Exploiting the Law of Large Numbers in Engineering

ENGR 102

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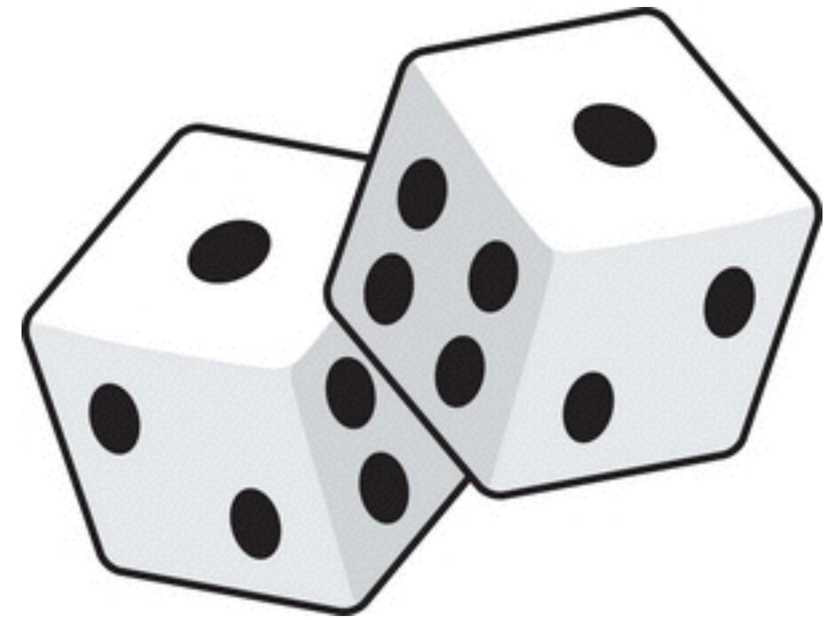
Overview

- Explore the concept of the Law of Large Numbers through "games of chance"
- How do we exploit the LLN in
  - design?
  - analysis?
- Humans and randomness
  - "Law of Averages" and Gambler's Fallacy

Game 1



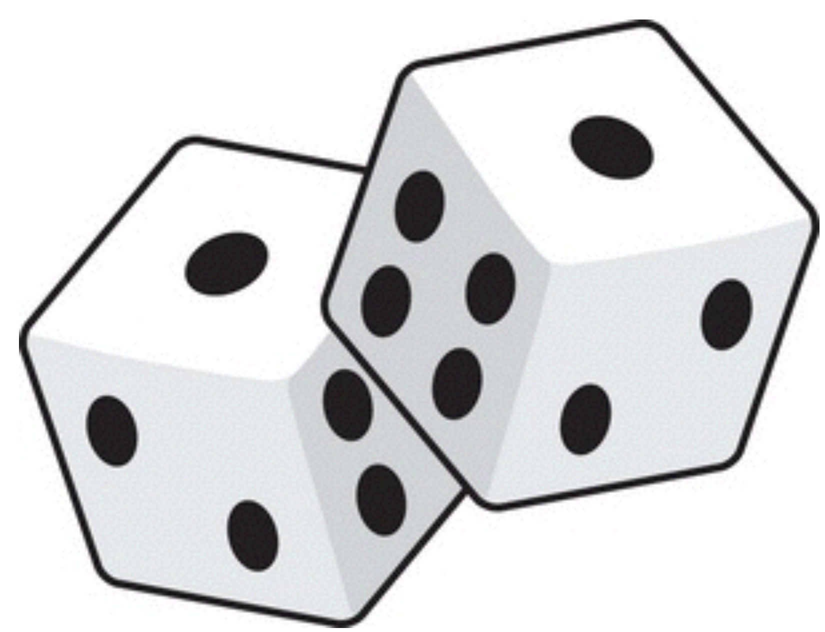
- Roll 10 fair dice
  - “win” is when a {1,2,3,4} is rolled
  - Your group has 10 pieces of candy
  - “bet” your candy as you like on some, none, or all of these rolls.
  - “even money” return (lose your candy or double it)

Game 1 - Results and Discussion

- Explain your strategy…
- Does this match the results?

