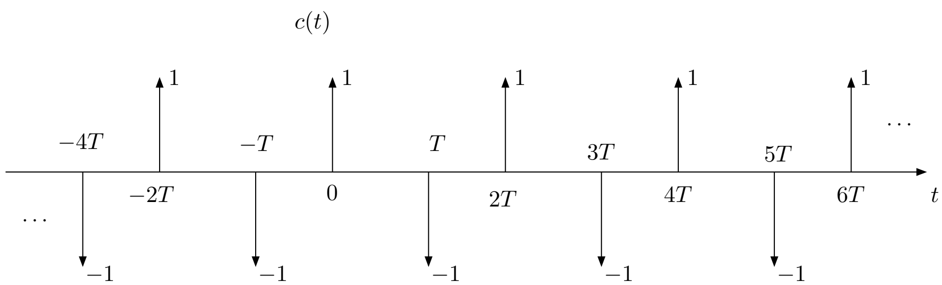

Figure 55: The sampling signal considered in Problem 4.20.

(b) It is desired to generate an output of the form $y(t) = \cos(30\pi t)$ for some amplitude A. Determine and sketch the Fourier transform of $x(t)$, $s(t)$, and $y(t)$.

(c) Specify and sketch a frequency response of the LTI system such that the output will be $y(t) = \cos(30\pi t)$.

Consider the system shown in Fig. 54 (with an example $X(j2\pi f)$ shown). In the following, assume that $1/T > 2W$. The signal $c(t)$ is shown in Fig. 55.

(a) Find and sketch the Fourier Transform of $c(t)$.

(b) Find and sketch the Fourier Transform of $y(t)$ in terms of the Fourier transform of $x(t)$.

(c) Consider the discrete time signal $z[n]$ that defines $y(t)$ in the sense that

$$y(t) = \sum_{n=-\infty}^{\infty} z[n]\delta(t - nT)$$

Determine and sketch the DTFT of $z[n]$ in terms of $X(j\omega)$.
If we define the discrete time signal $v[n] = x(nT)$, what is the relationship between $v[n]$ and $z[n]$? Express $z[n]$ in terms of $v[n]$.

4.21. Let $x(t) = \text{sinc}(t)$ and $c(t)$ as shown in Fig. 56. Consider $z(t) = x(t)c(t)$.

(a) Determine the Fourier Transform of $z(t)$ and sketch and label this FT.

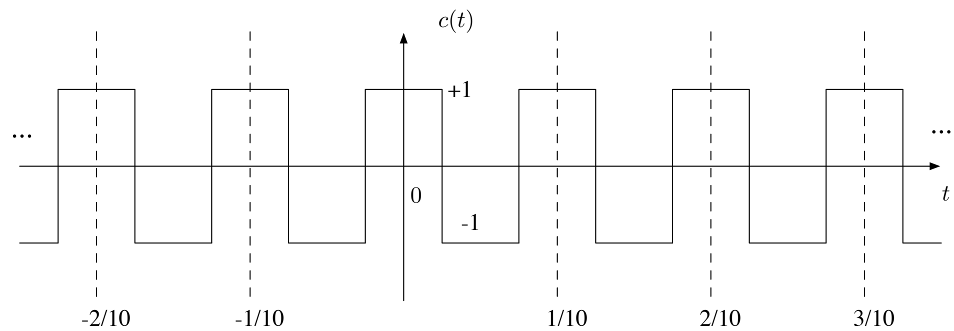

Figure 56: The square wave carrier considered in Problem 4.21.

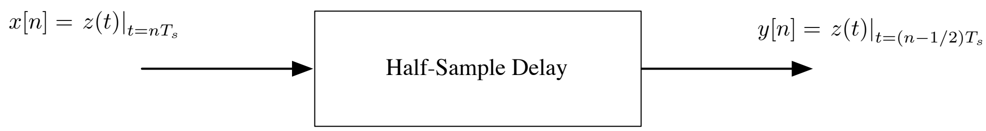

Figure 57: The half-sample delay considered in Problem 4.22.

(b) Let $y(t) = z(t)c(t)$ – determine this signal.

4.22. Consider delaying a discrete time signal by one half sample time. Specifically, consider the case when the discrete time signal $x[n]$ has been obtained by sampling the continuous time $x(t)$ without aliasing and the output of a half-sample delay is a discrete time signal $y[n]$ that is equivalent to sampling the same continuous time signal $x(t)$, delayed by one half sample time. This is illustrated in Fig. 57. Note that the signal "$x[n-1/2]$" is not defined, so it is non-trivial to implement the half-sample delay.

(a) Derive the details of the system that maps input $x[n]$ to output $y[n]$ as described above. Specifically, give a simple expression for $y[n]$ in terms of $x[n]$. Is this system casual? Stable? Memoryless? Linear? Time-Invariant?

(b) Discuss how you would implement or approximately implement the above system using only adders, multipliers and delays. Provide a block diagram for your proposed implementation.

4.23. Consider the digital communication system shown in Fig. 58. Here, each $a[n]$ is equal to either $+1$ or -1, depending on the n^{th} data bit to be sent and $p(t)$ is the pulse shaping filter.

(a) Give a simplified expression for $x(t)$. Considering the pulse $p(t)$ shown in Fig. 59, sketch $x(t)$ for $t \in [0, 5T]$ given the $p(t)$ and the data sequence $a[0] = +1$, $a[1] = -1$, $a[2] = -1$, $a[3] = +1$, $a[4] = +1$, $a[5] = -1$.

(b) The discrete time signal $z[n]$ can be written at $z[n] = a[n] * g[n]$ – find $g[n]$ in terms of $p(t)$. Sketch $g[n]$ for the pulse $p(t)$ in Fig. 59. Sketch $g[n]$ for the pulse $p(t)$ in Fig. 60.

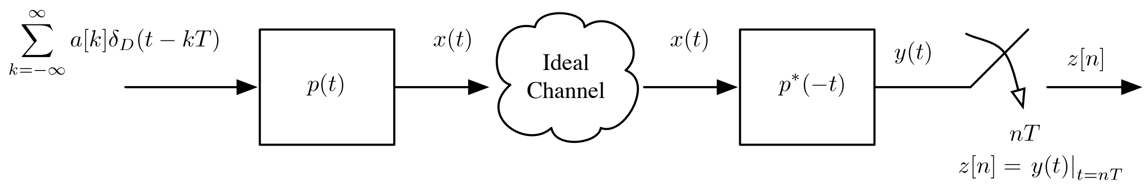

Figure 58: The system considered in Problem 4.23.

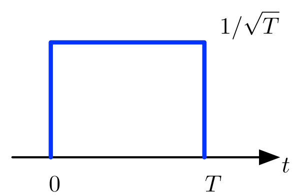

Figure 59: A duration T pulse considered in Problem 4.23.

