Complex Numbers and Functions

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Complex numbers and functions are widely used in systems engineering. While all "real world" signals are real valued, we will see that even real-valued signals have complex transforms when the most common engineering conventions are used. In communication systems design and analysis, real-valued signals centered around some carrier frequency are often modeled in terms of the corresponding complex baseband equivalent signal – *i.e.,* this complex signal is centered at zero frequency, but captures all information about the original signal. Similar phasor and complex impedance conventions are used in analyzing linear circuits. It is possible, in fact, to avoid using complex signals when treating only real-world, real-valued signals, but using complex signals significantly compacts the notation and mathematical expressions and also makes certain symmetries more apparent. Thus, students in this class should be comfortable with the basic concepts of complex numbers and functions as well as the mechanics of their manipulation.

Finally, while notational convenience is a major reason to adopt complex models, this convention is deeply ingrained in the field, both in academic references and engineering practice. For example, most Digital Signal Processing (DSP) chips have native support for complex variables and arithmetic.

1 Basic Definitions and Representations

An imaginary number is a real multiple of $j = \sqrt{-1}$. A complex number is the sum of a real and imaginary number. For example, the *complex number* z may be expressed as

$$z = a + jb \tag{1}$$

where a and b are real numbers. Complex numbers therefore include both (purely) real and (purely) imaginary numbers as a special case (*e.g.,* $b = 0$ and $a = 0$, respectively). It is useful to view a complex number in the complex plane defined by an abscissa ("x-axis") corresponding to the real line and an ordinate ("y-axis") corresponding to the imaginary axis. Thus, the complex number z in (1) is viewed as a point at (a, b) in the complex plane.

The *(complex) conjugate* of a complex number z, denoted z^*, is defined as

$$z^* = a - jb \tag{2}$$

where z is the complex number in (1). Another common notation for the conjugate is $\bar{z}$. Fig. 1 shows how z and z^* are related in the complex plane.