Game 2

- Same as Game 1 with one change
  - “win” is when a {1,2,3} is rolled



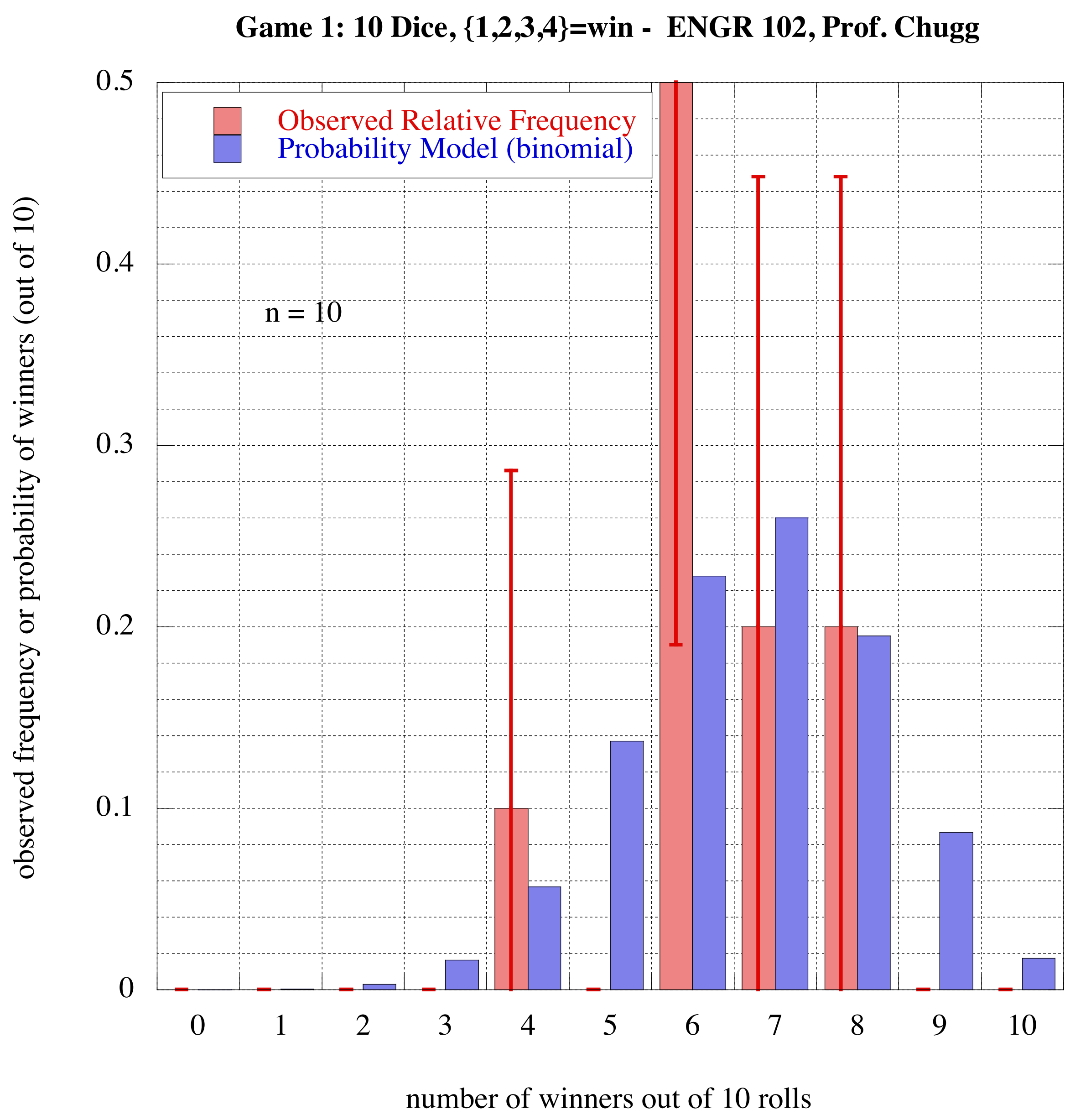
Game 2 - Results and Discussion

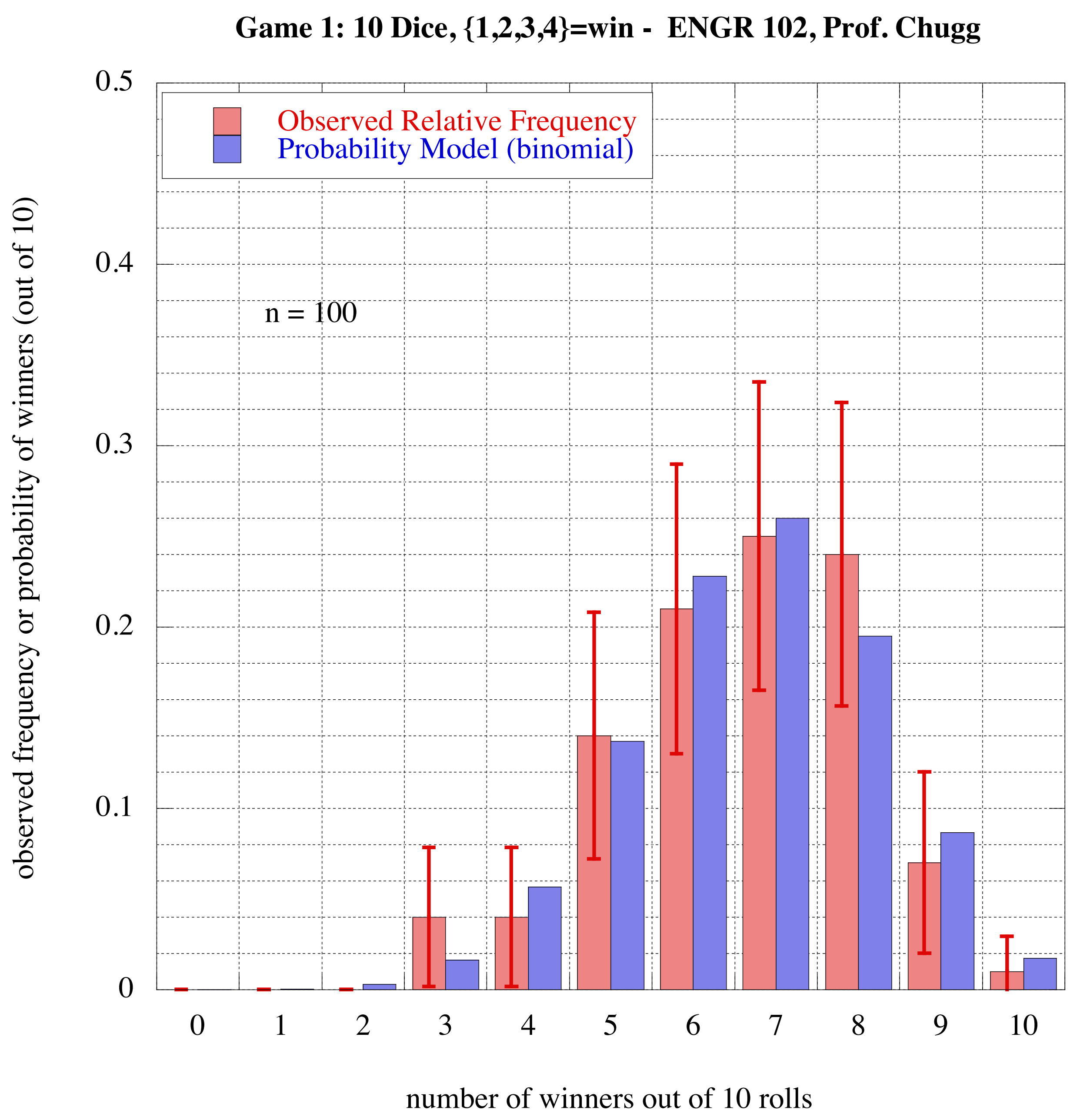
- Explain your strategy…
- Does this match the results?