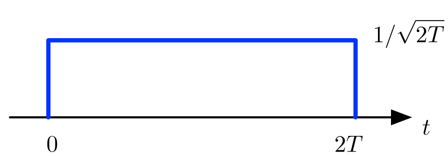

Figure 60: A duration $2T$ pulse $p(t)$ considered in Problem 4.23.

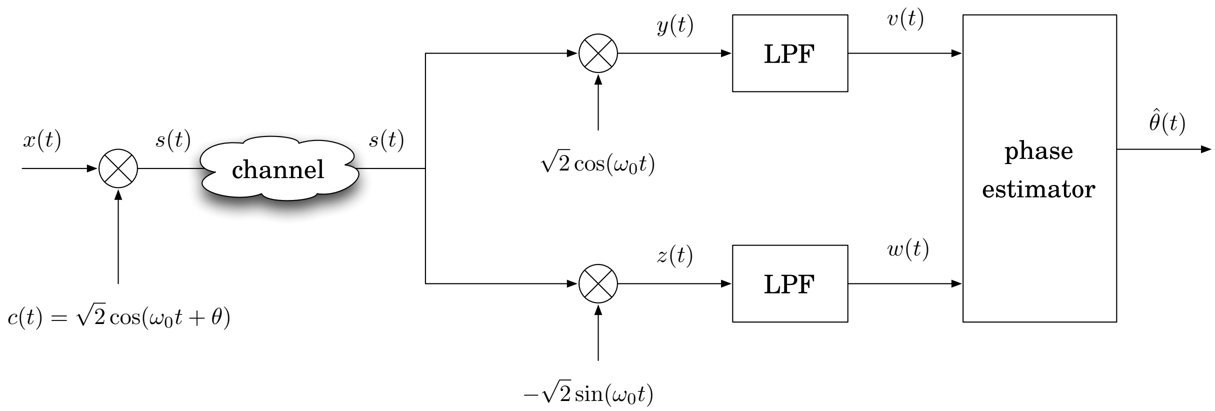

Figure 61: Block diagram for Problem 4.24.

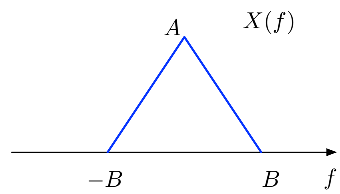

Figure 62: Example spectrum $X(f)$ to use for Problem 4.24.

(c) The pulse $p(t)$ is called a Nyquist pulse if $z[n] = a[n]$. In order for this to hold, conditions on $g[n]$ and $p[n]$ and $P(f)$ must hold. Specify those conditions.

(d) Explain why any pulse that lasts T seconds or less will be a Nyquist pulse. Also consider the following:

i. Is the pulse in Fig. 60 Nyquist?

ii. Can you find a Nyquist pulse with duration more than T seconds?

iii. Can you find a Nyquist pulse with duration more than T seconds that has $P(f) \neq 0$ for some values of $|f| > 1/(2T)$?

4.24. Consider the communication system shown in Fig. 61.

In this system, the information signal $x(t)$ is amplitude modulated by the carrier with some phase θ to produce $s(t)$. This passes through a channel to the receiver (modeled as an ideal channel). At the receiver, an In-phase and Quadrature mixer is used with zero phase – *i.e.,* not phase synched to the transmitter (also known as a freely oscillating I/Q mixer). This output is then low-pass filtered to reject double-frequency terms. The two outputs of this I/Q receiver are inputs to a box used to estimate the carrier phase θ. Specifically, this box is a memoryless system that maps $v(t)$ and $w(t)$ to an estimate of θ – this is $\hat{\theta}(t)$, an estimate of θ at time t.

Here, you can assume that the FT of $x(t)$ looks like:

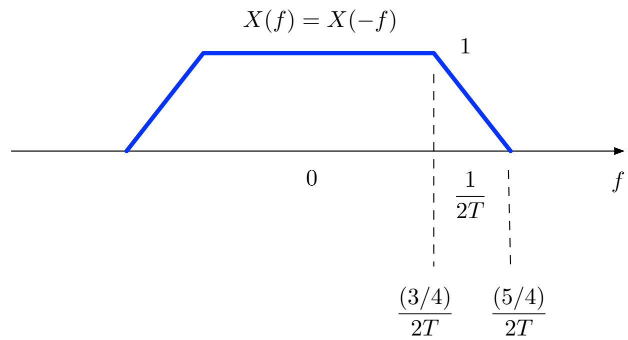

Figure 63: The frequency domain signal $X(f)$ in Problem 4.26.

Assume that $f_0 = \frac{\omega_0}{2\pi} >> B$. You can give your answers in terms of linear or angular frequency.

(a) Provide a labeled sketch of the Fourier transform of $c(t)$ and $s(t)$

(b) Provide a labeled sketch of the Fourier transform of $y(t)$ and $z(t)$.

(c) Provide a labeled sketch of the Fourier transform of $v(t)$ and $w(t)$. Express $v(t)$ and $w(t)$ in terms of $x(t)$ and θ.

(d) Determine the processing that defines a good phase estimator – *i.e.,* the define the memoryless system labeled phase estimator in the block diagram. Is this system stable? Linear?

4.25. (a) $z(t) = \text{rect}(2t) * \cos(2\pi(10)t)$. Find and sketch $Z(f)$.

(b) $z(t) = \text{rect}(2t)\cos(2\pi(10)t)$. Find and sketch $Z(f)$.

4.26. Consider the 60 Hz cosine wave $x(t) = \cos(2\pi 60t)$. This signal is (ideal impulse) sampled at a sampling frequency of 100 Hz.

(a) Sketch the Fourier Transform of the sampled signal.

(b) If this sampled signal is low-pass filtered on $[-1/(2T_s), +1/(2T_s)]$ what will be the output signal?

(c) If the sampled signal is band-passed filtered to produce $A\cos(2\pi f_0 t)$, what values of f_0 are possible?

4.27. Consider the frequency domain signal $X(f)$ shown in Fig. 63.

(a) Circle any of the statements below that are true:

- $x(t) > 0$ for all t.
- $x(t)$ takes on non-real values.
- $x(t)$ is real and even.
- $x(t)$ is real and odd.
- $x(t)$ is imaginary and Hermitian anti-symmtric.

(b) The signal $x(t)$ is sampled with sampling interval T using ideal impulse sampling to produce

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Provide a labeled sketch of $X_s(f)$.
- Provide a simplified expression for $X_s(f)$.
- Provide a simplied expression for $x_s(t)$ and sketch $x_s(t)$.

(c) Find an expression for $x(t)$ and simplify. Sketch the discrete time signal $y[n] = x(nT)$.