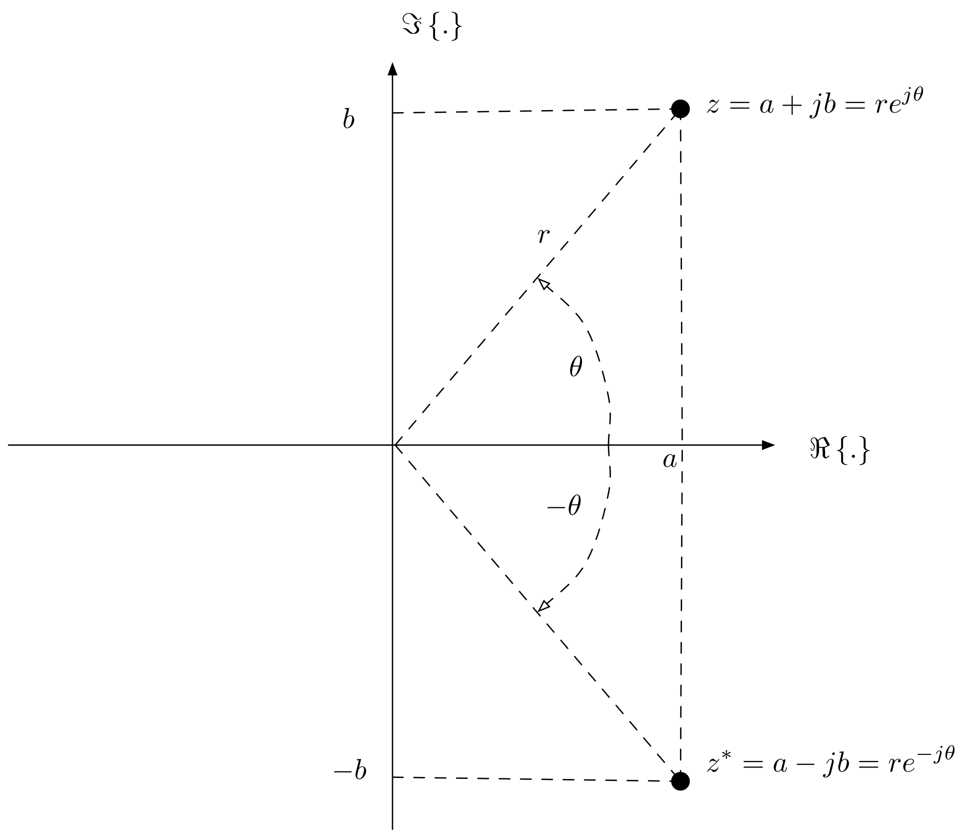

Figure 1: Geometric interpretation of z and z^*.

The *real part* of the complex number z in (1) is a and the *imaginary* part of z in (1) is b. Note that both the real and imaginary parts of z are real numbers. The real-part and imaginary-part operators take a complex number and return the corresponding real and imaginary parts, respectively. It is straightforward to show that these operators are defined as

$$\Re\{z\} = \frac{z+z^*}{2} \tag{3a}$$

$$\Im\{z\} = \frac{z-z^*}{2j} \tag{3b}$$

Since a complex number z can be viewed as a point in a plane, it can also be expressed in polar coordinates. In other words, the complex number in (1) can also be expressed as

$$z = re^{j\theta} \tag{4}$$

where r is a real, non-negative number and θ is a real number.[1] The geometrical interpretation of r and θ are given in (1) as well. Specifically, r is the length from the origin and θ is the angle from the real axis. In terms of the complex number z, r is written as $|z|$ and referred to as the *magnitude* of z. Similarly, θ is written as $\angle z$ and referred to as the *angle* or *argument* of z. Thus, any complex number can be written as

$$z = |z|e^{j\angle z} = \Re\{z\} + j\Im\{z\} \tag{5}$$

The magnitude of any complex number can be obtained by

$$|z|^2 = zz^* = (\Re\{z\})^2 + (\Im\{z\})^2 \tag{6}$$

The angle can be identified via

$$e^{j\angle z} = \frac{z}{|z|} \tag{7}$$

Conversion from Cartesian to magnitude-phase is performed via a standard rectangular-to-polar transformation

$$|z| = \sqrt{(\Re\{z\})^2 + (\Im\{z\})^2} \tag{8}$$

$$\angle z = \tan^{-1}(\Im\{z\}/\Re\{z\}) \tag{9}$$

The reverse conversion rule is

$$\Re\{z\} = |z|\cos\angle z \tag{10a}$$

$$\Im\{z\} = |z|\sin\angle z \tag{10b}$$

For example, consider $z = -4+4j$. This complex number has

$$\Re\{z\} = -4 \tag{11}$$

$$\Im\{z\} = +4 \tag{12}$$

$$|z| = 4\sqrt{2} \tag{13}$$

$$\angle z = 3\pi/4 \tag{14}$$

[1] It will be come clear that θ is only unique on an interval of length 2π, so one may consider the restriction that $\theta \in [0, 2\pi)$.

1.1 Useful Identities and Properties

All manipulations of complex number follow from the basic definitions above, but it is worth listing a few of the common identities used as below (the reader is encouraged to prove each of these). In each case, w and z are complex numbers and a is a real constant.

$$(wz)^* = w^* z^* \tag{15a}$$
$$|wz| = |w||z| \tag{15b}$$
$$\frac{1}{z} = \frac{1}{|z|} e^{-j\angle z} \tag{15c}$$
$$|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\Re\{wz^*\} \tag{15d}$$
$$= |z|^2 + |w|^2 \pm 2|w||z| \cos(\angle w - \angle z) \tag{15e}$$
$$\Re\{az\} = a\Re\{z\} \tag{15f}$$
$$\Im\{az\} = a\Im\{z\} \tag{15g}$$
$$|az| = |a||z| \tag{15h}$$
$$\angle(az) = \angle z + \frac{\pi}{2}(1 - \mathrm{sgn}(a)) \tag{15i}$$

where $\mathrm{sgn}(a)$ is $+1$ if $a \geq 0$ and -1 if $a < 0$ – note that this simply adds π to the angle if a is negative.