Analysis of these Games

- Primary question: is it better to
  - Put all your candy on one trial, OR
  - Split your money equally between all of the trails?
- What if we repeated this game 10 times? 100 times? 100,000 times?

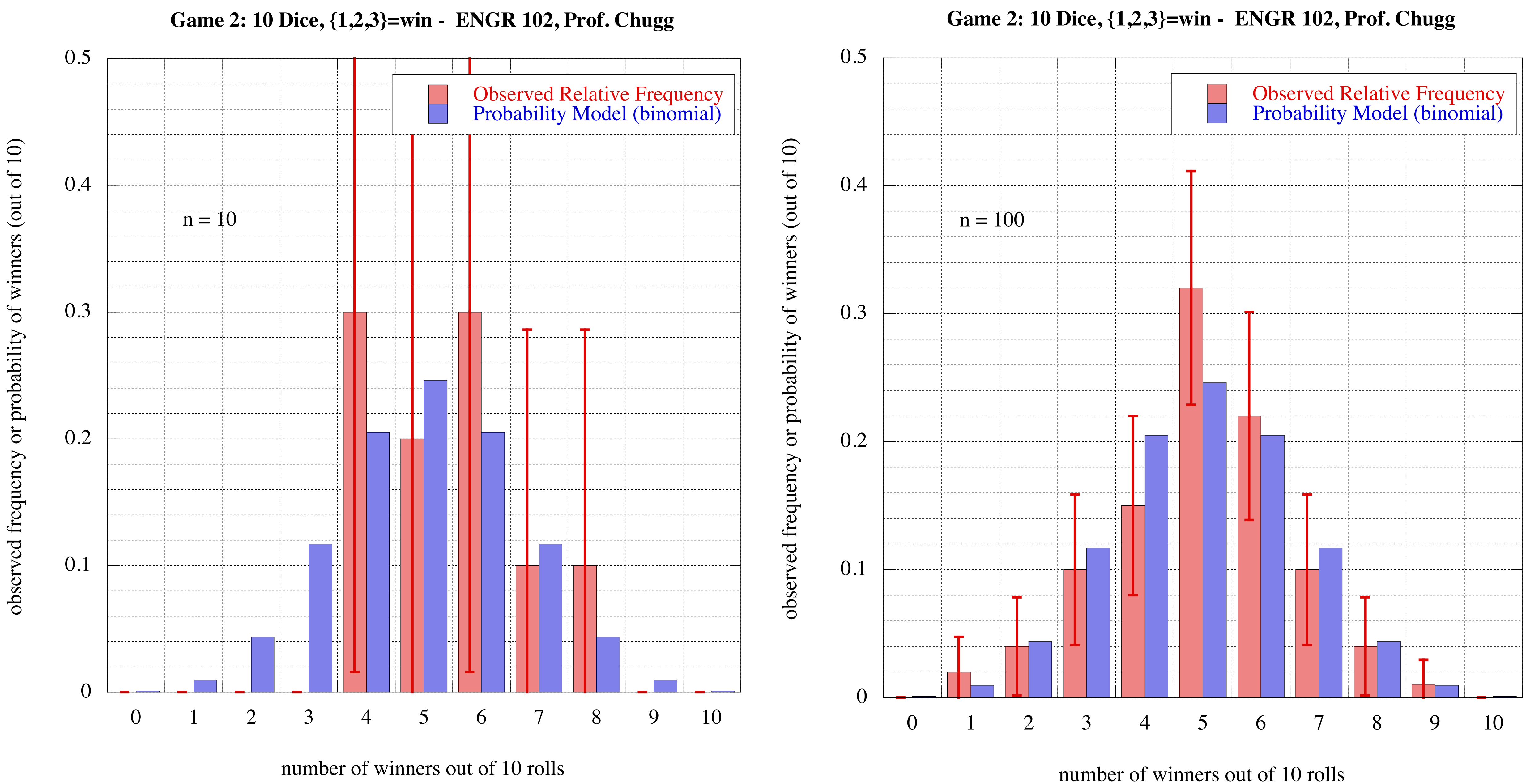
Analysis of these Game 1





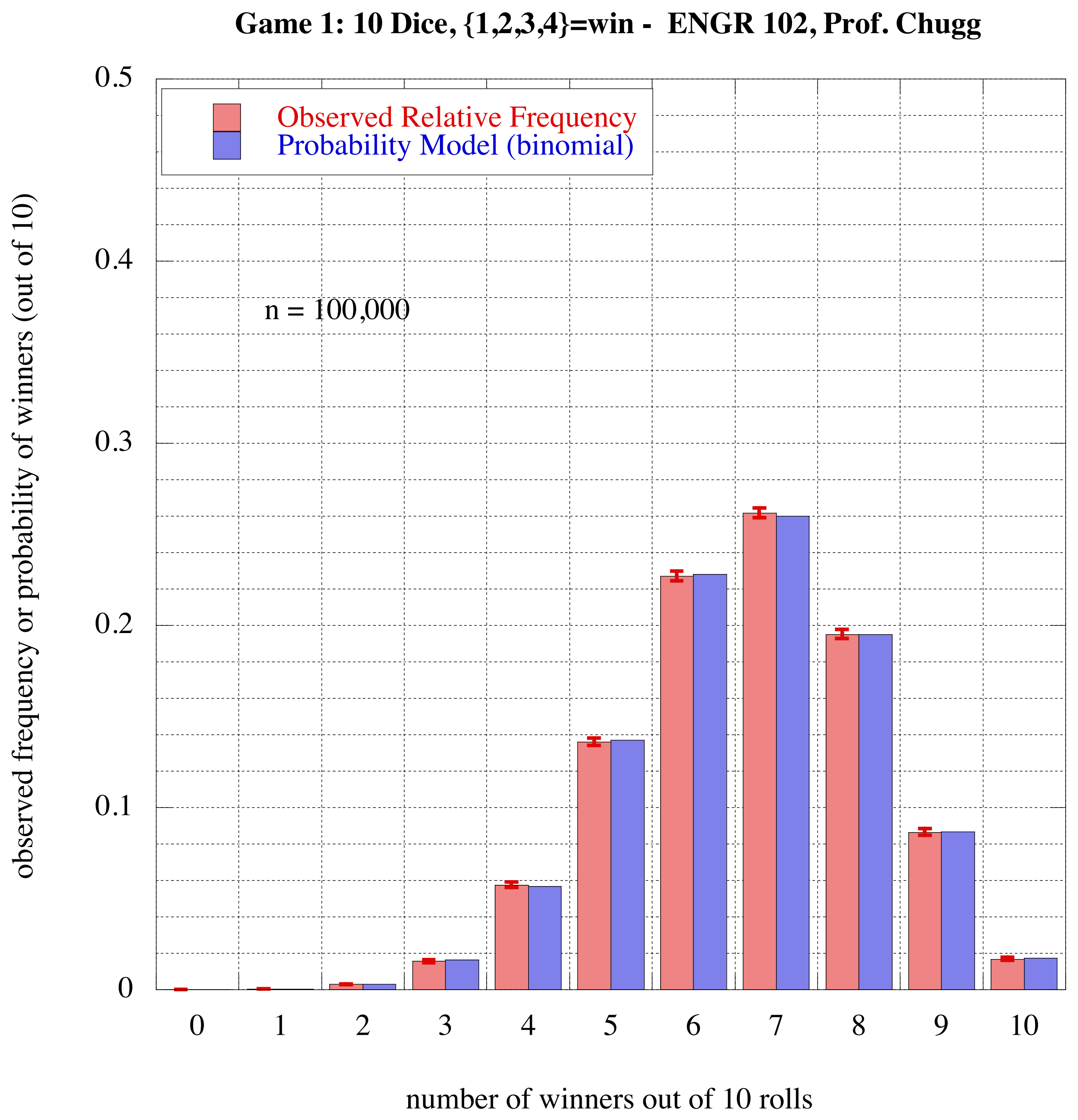
- Used a computer (Python) to play game many times