5 Discrete Time Fourier Analysis

5.1. Consider the following sum

$$\sum_{m=0}^{\infty} 2^{-m} \cos\left(\frac{\pi}{3}(n - m)\right)$$

This sum is equal to

$$A \cos(\Omega_0 n + \theta)$$

where $A > 0$, Ω_0, and θ are real numbers. Determine these values.

5.2. Using Fourier properties, evaluate the following integral:

$$\int_{-\pi}^{\pi} \frac{d\theta}{5/4 + \cos\theta}$$

5.3. Use Fourier properties to evaluate the following integral

$$\int_{-\pi}^{+\pi} \left(\frac{\sin(5\theta/2)}{\sin(\theta/2)}\right)^2 d\theta$$

5.4. Let $x[n]$ and $X(e^{j\Omega})$ be a DTFT pair. Determine the inverse DTFT of $\frac{d}{d\Omega}X(e^{j\Omega})$.

5.5. This problem deals with the discrete time LTI system governed by the difference equation

$$y[n] = \frac{1}{3}y[n-1] + x[n]$$

where $x[n]$ is the system input and $y[n]$ is the system output.

(a) Determine the impulse response and frequency response for this system. Is this system stable? Is it causal? Explain your reasoning.

(b) Determine and sketch the output of this system for the input $x[n] = \delta[n] - \delta[n-1]$.

(c) Determine and sketch the output of this system for the input $x[n] = \cos\left(\frac{\pi}{4}n\right)$.

5.6. Let $x[n] = 2^{-n}(\mathrm{u}[n] - \mathrm{u}[n-8])$. Find the DTFT $X(e^{j\Omega})$, the $N = 8$-point DFT $X_8[k]$, and the $N = 64$-point DFT $X_{64}[k]$ for this signal.

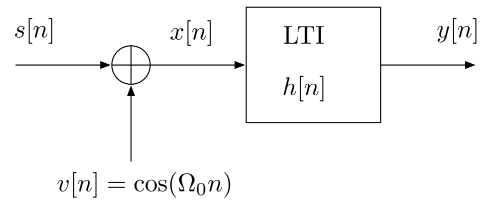

Figure 64: The tone interference system in Problem 5.11.

5.7. Consider a discrete time LTI system with frequency response given by

$$H(e^{j\Omega}) = \frac{2 - 3e^{-j\Omega} + 4e^{-j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega}}$$

Find the difference equation that governs this system and provide a block diagram for implementing the system using the fewest number of delay elements possible.

5.8. A causal, stable (discrete time) LTI system has frequency response

$$H(e^{j\Omega}) = \frac{4e^{j\Omega} - 8}{4e^{j\Omega} - 1}$$

Find the impulse response for this system and the difference equation that governs the relationship between input $x[n]$ and output $y[n]$.

5.9. Consider a moving average with input $x[n]$ and output $y[n]$ of the norm

$$y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3}$$

If the input to this system is $\sin\left(\frac{\pi}{4}n\right)$, what is the output?

5.10. Consider an LTI system with impulse response $h[n] = \text{sinc}^2(n/2) - \delta[n]$.

(a) Find and sketch the frequency response for this system.

(b) Find the output $y[n]$ given that the input is $\cos(\Omega_0 n)$ for the following values of Ω_0: 0, $\pi/4$, $\pi/2$, $10\pi/4$.

5.11. Consider the system in Fig. 64 where a signal of interest $s[n]$ is corrupted by an undesired tone interferer $v[n]$. The goal of this problem is to design a simple LTI filter $h[n]$ in an effort to eliminate the tone interferer. Specifically, the LTI system considered is governed by

$$y[n] = \alpha x[n] + x[n-1] + \alpha x[n-2]$$

where $x[n]$ is the system input, $y[n]$ is the system output, and α is a finite real number to be designed.

(a) Determine the frequency response and impulse response for the LTI system as a function of α. Is this system stable? Is it causal?

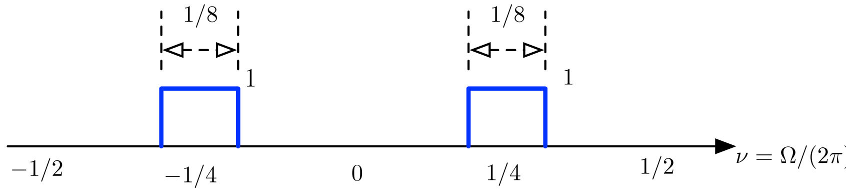

Figure 65: The desired band-pass filter characteristic in Problem 5.12.

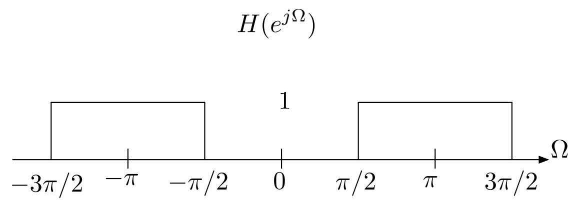

Figure 66: The desired high-pass filter characteristic in Problem 5.13.

(b) With the knowledge that the interferer is $v[n] = \cos(\Omega_0 n)$ for some known Ω_0, design the LTI system so that $h[n] * v[n] = 0$. Specifically, what value of α will achieve this perfect nulling of the tone interferer? Can perfect nulling be achieved for all Ω_0? Explain.

(c) Consider the specific example where $\Omega_0 = \pi/4$ and specify the choice of α that achieves perfect nulling. For this specific choice of α, sketch $h[n]$ and $|H(e^{j\Omega})|$.

(d) Again, for the special case of $v[n] = \cos((\pi/4)n)$, consider the case when the signal of interest is $s[n] = 10\sin((\pi/3)n)$. Determine the output of the LTI system, $y[n]$ for this case.

5.12. Design an FIR filter that approximates the filter with frequency response shown in Fig. 65. Specifically, find a causal, 7-tap filter with magnitude $|H(e^{j\Omega})|$ that well approximates the above frequency response.

5.13. It is desired to implement a discrete time high-pass filter with frequency response shown in Fig. 66. An approximation to this is sought using a FIR filter having impulse response with $h[n] = 0$ for all n with $|n| > 5$. What are good values for $h[n]$ for $|n| \leq 5$?

5.14. A first-order difference can be used to approximate a continuous-time differentiation operation in discrete time. Specifically, let $d[n]$ be the first-order difference of the input signal $x[n]$, defined by

$$d[n] = x[n] - x[n-1]$$

(a) The mapping from $x[n]$ to $d[n]$ is an LTI system.

i. Let $h[n]$ be the impulse response of this system, find and sketch $h[n]$.

ii. Find and sketch the magnitude of the DTFT of $h[n]$.

iii. Consider the input

$$x[n] = \begin{cases} 1 & n = 0, 1, 2, \dots 8 \\ 0 & \text{otherwise} \end{cases}$$

Sketch the output $d[n]$ for this input.