Unit magnitude complex numbers, which can be expressed as $z = e^{j\theta}$, are an important special case of complex numbers. These unit magnitude complex numbers are all located on the unit circle, with angle to the real axis given by θ. A special case of the relations in (3) and (10) yields

$$\Re\left\{e^{j\theta}\right\} = \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{16a}$$
$$\Im\left\{e^{j\theta}\right\} = \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{16b}$$
$$e^{j\theta} = \cos\theta + j\sin\theta \tag{16c}$$

The equation in (16c) is referred to as Euler's equation.

2 Complex Exponentials

A complex signal (or function), $z(t)$, is a signal that takes on complex values at a given time. Thus, for each value of t, the quantities $\Re\{z(t)\}$, $\Im\{z(t)\}$, $|z(t)|$, and $\angle z(t)$ are defined. Alternatively, thinking of t as varying, these can be seen as real-valued functions derived from the complex function $z(t)$. Complex functions of various sorts will be encountered in EE 301, but a particularly important type of complex signal is the complex exponential, which is reviewed in detail below.

2.1 Continuous Time Complex Exponentials

Consider the complex exponential signal

$$z(t) = Ce^{st} \tag{17}$$

where t is a real number and s and C are complex constants. In order to observe the qualitative behavior of this signal for various C and s, it is useful to express s in Cartesian coordinates as $s = \sigma + j\omega$ and C in mag-phase format so that

$$z(t) = |C|e^{j\angle C}e^{(\sigma+j\omega)t} \tag{18a}$$
$$= |C|e^{\sigma t}e^{j(\omega t+\angle C)} \tag{18b}$$
$$= |C|e^{\sigma t}\left[\cos(\omega t + \angle C) + j\sin(\omega t + \angle C)\right] \tag{18c}$$

It is helpful to consider different special cases of this signal to gain intuition on the general form above. First, consider the case of $C = 1$ and $\omega = 0$ so that

$$z(t) = e^{\sigma t} \tag{19}$$

This signal decays to right (*i.e.,* as t increases) if $\sigma < 0$. If $\sigma > 0$, then it decays to the left. In both cases, the signal diverges in the opposite direction of the decay. Also, note that the rate of decay increases as $|\sigma|$ increases.

Next consider the case of $\sigma = 0$ and $C = 1$ so that

$$z(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t \tag{20}$$

Thus, for this special case, the complex exponential oscillates at a frequency of ω rad/sec in a sinusoidal fashion in both real and imaginary parts.

This signal plays an important role in Fourier analysis, so it deserves a closer look. At any time t, $z(t)$ is a point on the unit circle in the complex plane and the expression in (20) is just that of (16c) with $\theta = \omega t$. Assuming $\omega > 0$, this point moves counter-clockwise around the unit circle as t increases. This concept is illustrated in Fig. 2. Note that for $\omega < 0$, the rotation is clockwise. The signal in (20) is at the point $(1, 0)$ in the complex plane when $t = 0$. It takes $T_0 = 2\pi/\omega$ seconds to make one cycle around the unit circle. Thus, the *linear frequency* of this signal is $f = 1/T_0 = \omega/(2\pi)$ cycles/sec, or Hertz (Hz). An equivalent measure of frequency is given by ω itself which measures the angular distance traveled per sec. Thus, ω is the *angular frequency* in rad/sec. Angular frequency is commonly used in the signal processing literature while linear frequency is often adopted in the communications field. Thus, the larger $\omega = |\Im\{s\}|$, the faster the oscillation.

When both σ and ω are nonzero, a combination of oscillation at frequency ω with exponential decay at rate σ is observed as can be seen in (18c). The multiplication by a complex constant C results in an amplitude gain by $|C|$ and a phase shift by $\angle C$. Fig. 3 summarizes the behavior of the general complex exponential by indicating the regions in the complex plane where the values of s result in qualitatively different signal behavior.