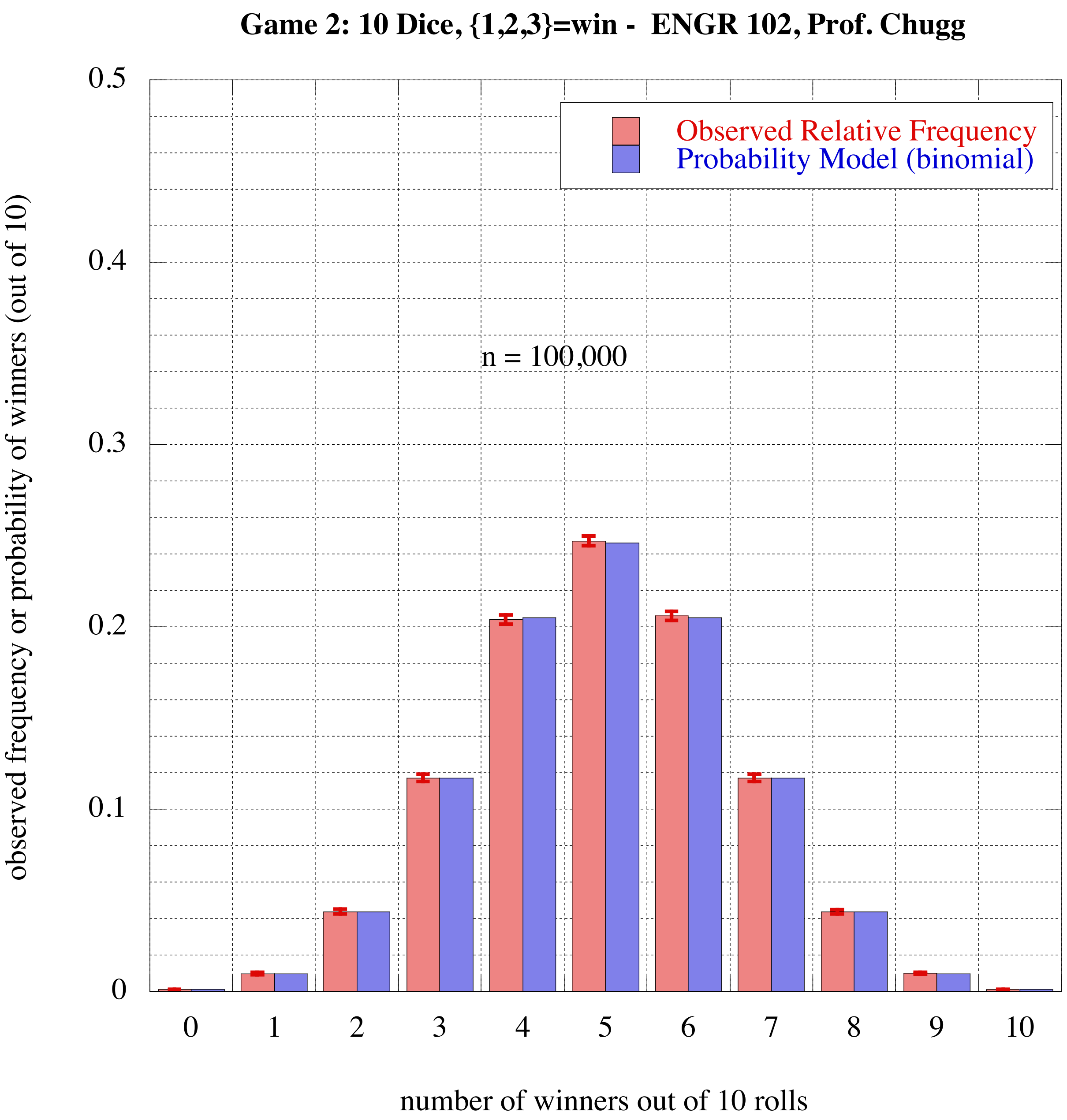
Analysis of these Game 2



- Used a computer (Python) to play game many times

Comparison of Games 1 & 2





- LLN implies that relative frequencies observed will converge to actual probabilities