(b) Consider a second order difference given by

$$y[n] = (x[n] - x[n-1]) - (x[n-1] - x[n-2]) = x[n] - 2x[n-1] + x[n-2]$$

The mapping from $x[n]$ to $y[n]$ is an LTI system.

i. Let $g[n]$ be the impulse response of this system, find and sketch $g[n]$.

ii. Find and sketch the magnitude of the DTFT of $g[n]$.

iii. Consider the input

$$x[n] = \begin{cases} 1 & n = 0, 1, 2, \dots 8 \\ 0 & \text{otherwise} \end{cases}$$

5.15. Consider $g[n] = a^n$ for $n = 0, 1, \dots N-1$ and zero otherwise. Find a closed form expression for the DFT of $g[n]$

5.16. Which of the following best gauges the computational complexity of the FFT relative to that of the corresponding direct N-point DFT computation?

(a) N

(b) $\log_2(N)/N$

(c) $2/N$

(d) $\log_2(N)$

5.17. Consider an LTI system with impulse response

$$\begin{aligned} h[n] =& \frac{1}{2}\delta[n] + \frac{1}{\pi}(\delta[n-1] + \delta[n+1]) - \frac{1}{3\pi}(\delta[n-3] + \delta[n+3]) \\ &+ \frac{1}{5\pi}(\delta[n-5] + \delta[n+5]) - \frac{1}{7\pi}(\delta[n-7] + \delta[n+7]) \end{aligned}$$

(a) Sketch the impulse response.

(b) Give a difference equation that characterizes this system in terms of an input signal $x[n]$ and $y[n]$.

(c) Explain why the DTFT of $h[n]$, $H(e^{j\Omega})$, will be real and even.

(d) Determine and plot the frequency response of this system. What can this system be used for?

(e) If the input is $x[n] = \cos\left(\frac{\pi}{4}n\right)$, determine the output $y[n]$.

(f) If the input is $x[n] = \cos\left(\frac{3\pi}{4}n\right)$, determine the output $y[n]$.

(g) Consider an LTI system with impulse response $h_c[n] = h[n-7]$, where $h[n]$ is as given above (*i.e.*, $h_c[n]$ is causal). What is $H_c(e^{j\Omega})$? Repeat the previous two parts using $h_c[n]$ in place of $h[n]$.

(h) Filter the data file using this LTI system and plot the original data ($x[n]$) and the filtered data ($y[n]$).

5.18. Consider a length $N = 4$ sequence $x[n]$ – *i.e.,* the signal comprises the four numbers $x[0]$, $x[1]$, $x[2]$, $x[3]$. Suppose you are given the following quantities:

$$E[0] = x[0] + x[2]$$
$$E[1] = x[0] - x[2]$$
$$O[0] = x[1] + x[3]$$
$$O[1] = x[1] - x[3]$$

Find the 4-point DFT of $x[n]$ in terms of these four numbers.

5.19. Consider a discrete time signal $x[n]$ that lasts only 2 samples with $x[0] = 7$ and $x[1] = 2$. Find the Discrete Fourier Transform (DFT) of $x[n]$ using the same convention used in Matlab.

5.20. Consider computing an $n = 512$ point Discrete Fourier Transform (DFT). Roughly how much more complex is it to compute the DFT directly instead of using the Fast Fourier Transform (FFT)?

5.21. Consider the LTI system with impulse response

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

Consider the $N = 32$ point DFT of this signal, $H[k]$. Find a closed form expression for $|H[k]|^2$ and provide numerical values for this quantity in dB for: $|H[0]|^2_{dB}$, $|H[12]|^2_{dB}$, and $|H[22]|^2_{dB}$.

5.22. Consider the signal $x[n] = -4\cos\left(2\pi\frac{7}{32}n\right)$. Consider the N-point DFT for $N = 4$, 16, 32, 64, and 1024. For which of these values of N will the DFT $\{X[k]\}_{k=0}^{N-1}$ comprise exactly 2 non-zero values? For those cases where this holds, specify the two values of $k \in \{0, 1. \dots N-1\}$ for which $X[k] \neq 0$.

5.23. Consider the signal $x[n] = -4\cos\left(\frac{42}{32}n\right)$. Consider the N-point DFT for $N = 4$, 16, 32, 64, and 1024. For which of these values of N will the DFT $\{X[k]\}_{k=0}^{N-1}$ comprise exactly 2 non-zero values? For those cases where this holds, specify the two values of $k \in \{0, 1. \dots N-1\}$ for which $X[k] \neq 0$.

5.24. Consider the stable LTI system with impulse response $h[n] = a^n \mathrm{u}[n]$. Provide the frequency response of this system $H(e^{j\Omega})$. Consider truncating this impulse response to N samples to form $g[n]$. Specifically, $g[n] = h[n]$ for $n = 0, 1, \dots N-1$ an zero otherwise. Find a closed form expression for the DFT of $g[n]$ Describe, quantitatively, how $|H(e^{j\Omega})|^2$ and $|G[k]|^2$ are related and how this difference will manifest in a plot of these quantities in dB.

5.25. Use a computer to compute the DFT of the following signals and plot $|X[k]|$ versus k. Do this for $N = 2, 4, 8, 16, 256$.

(a) $x[n] = \cos\left(2\pi\frac{3}{8}n\right)$

(b) $x[n] = \cos\left(2\pi\frac{1}{e}n\right)$

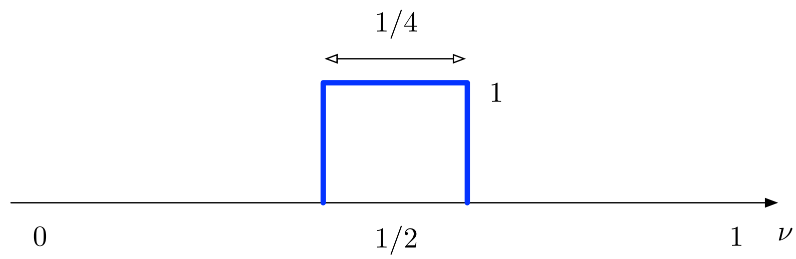

Figure 67: The frequency response for Problem 5.28

5.26. Consider the signal $h[n] = \delta[n] + 0.5\delta[n-1]$.

(a) Determine the DTFT $|H(e^{j\Omega})|^2$.

(b) Using a computer (e.g., Matlab), find the $N = 2, 4, 16$ point DFT of $h[n]$ – *i.e.,* $H_N[k]$. In computing this, you should assume that $h[n] = 0$ for $n = 2, 3, \ldots N-1$.

(c) Plot, on the same graph $10\log_{10}\left[|H(e^{j\Omega})|^2\right]$ and $10\log_{10}\left[|H_N[k]|^2\right]$ as a function of Ω. To plot $|H_N[k]|^2$, convert k to angular frequency via $\Omega_k = 2\pi k/N$.