2.2 Discrete Time Complex Exponentials

A discrete time complex exponential can also be defined as $w[n] = Ce^{sn}$, where C and s are complex constants. The convention in discrete time, however, is to consider e^s as another complex constant (*i.e.,* $z = e^s$) so that

$$w[n] = Cz^n \tag{21}$$

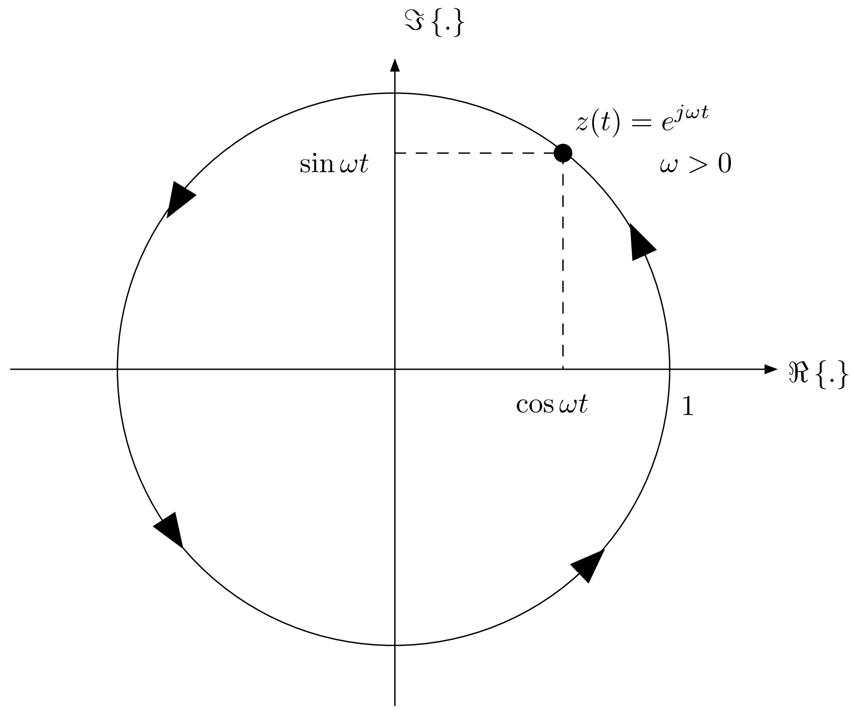

Figure 2: The signal $e^{j\omega t}$ spins counterclockwise around the unit circle at speed $\omega > 0$ rad/sec.

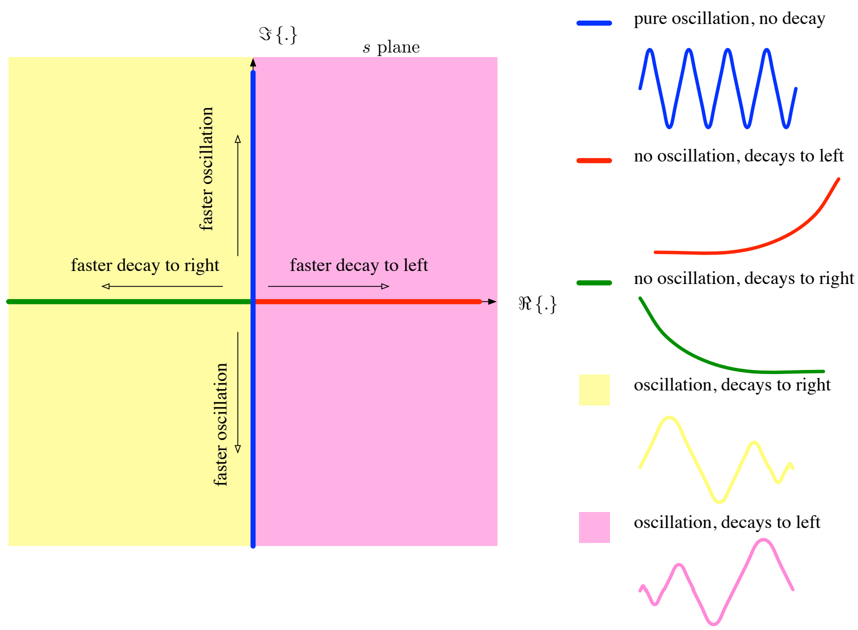

Figure 3: The qualitative behavior for the signal e^{st} varies as s moves in the complex plane.

