Fourier Series of a Real Signal

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February 24, 2021

For a periodic signal $x(t)$ we have the Fourier Series in terms of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\omega_k t} \tag{1}$$

$$X_k = \frac{1}{T} \int_T x(t) e^{-j\omega_k t} dt \tag{2}$$

$$\omega_k = k\omega_0 = k\left(\frac{2\pi}{T}\right) \tag{3}$$

If $x(t)$ is real-valued, then $x(t)$ also has a Fourier Series in terms of cosines and sines of these harmonic frequencies

$$x(t) = X_0 + \sum_{k=1}^{\infty} \left[X_k^c \cos(\omega_k t) + X_k^s \sin(\omega_k t)\right] \tag{4}$$

where X_0 is the same as in (2).

To see how these are related, let us conjugate (1) and use the fact that $x^*(t) = x(t)$

$$\begin{aligned}
x(t) &= x^*(t) && (5)\\
&= \left(\sum_{k=-\infty}^{\infty} X_k e^{j\omega_k t}\right)^* && (6)\\
&= \sum_{k=-\infty}^{\infty} X_k^* e^{-j\omega_k t} && (7)\\
&= \sum_{m=-\infty}^{\infty} X_{-m}^* e^{+j\omega_m t} && (8)\\
&= \sum_{k=-\infty}^{\infty} X_{-k}^* e^{j\omega_k t} && (9)
\end{aligned}$$

where we have made the change of variable $m = -k$ and then renamed m to k again in (9). Comparing (2) and (9) and realizing that this holds for any real, periodic $x(t)$, we can conclude that

$$x(t) \text{ real} \quad \Longleftrightarrow \quad X_k = X_{-k}^* \tag{10}$$

The result in (10) can be used to simplify (1) to

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\omega_k t} \tag{11}$$
$$= X_0 + \sum_{k=1}^{\infty} X_k e^{j\omega_k t} + X_{-k} e^{-j\omega_k t} \tag{12}$$
$$= X_0 + \sum_{k=1}^{\infty} \Re\left\{X_k e^{j\omega_k t}\right\} \tag{13}$$
$$= X_0 + \sum_{k=1}^{\infty} \left[2\Re\left\{X_k\right\}\cos(\omega_k t) - 2\Im\left\{X_k\right\}\sin(\omega_k t)\right] \tag{14}$$

where we have used the following facts

$$\left(X_k e^{j\omega_k t}\right)^* = \left(X_{-k}^* e^{j\omega_k t}\right)^* \tag{15}$$
$$= X_{-k} e^{-j\omega_k t} \tag{16}$$
$$\Re\left\{zw\right\} = \Re\left\{z\right\}\Re\left\{w\right\} - \Im\left\{z\right\}\Im\left\{w\right\} \tag{17}$$

Comparing (4) and (14) we can conclude that

$$X_k^c = 2\Re\left\{X_k\right\} \tag{18}$$
$$= X^* + X_k^* \tag{19}$$
$$= \frac{1}{T}\int_T x(t)e^{-j\omega_k t}dt + \left[\frac{1}{T}\int_T x(t)e^{-j\omega_k t}dt\right]^* \tag{20}$$
$$= \frac{1}{T}\int_T x(t)\left[e^{-j\omega_k t} + e^{-j\omega_k t}\right]dt \tag{21}$$
$$= \frac{2}{T}\int_T x(t)\cos(\omega_k t)dt \tag{22}$$
$$X_k^s = -2\Im\left\{X_k\right\} \tag{23}$$
$$= \frac{1}{j}(X_k^* - X_k) \tag{24}$$
$$= \frac{1}{j}\left\{\left[\frac{1}{T}\int_T x(t)e^{-j\omega_k t}dt\right]^* - \frac{1}{T}\int_T x(t)e^{-j\omega_k t}dt\right\} \tag{25}$$
$$= \frac{1}{j}\frac{1}{T}\int_T x(t)\left[e^{j\omega_k t} - e^{-j\omega_k t}\right]dt \tag{26}$$
$$= \frac{2}{T}\int_T x(t)\sin(\omega_k t)dt \tag{27}$$

This coefficient expressions can also be found using orthogonality of the cosines and sines at these harmonic frequencies.

In summary we have

$$x(t) = X_0 + \sum_{k=1}^{\infty} \left[X_k^c \cos(\omega_k t) + X_k^s \sin(\omega_k t) \right] \tag{28}$$

$$X_0 = \frac{1}{T} \int_T x(t) dt \tag{29}$$

$$X_k^c = 2\Re\left\{X_k\right\} = \frac{2}{T} \int_T x(t) \cos(\omega_k t) dt \tag{30}$$

$$X_k^s = -2\Im\left\{X_k\right\} = \frac{2}{T} \int_T x(t) \sin(\omega_k t) \tag{31}$$

Therefore, if we have a real periodic signal $x(t)$ we could always use this real FS expansion. This real form is often useful when you would like to plot a truncated FS – *i.e.,* it is often easier to work with sines and cosines than complex exponentials. From this form it is clear that anytime $x(t)$ is real and even, it can be written as a linear combination of cosines. When $x(t)$ is real and odd, it can be written in terms of sines.