(d) Repeat the previous parts of this problem for the signal $h[n] = (0.95)^n \mathrm{u}[n]$ – except plot the results for $N = 4, 16, 128$. Note that since $h[n]$ is infinite length, to compute the N-point DFT, truncate the signal to zero outside $n = 0, 1, \ldots N-1$. Discuss you results – *i.e.,* how are the DTFT and the DFT related in this case?

5.27. Consider computing an $N = 2^{16}$ point DFT. What is the computational complexity of using the FFT for this computation for this task relative to the complexity of direct DFT computation?

$$\frac{\text{FFT Computational Complexity}}{\text{DFT Computational Complexity}} =$$

5.28. A discrete time LTI system has frequency response as shown in Fig. 67 (plotted against normalized, linear frequency).

How would you characterize this system?

(a) Low-pass filter

(b) Band-pass filter

(c) High-pass filter

If the input to this system is $x[n] = \cos^2(2\pi(0.2)n)$, what is $y[n]$

5.29. A continuous time audio signal $x(t)$ is sampled at 44 kHz to produce a discrete time signal $y[n]$. It is desired to perform a bandpass filter (BPF) operation centered around 11 kHz in the original signal. What should be the center frequency in a discrete-time BPF to accomplish this?

(a) $\Omega_0 = \pi/4$ rad/sample

(b) $\nu_0 = \pi/2$ sample^{-1}

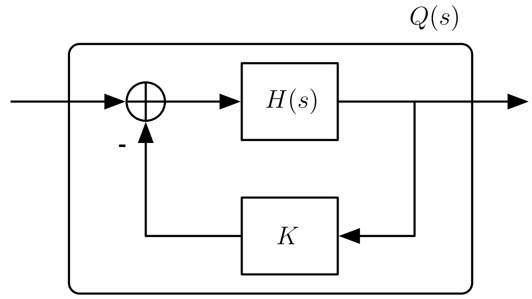

Figure 68: The proportional feedback controller in Problem 6.1.

(c) $\nu_0 = 1/4$ sample^{-1}

(d) $\nu_0 = 4$ sample^{-1}

6 Laplace and Z Transforms, Feedback Control

6.1. Consider a causal, LTI system for which the Laplace transform of the impulse response is

$$H(s) = \frac{1}{(s-3)(s+4)}$$

(a) Plot in the s-plane, the location of the poles and zeros of $H(s)$ and indicate the region of converge (ROC) for this transform. s this system stable? Does the frequency response of this system exist? Explain your reasoning.

(b) A simple proportional feedback controller is suggested for this system, as diagramed in Fig. 68. Determine the closed-loop response for this system $Q(s)$ and the poles of $Q(s)$.

(c) Determine the range of values of K for which the closed-loop system is stable. It is desired that the closed-loop system be stable and exhibit no oscillation in its impulse response. Determine the range of values of K for which this condition is met.

6.2. The impulse response of a stable, LTI system, $h(t)$, has a Laplace transform with poles and as diagramed in Fig. 69 in the s-plane. Reproduce this and show the region of convergence for this Laplace transform. Is $h(t)$ right-sided, left-sided, two-sided, or is there not enough information provided to specify? Is $h(t)$ purely imaginary, does it have nonzero real and imaginary parts, or is there not enough information provided to answer?

6.3. Consider a real, second-order (continuous time) causal system with impulse response $h(t)$. Fig. 70 shows four possible pole-plots in the s-plane for $H(s)$ and, to the right, four possible impulse response functions. Describe the best correspondence between the pole-plots on the left with $h(t)$ on the right by connecting pairs with lines.

6.4. Consider a standard RC circuit with input source voltage $x(t)$ and output voltage $y(t)$ measured across the capacitor with a resistance of 1 kilo-Ohm and a capacitance of 25 micro-Farad as shown in Fig. 71. What is the transfer function for this system? For the input $x(t) = \cos(4t)$, determine the output and the ratio of the output power to input power in dB.

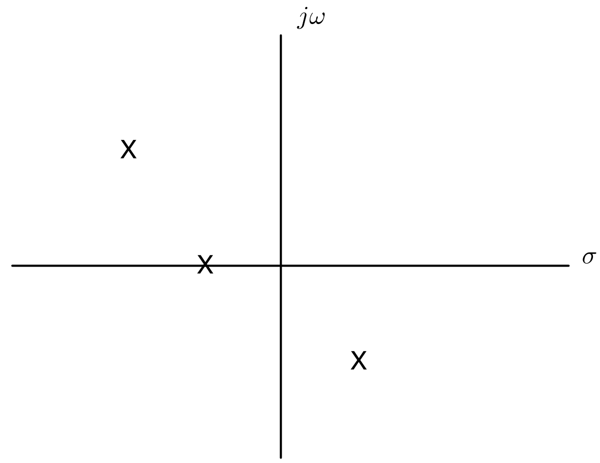

Figure 69: The poles for the Laplace transform of $h(t)$ in Problem 6.2.

6.5. Consider the causal LTI system with transfer function

$$H(s) = \frac{1}{s-2}$$

which is to be controlled using feedback control with controller transfer function $A + (B/s)$, where A and B are constants. This is illustrated in Fig. 72.

(a) Give a simplified expression for the closed-loop system response $Q(s)$. What are the poles and zeros of $Q(s)$?

(b) What is the range for the parameters A and B for which the closed loop system is stable? What is the range for the parameters A and B for which the closed loop system is stable and exhibits no oscillatory behavior in its impulse response?

(c) Consider the case of $A = 6$ and sketch the root-locus diagram for the closed loop system as the parameter B increases from $-\infty$ to $+\infty$.

(d) It is desired to simulate the closed loop system using a circuit of operational amplifiers with each configured as a adder, multiplier, or integrator. Provider a block diagram for implementing $Q(s)$ using these 3 types of blocks where the minimum number of integrators is used.

6.6. Consider the system implemented as illustrated in Fig. 73. Here the subsystem labeled I is an integrator – *i.e.,* if the input is $v(t)$, the output is

$$w(t) = \int_{-\infty}^{t} v(\tau) d\tau$$

(a) Is this system linear? Time-Invariant? Causal? Provide the differential equation that relates the input $x(t)$ and output $y(t)$ and the system response $H(s)$.