is a general complex, discrete time, exponential. In this case it is helpful to express both C and z in magnitude phase format to determine the behavior of the signal for various values of these constants. Specifically, let $z = re^{j\Omega}$ so that

$$w[n] = |C|e^{j\angle C}\left(re^{j\Omega}\right)^n \tag{22a}$$
$$= |C|r^n e^{j(\Omega n + \angle C)} \tag{22b}$$
$$= |C|r^n\left[\cos(\Omega n + \angle C) + j\sin(\Omega n + \angle C)\right] \tag{22c}$$

Again, it is useful to consider special cases to gain intuition about the behavior of the general complex exponential. First, consider the case of $C = 1$ and $\Omega = 0$ so that $w[n] = r^n$ where $r = |z| \geq 0$. Note that for $0 < r < 1$, this signal decays exponentially to the right and for $r > 1$, it decays exponentially to the left. As r approaches 1 from either the left or the right, the rate of this decay decreases and for $r = 1$, the signal is a constant that does not decay.

Next consider the case of $\Omega = \pi$ and $C = 1$ so that

$$w[n] = r^n \cos(\pi n) = (-1)^n r^n = (-r)^n \tag{23}$$

This signal has behavior similar to the signal of $\Omega = 0$, except the sign of the signal alternates at each sample time. Specifically, for $0 < r < 1$ (*i.e.,* $-1 < z < 0$), this signal decays exponentially to the right and for $r > 1$ (*i.e.,* $-z < -1$), it decays exponentially to the left, alternating signs in each case.

Next consider the case of $C = 1$ and $r = 1$, so that

$$w[n] = e^{j\Omega n} = \cos \Omega n + j \sin \Omega n \tag{24}$$

which is analogous to the continuous time signal in (20). Since n only takes integer values, the angle of $w[n]$ increments by Ω at each sample time. For example, if $\Omega = \pi/4$, then the angle will be 0 at $n = 0$, $\pi/4$ at $n = 1$, $\pi/2$ for $n = 2$, etc. Note that Ω is therefore a measure of speed of angular rotation in radians/sample. In fact, Ω is called *normalized angular frequency* or *discrete time angular frequency* and has units of rad/sample, often shortened to radians. As in continuous time, one may also consider a linear frequency variable that measures the fraction of a cycle incremented at each sample. Specifically, define $\nu = \Omega/(2\pi)$ as the *normalized linear frequency* with units of cycles/sample (often referred to as unit-less).

Discrete time frequency differs dramatically from continuous time frequency. One important difference is that while continuous time frequency is unique for any real ω, discrete time frequency is only unique for Ω on an interval of length 2π. This is because

$$e^{j(\Omega+2\pi)n} = e^{j\Omega n} e^{j2\pi n} = e^{j\Omega n} \tag{25}$$

so that the frequencies Ω and $\Omega + k(2\pi)$ are the same in discrete time. This means, for example, that $\Omega = 9\pi/4$ is equivalent to $\Omega = \pi/4$. Consider $w[n] = e^{(9\pi/4)n}$. For $n = 0$, this has angle 0. For $n = 1$ it has angle $9\pi/4$ which is equivalent to $\pi/4$. Incrementing n by one causes an increase of $2\pi + \pi/4$ in the angle which is equivalent to an increment of $\pi/4$. One may imagine that for $\Omega = 9\pi/4$, the signal makes an "extra lap" around the unit circle between sample times, but the net effect is the same for the discrete time signal value. This concept is illustrated in Fig. 4, where $\cos((9\pi/4)t)$ and $\cos((\pi/4)t)$ are plotted versus the continuous variable t and the values for integer