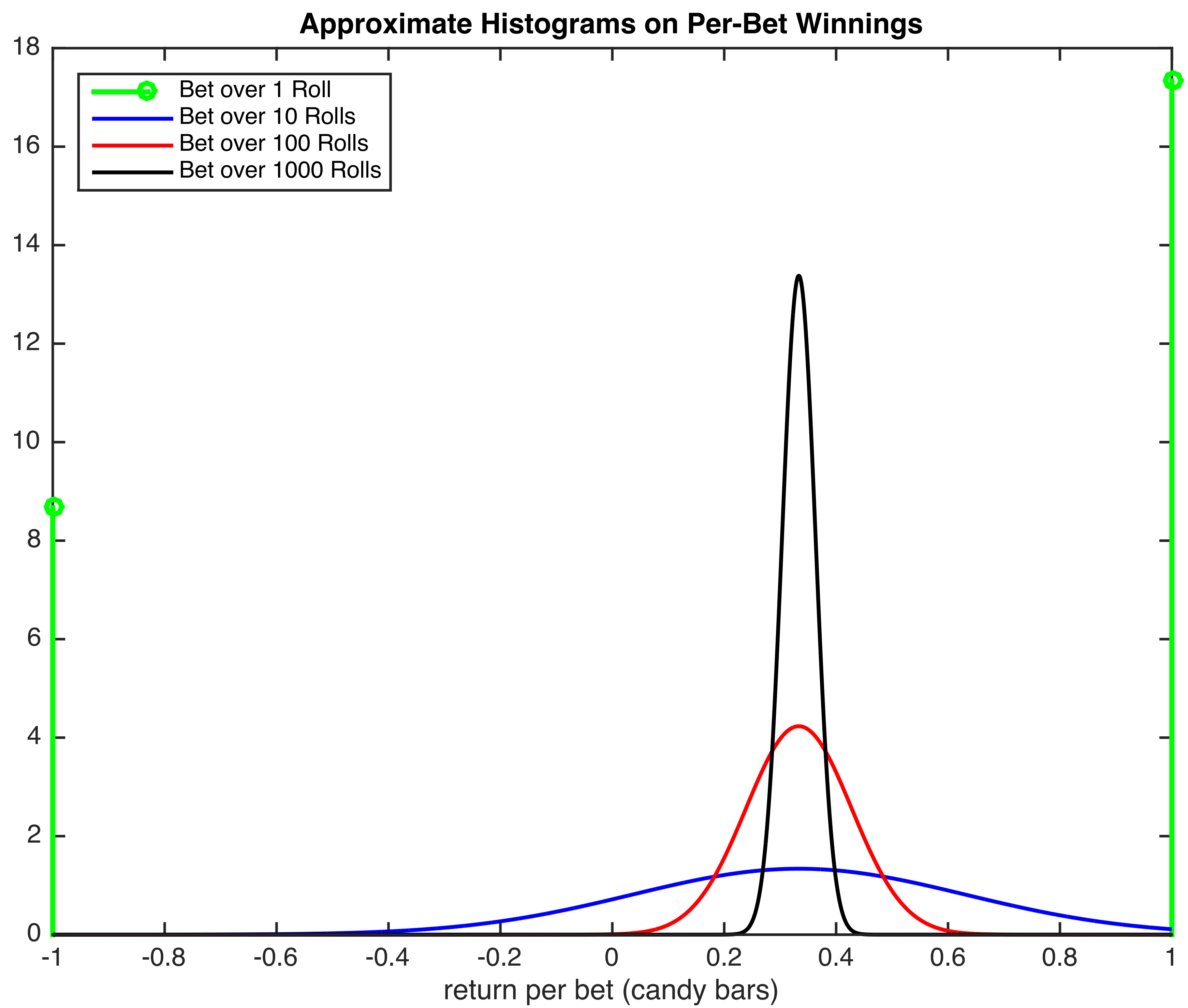
Analysis of these Games

- What if we equally spread our bets over K dice rolls?
  - We used K=10
  - What happens as K increases?

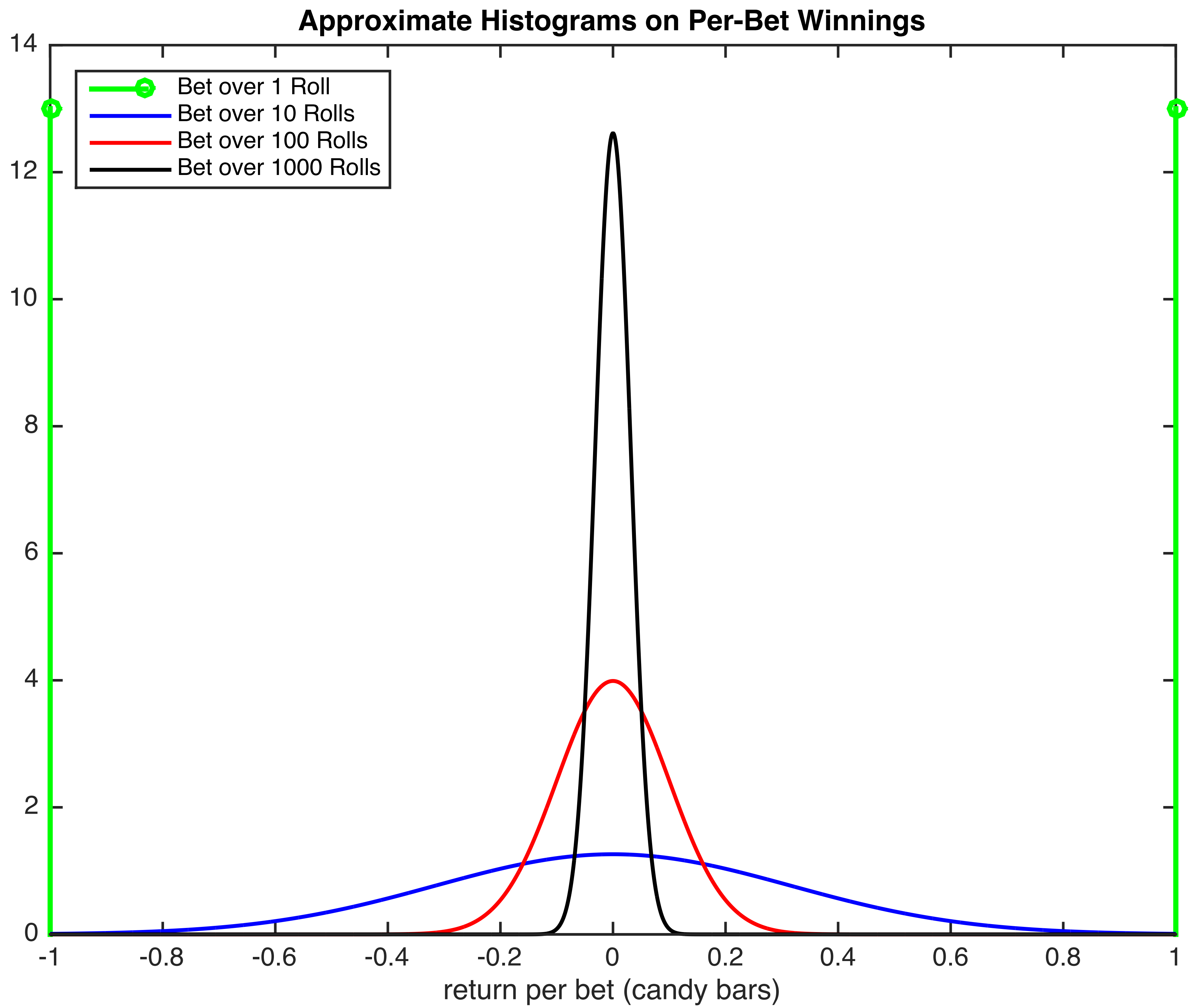
$$B = \frac{\text{net candy bars won}}{\text{number of candy bars bet}}$$