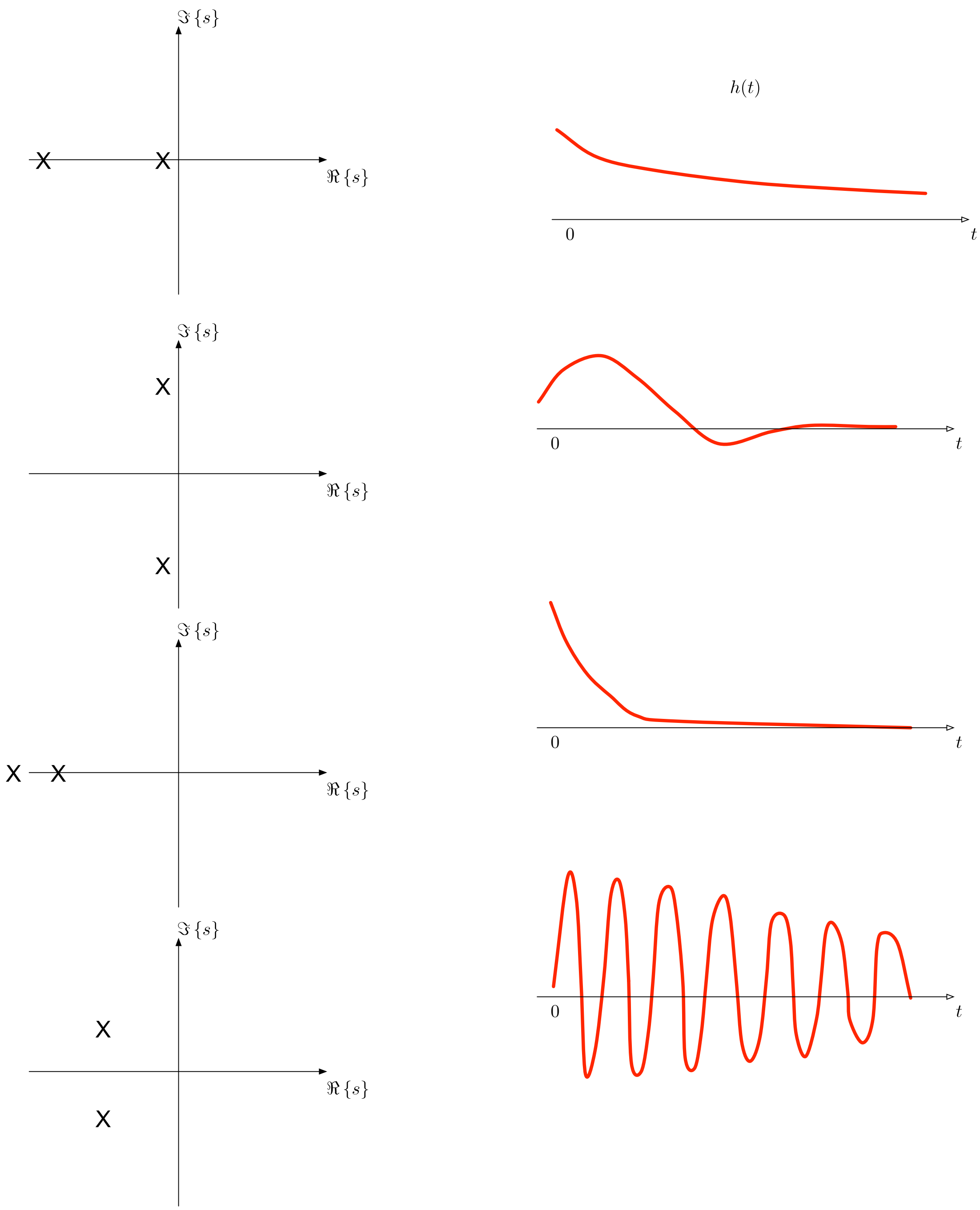

Figure 70: Determine the correspondence between the pole plot and the impulse response in Problem 6.3.

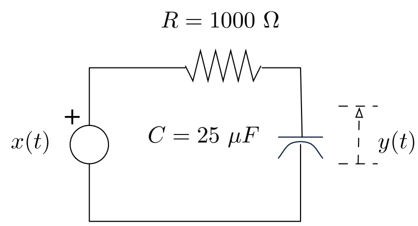

Figure 71: TheRC circuit considered in Problem 6.4.

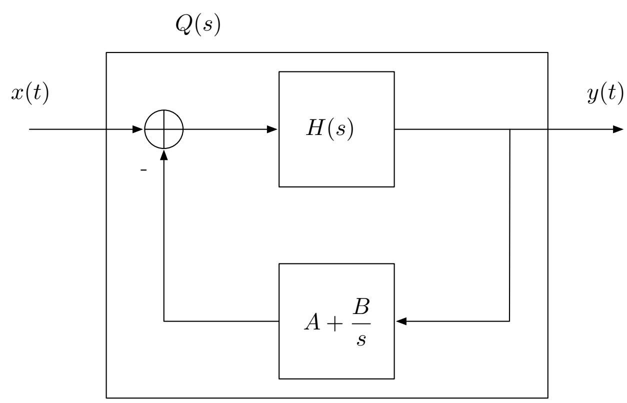

Figure 72: The proportional-integral control system in Problem 6.5.

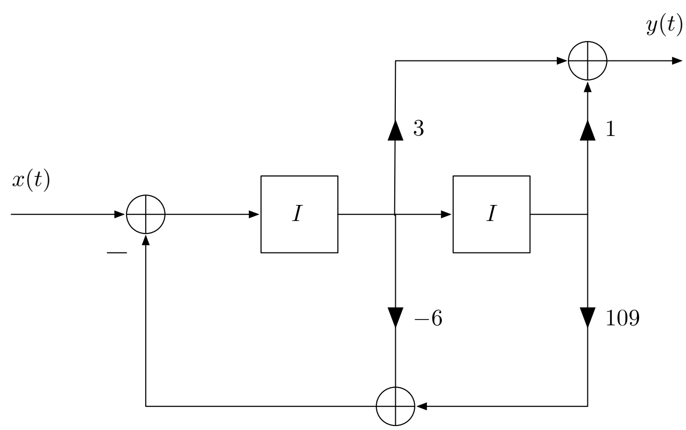

Figure 73: A block diagram implementation of the system in Problem 6.6.

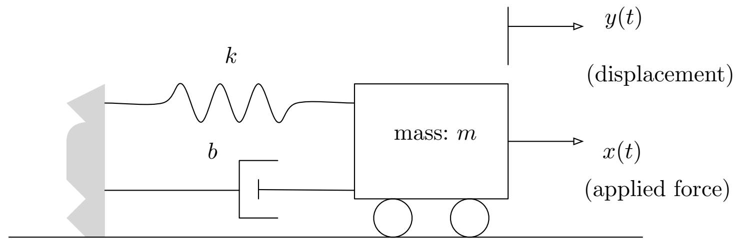

Figure 74: The spring-dampner system in Problem 6.7.

(b) What are the poles and zeros of $H(s)$? Draw a pole-zero diagram in the s-plane and show the region of convergence for $H(s)$ on the same diagram. Make sure to label your diagram. Is this system stable?