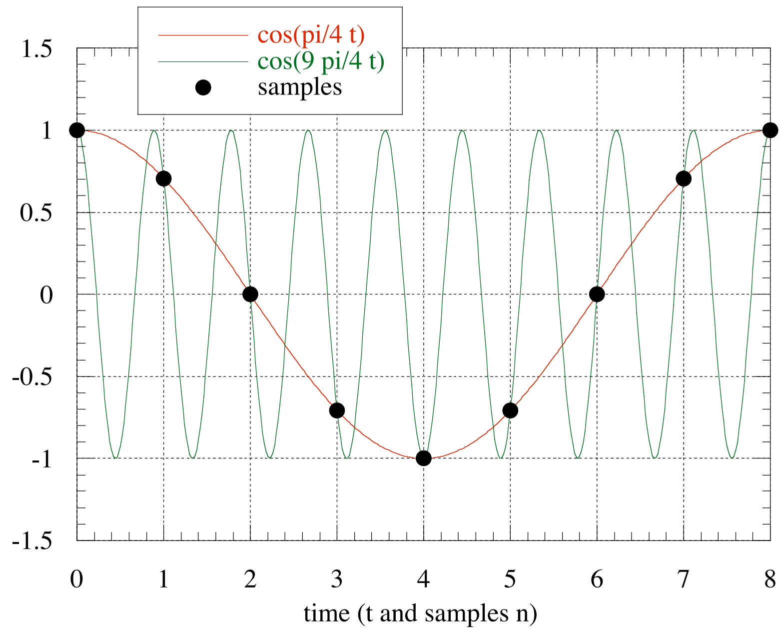

Figure 4: Normalized frequency Ω is unique only an interval of length 2π (*e.g.,* $[-\pi, +\pi)$ because of this aliasing effect.

values of $t = n$ have been noted to be the same. This concept foreshadows aliasing or the strobe-light effect where a higher frequency appears as a lower frequency when sampled at a specific rate.[2] Another consequence of this is that the highest normalized frequency is $\Omega = \pi$, which is the case of alternating signs at each sample. For example, $\Omega = 5\pi/4$ is equivalent to $\Omega = -3\pi/4$ and results in a slower rate of oscillation than $\Omega = \pi$.

Another difference between the continuous time signal $z(t) = e^{j\omega t}$ and the discrete time signal $w[n] = e^{j\Omega n}$ is that the former is periodic for any choice of ω, but the latter is not period for every Ω. Specifically, $z(t + T_0) = z(t)$ where $T_0 = 2\pi/\omega$ is the period (*i.e.,* the time required to complete one cycle of the unit circle). The condition for $w[n]$ to be periodic is

$$w[n] = e^{j\Omega n} = w[n + N_0] = e^{j\Omega(n+N_0)} = e^{j\Omega n} e^{j\Omega N_0} \tag{26}$$

which requires

$$e^{j\Omega N_0} = 1 \tag{27}$$

or, equivalently, $\Omega N_0 = m(2\pi)$ for some integer m. It follows that $w[n] = e^{j\Omega n}$ is periodic if and only if Ω is a rational multiple of 2π. For example, $\Omega = \pi/4$ yields a period of $N_0 = 8$ while $\Omega = 1$

[2]This is also referred to as the "wagon-wheel effect" from the observation in old western movies of the wagon wheel slowing or stopping on the film even as the wagon speeds along. This is because the film is taking a fixed number of pictures per second and only catches the position of the wheel at those instants. For example, if the speed of the wheel is just right, the wheel position can be the same at each snapshot giving the illusions of a stationary wheel even though it is spinning quite rapidly in reality.