- If we fix K and repeat this game many times, what would a histogram of B look like?

Game 1: Histogram of B



Game 2: Histogram of B



Bell Curve: aka, Gaussian or Normal

Gauss and his bell curve were on the German 10 Mark bill (pre-Euro)

Intuitive Takeaways

- Law of Large Numbers
  - *The more trials we average over, the less variation we will see — in the limit, this converges*
- Central Limit Theorem
  - *For large n, the histogram looks like a bell curve around the eventual limit*
- Stats
  - *Our error in estimation improves like* $\frac{1}{\sqrt{N}}$

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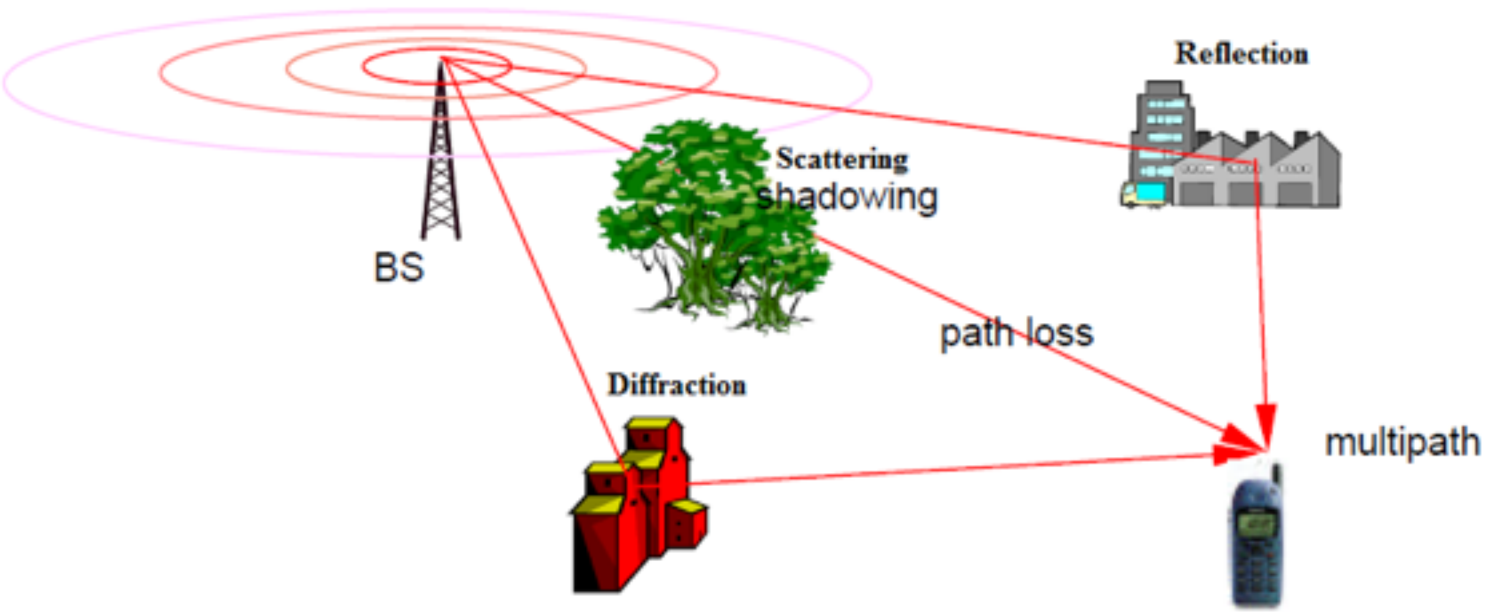
Diversity: Design Exploiting the LLN