6.7. Consider the mechanical system shown in Fig. 74. This system is a simple model for an automobile suspension. It includes a spring, with spring coefficient k, and a viscous dampener, with parameter b. The force exerted by the spring is k times the amount it has been stretched while the dampener exerts a force against movement with magnitude b times the speed of movement. Simple application of Newton's second law of motion (*i.e.,* $F = ma$) implies that

$$x(t) - ky(t) - by'(t) = my''(t)$$

Note the b, m, and k are all positive constants.

Instead of building and testing a mechanical system, an electrical system is to be used to simulate the mechanical system.

(a) Give an expression for the Laplace transform of the impulse response for this system and find all poles. For what values of m, b, and k is this system stable?

(b) Provide a block diagram for simulating this system using only multipliers, adders, and integrators.

(c) Find the impulse response for this system when $m = 1$, $k = 4$, and $b = 8$. It is desired that the automobile not bounce up and down when responding to an abrupt bump. What condition of m, b, and k will ensure this characteristic? If the above condition is not met, what will be the frequency of the automobile bouncing?

6.8. A discrete time, causal infinite-length impulse response (IIR) LTI system that is initially at rest is stable if

(a) All the zeros are outside the unit circle

(b) All the poles are outside the unit circle

(c) All the poles are inside the unit circle

(d) The same number of poles are inside as outside the unit circle

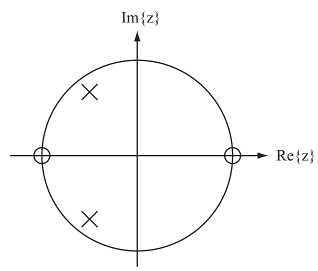

Figure 75: The pole-zero diagram in Problem 6.10.

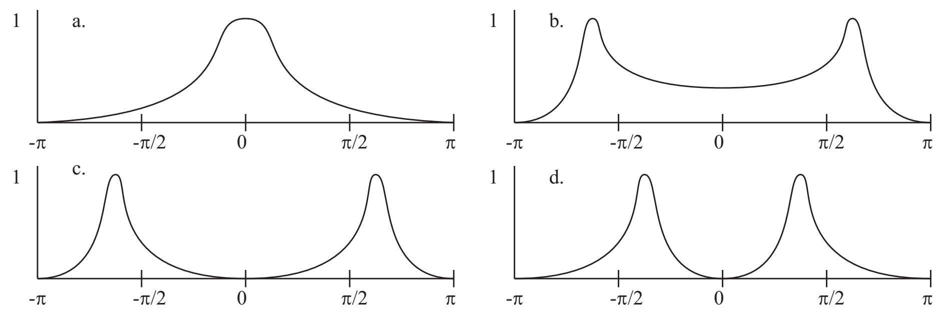

Figure 76: Possible frequency responses for Problem 6.10.

6.9. Let $h[n] = (-1/3)^n \mathrm{u}[n]$. Find the Z-transform $H(z)$, including the region of convergence. Sketch the poles in the Z-plane (label the location and include the unit circle). Find and sketch the magnitude of $H(e^{j\Omega})$.

6.10. For the pole-zero diagram shown in Fig. 75, which of the curves in Fig. 76 best matches the magnitude of the frequency response the system will have?

6.11. Consider a stable discrete time LTI system with Z-transform having the pole-zero plot as shown in Fig. 77. Which of the plots in Fig. 78 best matches the magnitude of the frequency response for this system?

6.12. Consider the system implemented as illustrated in Fig. 79.

(a) Is this system linear? Causal? Time-Invariant? Provide the difference equation that relates the input $x[n]$ and output $y[n]$ and the system response $H(z)$.

(b) What are the poles and zeros of $H(z)$? Draw a pole-zero diagram in the z-plane and show the region of convergence for $H(z)$ on the same diagram. Make sure to include the unit circle and label your diagram. Is this system stable?

6.13. Find the Z transform of the impulse response

$$h[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{3}n\right)$$

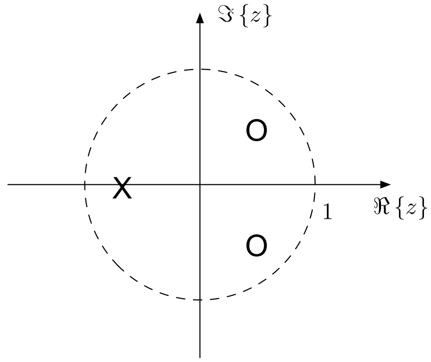

Figure 77: The pole-zero diagram in Problem 6.11.

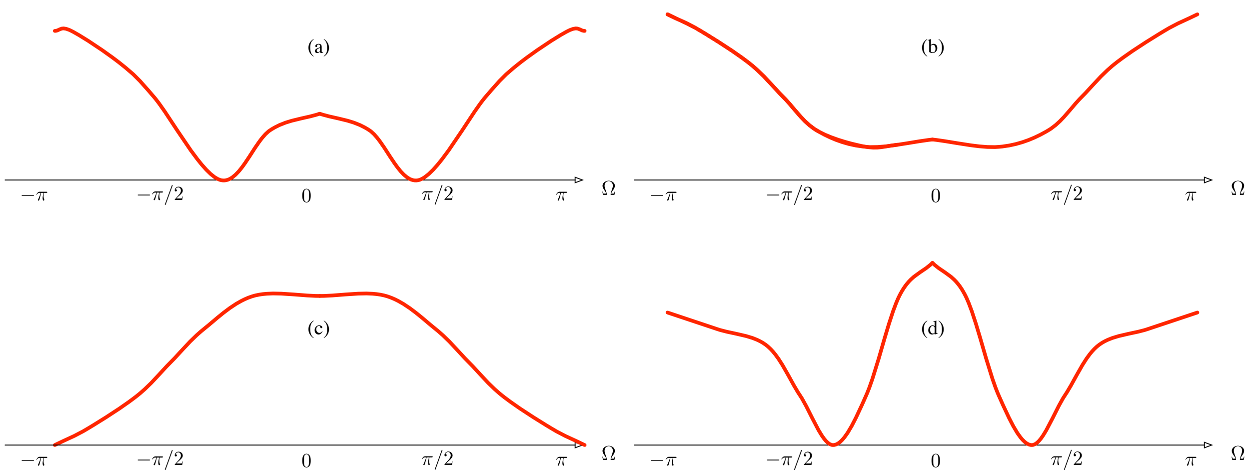

Figure 78: Possible frequency responses for Problem 6.11.