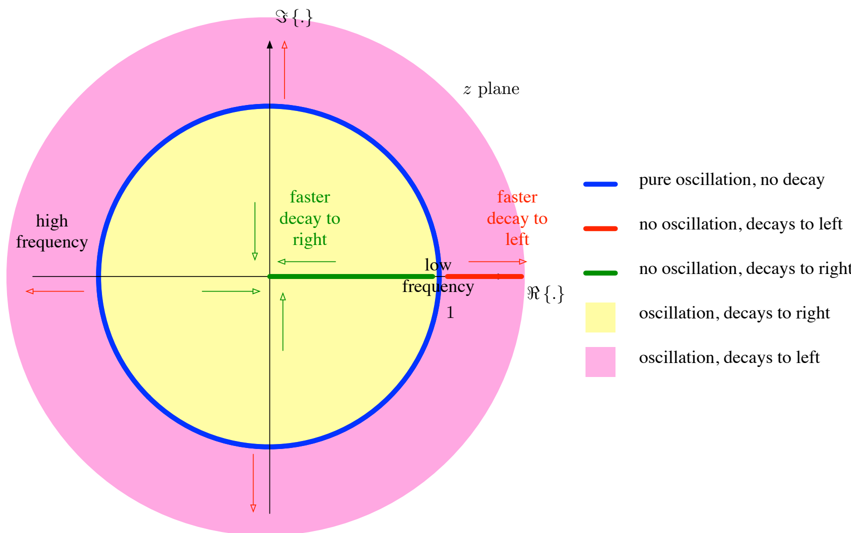

Figure 5: The qualitative behavior for the signal z^n varies as z moves in the complex plane.

results in an aperiodic signal. This may seem intuitively odd, but the explanation is that as the phase of the signal is incremented around the unit circle, the signal laps the unit circle again an again, but never lands on the same point twice. For example, with $\Omega = 1$, the point e^j only occurs at $n = 1$ and is never revisited.

Finally, the general case of the complex exponential in (22) is a mixture of oscillation and exponential decay. If the value of z lies within the unit circle, the signal will decay to the right while it decays to the left of $|z| > 1$. The frequency of oscillation is $\Omega = \angle z$ with the special cases of $\Omega = 0$ and $\Omega = \pi$ considered above. The rate of exponential decay is faster as the value of z becomes further from the unit circle and when z is on the unit circle, no decay occurs – the signal oscillates forever at frequency $\Omega = \angle z$. This behavior is summarized in Fig. 5. Note that the same behaviors occur in both Figs. 3 and 5, but the regions differ due to the different convention for defining exponential signals in continuous and discrete time.

2.3 Symmetry Measures for Signals

It is useful to characterize odd and even signals, defined as follows

$$z(t) = z(-t) \qquad \text{Even Function} \tag{28a}$$
$$z(t) = -z(-t) \qquad \text{Odd Function} \tag{28b}$$

Prototypical examples of even and odd functions are cosines and sines, respectively. In general, functions may be neither even nor odd, but it is often useful to exploit the even or odd properties of signals, for example, to simplify integrals. For example, the integral of an odd function over a symmetric interval around the origin is zero.

Much like the real and imaginary operators, it is useful to define operators to find the even and odd parts of a given signal. These operators are defined by

$$\mathbb{EV}\{z(t)\} = \frac{z(t) + z(-t)}{2} \tag{29a}$$

$$\mathbb{ODD}\{z(t)\} = \frac{z(t) - z(-t)}{2} \tag{29b}$$

Note that this provides an even-odd decomposition of an arbitrary signal

$$z(t) = \mathbb{EV}\{z(t)\} + \mathbb{ODD}\{z(t)\} \tag{30}$$

Also note that the even part of any even function is that function itself and the even part of an odd function is zero with similar properties holding for the odd part operator.