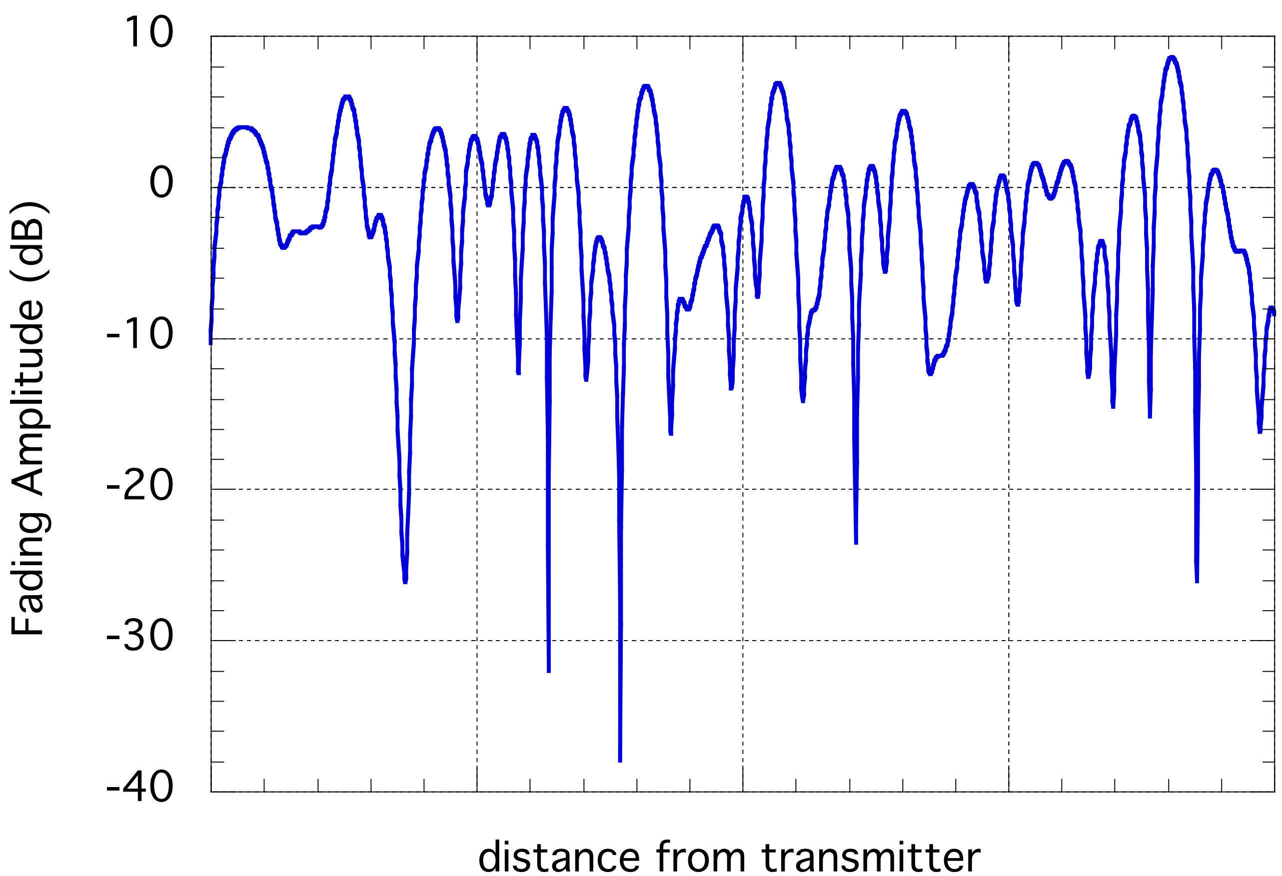
- The "spread it out" betting strategy is an example of diversity
  - Diversity: spread risks out over many (independent) random quantities
- Used in many engineering designs
  - Finance: mutual funds
  - Wireless Communications: multiple antennas
  - Fault tolerant systems (e.g., data center storage)