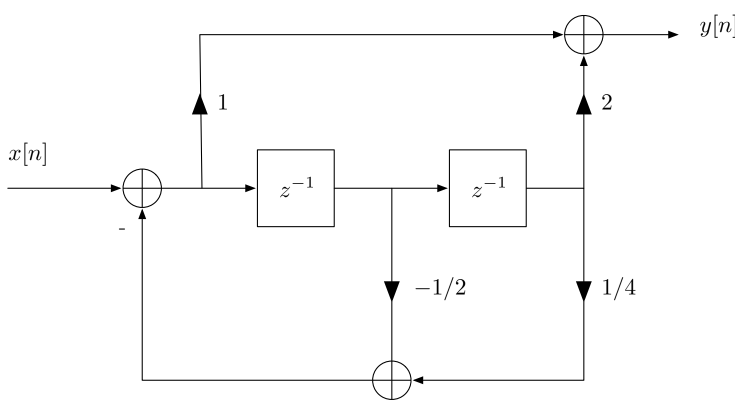

Figure 79: A block diagram implementation of the system in Problem 6.12.

Show the location of the poles of $H(z)$ in the Z-plane.

6.14. Consider a stable, discrete time LTI system with system response

$$H(z) = \frac{z^2}{z+3}$$

Sketch the ROC and indicate the location of all poles and zeros. Determine the properties that are true for $h[n]$:

(a) Is $h[n]$ right-sided, left-sided, two-sided, or is there not enough information provided to determine?

(b) Is $h[n]$ purely imaginary, complex (non-zero real and imaginary parts), or is there not enough information provided to determine?

(c) Is $h[n]$ causal, anti-causal, neither causal nor anti-causal, or is there not enough information provided to determine?

(d) Is $h[0]$ non-zero, zero, or s there not enough information provided to determine?

6.15. Consider a stable, discrete time LTI system with system response

$$H(z) = \frac{z^2}{z-(1/3)}$$

Sketch the ROC and indicate the location of all poles and zeros.

Determine the properties that are true for $h[n]$:

(a) Is $h[n]$ right-sided, left-sided, two-sided, or is there not enough information provided to determine?

(b) Is $h[n]$ purely imaginary, complex (non-zero real and imaginary parts), or is there not enough information provided to determine?

(c) Is $h[n]$ causal, anti-causal, neither causal nor anti-causal, or is there not enough information provided to determine?

(d) Is $h[0]$ non-zero, zero, or s there not enough information provided to determine?

6.16. Consider the causal LTI system with transfer function

$$H(z) = \frac{1}{1+2z^{-1}+4z^{-2}}$$

which is to be controlled using feedback control with controller transfer function $G(z) = 2Kz^{-1}$, where K is a constant. This is illustrated in Fig. 80.

(a) Give a simplified expression for the closed-loop system response $Q(z)$. What are the poles and zeros of $Q(z)$?

(b) What is the range for the parameter K for which the closed loop system is stable? What is the range for the parameter K for which the closed loop system is exhibits oscillatory behavior in its impulse response at a frequency other than $\Omega_0 = \pi$?

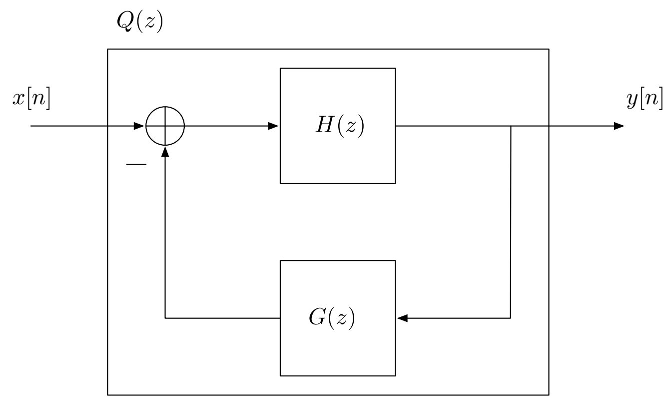

Figure 80: The feedback control system considered in Problem 6.16.

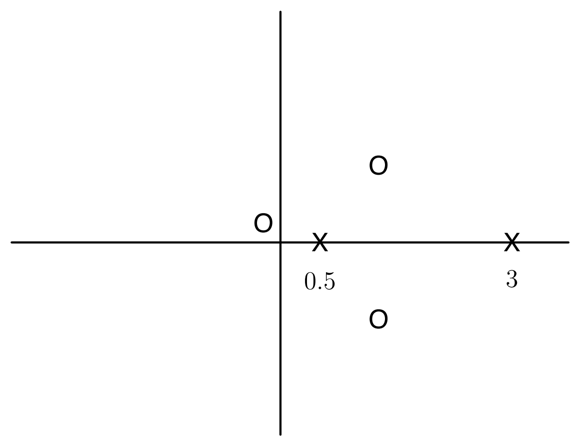

Figure 81: The poles and zeros in the Z-plane for the system in Problem 6.17.

(c) Sketch the root-locus diagram for the closed loop system as the parameter K varies from $-\infty$ to $+\infty$.

6.17. The impulse response of a stable, LTI, discrete time system, $h[n]$, has a Z-transform with poles and zeros as diagramed in the Z-plane in Fig. 81. Reproduce this diagram and add the region of convergence for this Z-transform. Is $h[n]$ right-sided, left-sided, two-sided, or is there not enough information provided to answer this?

6.18. A continuous time, causal LTI system has transfer function

$$H(s) = \frac{5}{s^2 + 2s + 5}$$

Determine/provide/answer the following:

(a) Sketch the ROC for $H(s)$ and show the pole and zero locations.

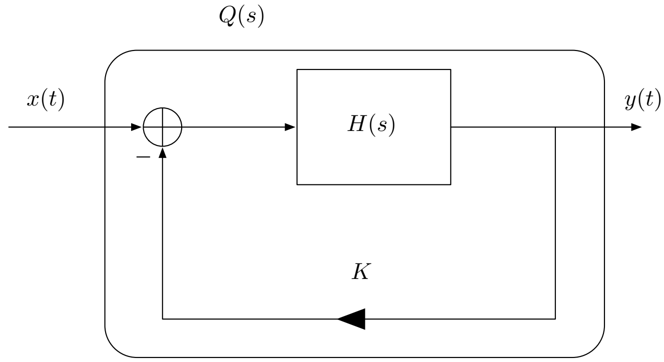

Figure 82: Proportional control feedback system for Problem 6.19.

(b) The differential equation that governs the input $x(t)$ and output $y(t)$ of this system.

(c) A block diagram implementing this system using only integrators, signal adders, and signal gains.

(d) Is this system stable?

(e) What is the output $y(t)$ for when $x(t) = \sin(\sqrt{5}t)$?

(f) Does this system exhibit resonance?

6.19. Consider the causal, LTI system with transfer function

$$H(s) = \frac{3}{s+2} + \frac{2}{s-2}$$

(a) Provide a of the ROC for $H(s)$ indicating the location of all poles and zeros.

(b) A proportional feedback control loop is to be applied to this system as shown in Fig. 82. Determine the closed loop transfer function $Q(s)$. For what values of the real number K is the closed loop system stable? For what values is the closed loop system unstable?