The concepts of even and odd are meaningful for complex signals but most useful for real signals. An analogous property for complex signals is *Hermitian Symmetry*[3] defined by

$$z(t) = z^*(-t) \qquad \text{Hermitian Symmetric (HS) Function} \tag{31a}$$

$$z(t) = -z^*(-t) \qquad \text{Hermitian Anti-Symmetric (HAS) Function} \tag{31b}$$

The HS and HAS part operators can therefore be defined as

$$\mathbb{HS}\{z(t)\} = \frac{z(t) + z^*(-t)}{2} \tag{32a}$$

$$\mathbb{HAS}\{z(t)\} = \frac{z(t) - z^*(-t)}{2} \tag{32b}$$

Note that this provides an HS-HAS decomposition of an arbitrary signal

$$z(t) = \mathbb{HS}\{z(t)\} + \mathbb{HAS}\{z(t)\} \tag{33}$$

Also note that the HS-part of a HS signal is that signal itself and the HS-part of a HAS function is zero.

Hermitian symmetry is a compact way of capturing symmetry properties of complex signals. For example, we shall see that when a real time domain signal is viewed in the frequency domain, it is Hermitian Symmetric.

3 Exercises

1. Express $z = 4 + j2$ is magnitude-phase form. Show z, z^*, $1/z$ and $1/z^*$ in the complex plane.
2. Express $z = \frac{1}{2}e^{-j\pi/3}$ in Cartesian coordinates. Show z, z^*, $1/z$ and $1/z^*$ in the complex plane.
3. Simplify $\sqrt{j}$.
4. Consider the quantity $e^{j\frac{2\pi}{N}kn}$ for k and n both taking values in $\{0, 1, 2, \ldots N-1\}$.
   (a) Show that $e^{j\frac{2\pi}{N}kn} = e^{j\frac{2\pi}{N}(kn)_{\mathrm{mod}\,N}}$ where $(kn)_{\mathrm{mod}\,N}$ is kn modulo N.

[3] Also referred to as conjugate symmetry.

(b) For $N = 6$ how many distinct values of $e^{j\frac{2\pi}{N}kn}$ are there? Make a table of the 36 entries for all k and n and enter the simplified value of $e^{j\frac{2\pi}{N}kn}$.

5. Simplify and plot $\Re\left\{(2+j)e^{(-4+3j)t}/(1-j)\right\}$.

6. Verify equations (30) and (33) hold in general.

7. Show that if $z(t)$ is even, then

$$\int_{-T}^{+T} z(t)dt = 2\int_{0}^{T} z(t)dt$$

8. Show that for $z(t)$ odd,

$$\int_{-T}^{+T} z(t)dt = 0$$

9. Suppose $x(t)$ is even and $y(t)$ is odd. Find the even and odd parts of $z(t) = x(t)y(t)$ and simplify $\int_{-T}^{+T} z(t)dt$.

10. Show that $z(t)$ is HS if and only if $|z(t)|$ is even and $\angle z(t)$ is odd.

11. Show that $z(t)$ is HS if and only if $\Re\{z(t)\}$ is even and $\Im\{z(t)\}$ is odd.

12. If a signal is real and HS, what can be said about it?

13. If a signal is imaginary and HS, what can be said about it?

14. Consider the following function

$$g(\Omega) = \frac{1}{1+(1/2)e^{-j\Omega}}$$

(a) Is this function HS? Is it HAS?
(b) Find $|g(\Omega)|^2$
(c) Find $\angle g(\Omega)$

15. Consider the following function

$$g(\Omega) = \frac{1}{1+(1/2)(1+j)e^{-j\Omega}}$$

(a) Is this function HS? Is it HAS?
(b) Find $|g(\Omega)|^2$
(c) Find $\angle g(\Omega)$

16. Consider the following function

$$g(\omega) = \frac{1}{10+j\omega}$$

(a) Is this function HS? Is it HAS?

(b) Find $|g(\omega)|^2$

(c) Find $\angle g(\omega)$

17. Consider the following function

$$g(\omega) = \frac{1}{(10 - 10j) + j\omega}$$

(a) Is this function HS? Is it HAS?

(b) Find $|g(\omega)|^2$

(c) Find $\angle g(\omega)$

18. Find the HS and HAS parts of $z(t) = e^{j\omega t}$.