Example: Wireless Communications

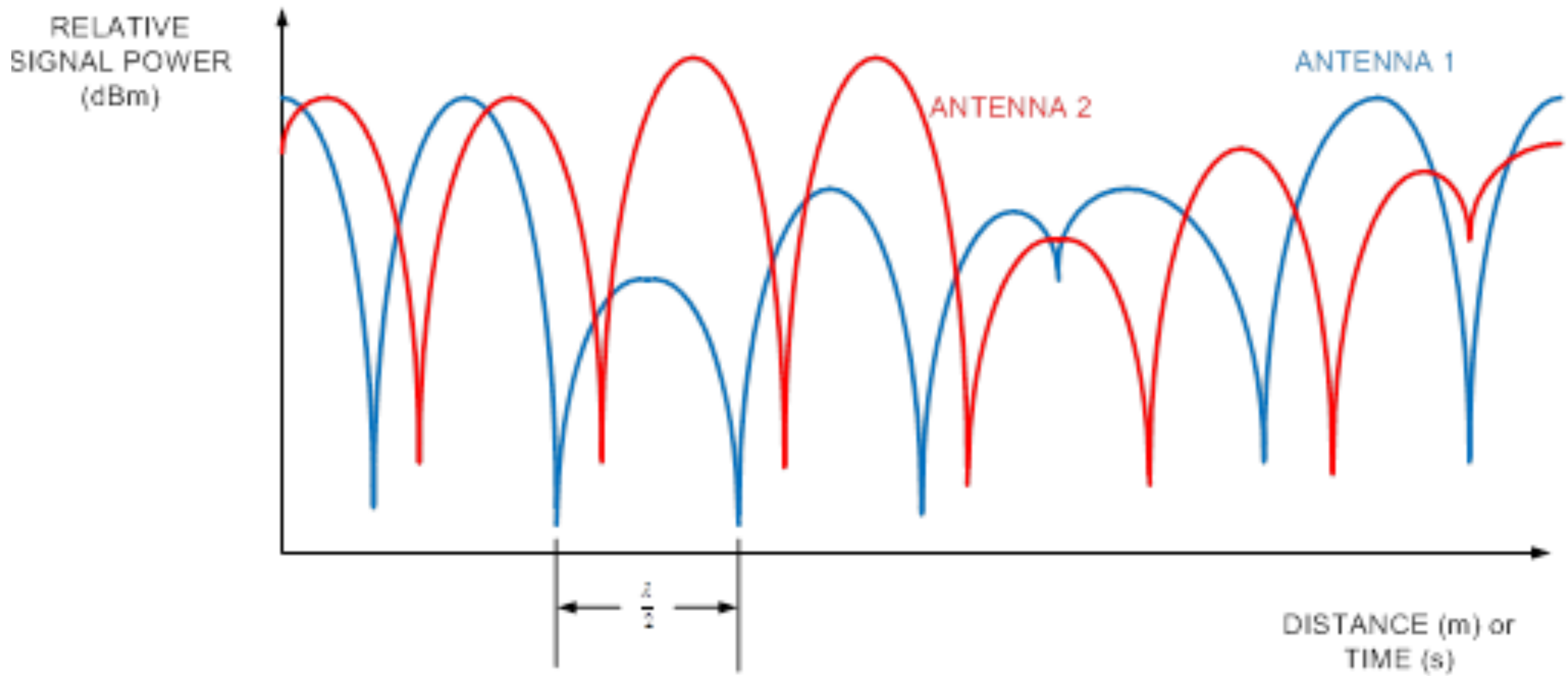


Multipath propagation

Signal level fluctuates greatly with distance from the transmitter



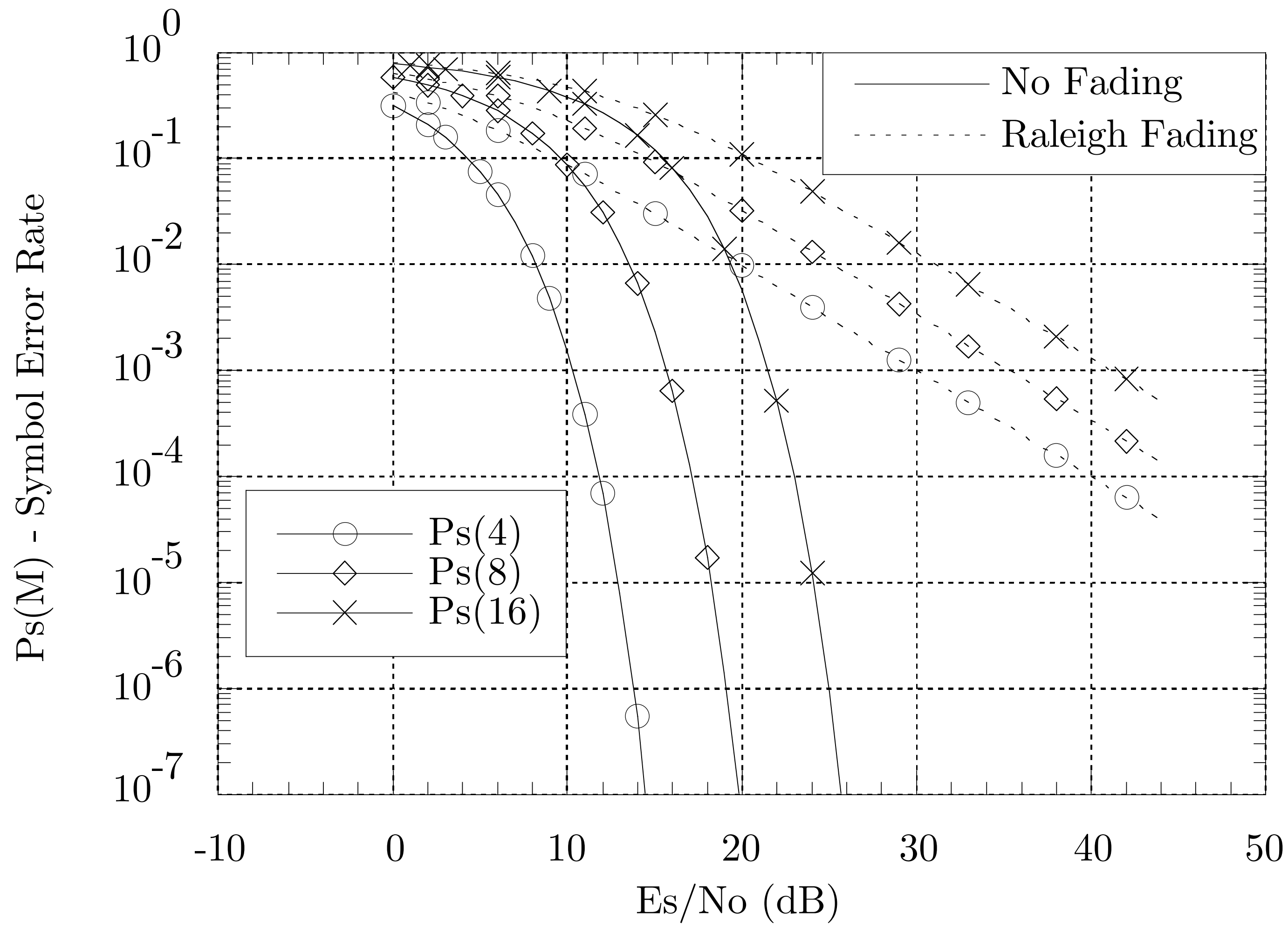
Example: Wireless Communications



2 Antennas for Spatial Diversity

Other approaches to get diversity?

Example: Wireless Communications



- **Intuition:** worst case dominates!

$$\alpha 10^{-1} + (1-\alpha)10^{-6} \cong \alpha 10^{-1} \gg 10^{-6}$$

Fading is a KILLER… Diversity is only way to mitigate it

Example: Calibrating Clocks

Two Digital Clocks (do not display seconds)

	clock A	clock B
Sometimes when you check, Clock A leads Clock B	12:26	12:25
Sometimes when you check, Clock A agrees with Clock B	3:14	3:14

Example: Calibrating Clocks

Engineering Problem:

Design:

Can you use the idea of the Law of Large Numbers to estimate the offset, in seconds, by which Clock A leads Clock B from a sequence of randomly sampled observations (i.e., each observation is "agree" or "disagree")

Analysis:

Can you assess the accuracy of your calibration method (maybe by running experiments and/or simulating experiments on a computer)?

Homework 1

1. Investigate the Clock Calibration problem on the previous slide
2. Describe an example of the Law of Large Numbers, the diversity design principle, and/or the Central Limit Theorem that you encountered during the week.